J. of Difference Equations and Applications, 2002, Vol. 8(1), pp. 61 – 73

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Geometric Transient Solutions of Autonomous Scalar Maps

SHANDELLE M. HENSON [a,*] and J. M. CUSHING [b,†]

[a]*Department of Mathematics, College of William and Mary, Williamsburg, VA 23187;* [b]*Department of Mathematics and Program in Applied Mathematics, University of Arizona, Tucson, AZ 85721*

(Received 21 November 2000; In final form 6 December 2000)

Smooth autonomous scalar maps with locally asymptotically stable equilibria have families of asymptotically constant solutions which decay geometrically to the equilibria. Locally, all transients converging to the equilibria have this form.

Keywords: Difference equations; Bifurcation; Equilibria; Stability; Geometric convergence

1. THE PROBLEM

Consider the autonomous scalar map

$$x(t+1) = f(x(t)) \tag{1}$$

with locally asymptotically stable equilibrium (fixed point) x_e. Under what conditions does (1) have what we shall call *geometric transient* solutions, that is, solutions of the form

$$x(t) = \sum_{n=0}^{\infty} c_n r^{nt}, \quad c_0 = x_e$$

*Corresponding author. e-mail: henson@math.wm.edu
†e-mail: cushing@math.arizona.edu

ISSN: 1023-6198 DOI: 10.1080/10236190290000999

for real coefficients c_i and some fixed real number r satisfying $0 < |r| < 1$? Under what conditions do all solutions starting sufficiently close to x_e have this form?

Let $]a, b[$ denote an open interval. We will reserve the notation (x, y) for ordered pairs. Let R denote the set of real numbers, R_+ the set of positive real numbers, and $N = \{1, 2, 3, \dots\}$. In our applications, r is positive; we assume $r \in]0, 1[$.

Fix $r \in]0, 1[$, and let H_r be the set of all real sequences of the form $\{\sum_{n=0}^{\infty} c_n r^{nt}\}_{t=0}^{\infty}$ such that $\sum_{n=0}^{\infty} |c_n|$ converges. H_r is a Hilbert space under the inner product

$$\left\langle \left\{ \sum_{n=0}^{\infty} c_n r^{nt} \right\}_{t=0}^{\infty}, \left\{ \sum_{n=0}^{\infty} d_n r^{nt} \right\}_{t=0}^{\infty} \right\rangle = \sum_{n=0}^{\infty} c_n d_n$$

with norm $\|\cdot\|$ defined by

$$\left\| \left\{ \sum_{n=0}^{\infty} c_n r^{nt} \right\}_{t=0}^{\infty} \right\|^2 = \sum_{n=0}^{\infty} c_n^2.$$

We seek solutions of (1) in H_r.

If all initial conditions sufficiently close to the stable equilibrium x_e give rise to solutions in H_r, we will say x_e is *locally H_r-stable* according to the following definition.

DEFINITION 1 Let $r \in]0, 1[$. A locally asymptotically stable equilibrium x_e of the map $x(t+1) = f(x(t))$ is ***locally H_r-stable*** iff $\exists\ \delta > 0$ such that $\forall\ x_0$ we have

$$|x_0 - x_e| < \delta \Rightarrow \{x_t\}_{t=0}^{\infty} \in H_r.$$

We study equilibrium H_r-stability of Eq. (1) in the context of bifurcation theory. Specifically, we consider a general one-parameter family of maps of the form

$$x(t+1) = f(a, x(t)), \quad f(a, 0) = 0 \tag{2}$$

for $a \in R$ and study the H_r-stability of both the trivial equilibrium $x_e = 0$ and nontrivial equilibria $x_e(a)$ which depend on a. This bifurcation theoretic approach has been used to study similar questions about other types of equations. For example, the bifurcation of asymptotically periodic solutions (including equilibria) was studied for systems of Volterra type integral equations in [4] and for systems of discrete renewal difference equations in [3]. These results do not apply, however, to one dimensional maps of the form (2).

We consider Eq. (2) under the assumption (A1) below.

(A1) $f: R_+ \times R \longrightarrow R$ has a Maclaurin series expansion in x which converges for all $x \in R$, and satisfies $f(a,0) = 0$ and $f_x(a,0) = a$ for all $a \in R_+$. Thus,

$$x(t+1) = f(a, x(t)) = ax(t) + h(a, x(t)) \tag{3}$$

where

$$h(a,x) = \sum_{i=2}^{\infty} \frac{1}{i!} \frac{\partial^i f}{\partial x^i}(a,0)x^i.$$

An ordered pair $(a, \mathbf{x}) \in R_+ \times H_r$ is a "solution pair" if the sequence $\mathbf{x} = \{x(t)\}_{t=0}^{\infty} \in H_r$ is a solution of (3) associated with parameter value a. An ordered pair $(a, \mathbf{x}_e)$ is an "equilibrium pair" if $\mathbf{x}_e = \{x_e\}_{t=0}^{\infty}$ where x_e is a fixed point of (3).

Note that $(a, \mathbf{0})$, where $\mathbf{0} = \{0\}_{t=0}^{\infty}$, is an equilibrium pair for all values of $a \in R_+$. From the point of view of bifurcation theory, this constitutes a continuum (called the "trivial branch") of equilibria in $R_+ \times H_r$. The zero equilibrium is locally asymptotically stable for $a < 1$ and unstable for $a > 1$. Given (A1), bifurcation theory also guarantees the existence of a continuum of nontrivial equilibrium pairs $(a, \mathbf{x}_e)$ bifurcating (transcritically) from the trivial branch at the equilibrium pair $(1, \mathbf{0})$ [5]. The direction of bifurcation and the stability of the equilibria along this branch depend on the properties of f. For some of the most common maps of interest in both applied and pure studies the branch of positive equilibria is locally asymptotically stable near the bifurcation point and bifurcates "to the right", *i.e.*, corresponds to $a > 1$. Many maps of interest, such as the logistic map $f(a,x) = ax(1-x)$ and the Ricker map $f(a,x) = axe^{-x}$ fall into this category. In this paper we restrict our attention to such maps. Therefore, we assume:

(A2) There is an $a_{cr} > 1$ such that for each $a \in]1, a_{cr}[$ there exists a positive equilibrium solution $x_e(a)$ of (3) with $0 < f_x(a, x_e(a)) < 1$ such that $x_e(a)$ is continuous in a and $\lim_{a \to 1^+} x_e(a) = 0$.

Our goal is to show that the locally asymptotically stable equilibria corresponding to $a \in]0, 1[$ and $a \in]1, a_{cr}[$ are all locally H_r-stable for an appropriate choice of r.

2. LINEAR THEORY

Fix r, $a \in]0, 1[$. Consider the homogeneous and nonhomogeneous linear equations

$$x(t+1) = ax(t) \tag{H}$$

$$x(t+1) = ax(t) + b(t) \tag{NH}$$

where the sequence defined by

$$b(t) = \sum_{n=0}^{\infty} d_n r^{nt}$$

is a member of H_r. We have the following Fredholm Alternative:

THEOREM 2 *Either $a \neq r^n$ for any $n \in N$, in which case (H) has no nontrivial solution in H_r and (NH) has a unique solution in H_r generated by the initial condition*

$$x(0) = \sum_{n=0}^{\infty} \frac{d_n}{r^n - a};$$

or else $a = r^m$ for some $m \in N$, in which case (H) has nontrivial solutions in H_r and (NH) has a solution in H_r if and only if $d_m = 0$.

Proof The solution of (H) is $x(t) = a^t x(0)$, and the solution of (NH) is

$$\begin{aligned} x(t) &= a^t x(0) + \sum_{i=0}^{t-1} a^{t-1-i} b_i \\ &= a^t x(0) + \sum_{i=0}^{t-1} a^{t-1-i} \sum_{n=0}^{\infty} d_n r^{ni} \\ &= a^t x(0) + a^{t-1} \sum_{n=0}^{\infty} d_n \sum_{i=0}^{t-1} \left(\frac{r^n}{a} \right)^i \end{aligned}$$

Suppose $a \neq r^n$ for all $n \in N$. In this case (H) has no nontrivial solution in H_r, and the solution

$$
\begin{aligned}
x(t) &= a^t x(0) + a^{t-1} \sum_{n=0}^{\infty} d_n \sum_{i=0}^{t-1} \left(\frac{r^n}{a} \right)^i \\
&= a^t x(0) + a^{t-1} \sum_{n=0}^{\infty} d_n \frac{1 - (r^n/a)^t}{1 - (r^n/a)} \\
&= a^t \left(x(0) - \sum_{n=0}^{\infty} \frac{d_n}{r^n - a} \right) + \sum_{n=0}^{\infty} \frac{d_n}{r^n - a} r^{nt}
\end{aligned}
$$

of (NH) is in H_r if and only if

$$
x(0) = \sum_{n=0}^{\infty} \frac{d_n}{r^n - a}
$$

Now suppose $a = r^m$ for some $m \in N$. Then the solution $x(t) = a^t x(0)$ of (H) is a member of H_r for every $x(0) \in R$, and the solution

$$
\begin{aligned}
x(t) &= r^{mt} x(0) + r^{m(t-1)} \sum_{n=0}^{\infty} d_n \sum_{i=0}^{t-1} \left(\frac{r^n}{r^m} \right)^i \\
&= r^{mt} x(0) + r^{m(t-1)} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} d_n \frac{1 - (r^n/r^m)^t}{1 - (r^n/r^m)} + t d_m r^{m(t-1)} \\
&= r^{mt} \left(x(0) - \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{d_n}{r^n - r^m} \right) + \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{d_n}{r^n - r^m} r^{nt} + t d_m r^{m(t-1)}
\end{aligned}
$$

of (NH) is in H_r if and only if $d_m = 0$. ■

Now define the linear operator $L : H_r \longrightarrow H_r$ by

$$
L\{x(t)\}_{t=0}^{\infty} = \{x(t+1) - a x(t)\}_{t=0}^{\infty}
$$

so that (H) and (NH) become

$$
L\mathbf{x} = \mathbf{0} \tag{4}
$$

$$
L\mathbf{x} = \mathbf{b} \tag{5}
$$

for $\mathbf{x}, \mathbf{b} \in H_r$, where $\mathbf{x} = \{x(t)\}_{t=0}^{\infty}$ and $\mathbf{b} = \{b(t)\}_{t=0}^{\infty}$.

If the kernel of L (denoted *ker* L) is nontrivial, that is, if $a = r^m$ for some $m \in N$, then it is spanned by $\mathbf{v} = \{r^{mt}\}_{t=0}^{\infty} \in H_r$. From the Fredholm Alternative above, we see that the range of L (denoted *ran* L) is $(ker\, L)^{\perp}$. Thus, the restriction of L to *ran* L is a bijection on *ran* L, so L has a right inverse L^{-1} on *ran* L. Define the projection $P: H_r \longrightarrow H_r$ onto *ran* L by

$$P\mathbf{b} = \mathbf{b} - \langle \mathbf{b}, \mathbf{v} \rangle \mathbf{v}$$

Then $I - P$ is a projection onto *ker* L and $ker\,(I - P) = ran\, L$. Note that every element $\mathbf{b} \in H_r$ can be expressed uniquely as $P\mathbf{b} + (I - P)\mathbf{b}$ by means of the splitting $H_r = (ker\, L)^{\perp} \oplus ker\, L = ran\ L \oplus ker\, L$.

The Fredholm Alternative can be restated in H_r as follows.

THEOREM 3 *Let* $\boldsymbol{b} \in H_r$. *Either* $ker\ L = \{\mathbf{0}\}$, *in which case* (5) *has a unique solution* $L^{-1}\boldsymbol{b} \in H_r$; *or else* $ker\ L \neq \{\mathbf{0}\}$, *in which case* (5) *has a solution in* H_r *if and only if* $(I - P)\boldsymbol{b} = \mathbf{0}$.

From a bifurcation point of view we say the homogeneous equation (4) has a "vertical bifurcation" of nontrivial solutions in H_r from the trivial (zero) solution whenever the bifurcation parameter $a = r^m$ for some $m \in N$. It is precisely at these critical values of a that we look for bifurcations in the associated nonlinear equation.

3. NONLINEAR THEORY

Consider the nonlinear equation

$$x(t+1) = f(a, x(t)) = ax(t) + h(a, x(t)) \tag{6}$$

for fixed $r \in \,]0, 1[$. We first study solutions in H_r which bifurcate from the trivial equilibrium at the critical values r^m of the parameter a.

3.1. Bifurcations from Zero

Assume (A1). Fix $m \in N$. Write the nonlinear equation as

$$x(t+1) - r^m x(t) = (a - r^m)x(t) + h(a, x(t)).$$

Define the linear operator $L: H_r \longrightarrow H_r$ by

$$L\{x(t)\}_{t=0}^{\infty} = \{x(t+1) - r^m x(t)\}_{t=0}^{\infty}$$

and the nonlinear operator $B: H_r \longrightarrow H_r$ by

$$B(a, \{y(t)\}_{t=0}^{\infty}) = \{(a - r^m)y(t) + h(a, y(t))\}_{t=0}^{\infty}.$$

Note that L has a nontrivial kernel spanned by $\mathbf{v} = \{r^{mt}\}_{t=0}^{\infty}$.

To solve the operator equation

$$L\mathbf{x} = B(a, \mathbf{x})$$

for nontrivial $\mathbf{x} \in H_r$ near $a = r^m$, we use the Liapunov-Schmidt method [1, 2] and propose the Ansatz solution

$$\begin{aligned} \mathbf{x} &= \varepsilon \mathbf{v} + \varepsilon \mathbf{w}(\varepsilon) \\ a &= r^m + \lambda(\varepsilon) \end{aligned}$$

where ε is a small real parameter, $\lambda(0) = 0$, $\mathbf{w}(0) = \mathbf{0}$, $\mathbf{w}(\varepsilon) = \{w(t, \varepsilon)\}_{t=0}^{\infty} \in H_r$, and $\langle \mathbf{v}, \mathbf{w}(\varepsilon) \rangle = 0$ for all ε. Note that

$$\begin{aligned} w(t, \varepsilon) &= w_0(\varepsilon) + w_1(\varepsilon) r^t + w_2(\varepsilon) r^{2t} + \cdots \\ w_m(\varepsilon) &= 0 \quad \text{for all } \varepsilon \end{aligned} \tag{7}$$

since $\langle \mathbf{v}, \mathbf{w}(\varepsilon) \rangle = 0$. Substitution of the Ansatz into the operator equation gives

$$L(\varepsilon \mathbf{v} + \varepsilon \mathbf{w}(\varepsilon)) = B(r^m + \lambda(\varepsilon), \varepsilon \mathbf{v} + \varepsilon \mathbf{w}(\varepsilon)). \tag{8}$$

Since B is $O(\varepsilon^2)$, the corresponding equations for first and higher order terms in ε are

$$L\mathbf{v} = \mathbf{0} \tag{9}$$

$$L\mathbf{w}(\varepsilon) = T(\varepsilon, \lambda, \mathbf{w}) \tag{10}$$

where $T(\varepsilon, \lambda, \mathbf{w}) = (1/\varepsilon) B(r^m + \lambda(\varepsilon), \varepsilon \mathbf{v} + \varepsilon \mathbf{w}(\varepsilon))$ is $O(\varepsilon)$.

Apply first the projection P, and then $I - P$, to both sides of (10) to obtain

$$\begin{aligned} L\mathbf{w}(\varepsilon) &= PT(\varepsilon, \lambda, \mathbf{w}) \\ \mathbf{0} &= (I - P)T(\varepsilon, \lambda, \mathbf{w}). \end{aligned}$$

Since $PT(\varepsilon,\lambda,\mathbf{w})\in ran\,L$, we may invert L to obtain

$$\mathbf{0}=\mathbf{w}(\varepsilon)-L^{-1}PT(\varepsilon,\lambda,\mathbf{w}) \tag{11}$$

$$\mathbf{0}=(I-P)T(\varepsilon,\lambda,\mathbf{w}). \tag{12}$$

Because of the uniqueness of representation in the direct sum $H_r=ran\,L\oplus ker\,L$, the system of Eqs. (11)–(12) is equivalent to (8). The goal is to show the existence and uniqueness of $\lambda(\varepsilon)$ and $\mathbf{w}(\varepsilon)$ for small ε.

Define the operator $\Gamma: R\times R\times H_r\longrightarrow H_r$ by

$$\Gamma(\varepsilon,\lambda,\mathbf{w})=(I-P)T(\varepsilon,\lambda,\mathbf{w}).$$

Then $\Gamma(0,0,\mathbf{0})=\mathbf{0}$ since T is $O(\varepsilon)$. The Fréchet derivative of Γ with respect to λ evaluated at $(0,0,\mathbf{0})$ is the linear operator

$$\Delta\lambda\longmapsto\{r^{mt}\Delta\lambda\}_{t=0}^{\infty}$$

which is nonsingular. Hence, the Implicit Function Theorem [1] allows us to solve Eq. (12) uniquely for $\lambda=\lambda(\varepsilon,\mathbf{w})$ where $\lambda(0,\mathbf{0})=0$. Substitution of $\lambda(\varepsilon,\mathbf{w})$ into Eq. (11) leads to

$$\mathbf{0}=\mathbf{w}-L^{-1}T(\varepsilon,\lambda(\varepsilon,\mathbf{w}),\mathbf{w}).$$

Another straightforward application of the Implicit Function Theorem yields the existence of a solution $\mathbf{w}=\mathbf{w}(\varepsilon)$ of this equation which is infinitely differentiable in ε and for which $\lim_{\varepsilon\to 0}\mathbf{w}(\varepsilon)=\mathbf{0}$.

Thus, we have shown the following result.

THEOREM 4 *Assume (A1). For fixed $m\in N$ and $r\in\,]0,1[$ and for all sufficiently small ε, there exists a unique continuum $(a(\varepsilon),\boldsymbol{x}(\varepsilon))\in R_+\times H_r$ of nontrivial solution pairs of $L\boldsymbol{x}=B(a,\boldsymbol{x})$ bifurcating from $(r^m,\mathbf{0})$, i.e., satisfying $\lim_{\varepsilon\to 0}(a(\varepsilon),\boldsymbol{x}(\varepsilon))=(r^m,\mathbf{0})$.*

For each ε, a pair $(a(\varepsilon),\mathbf{x}(\varepsilon))=(a(\varepsilon),\{x(t,\varepsilon)\}_{t=0}^{\infty})$ from the continuum in Theorem 4 is a solution pair of the difference Eq. (6), *i.e.*, $\{x(t,\varepsilon)\}_{t=0}^{\infty}\in H_r$ solves (6) with parameter value $a=a(\varepsilon)$.

3.1.1. Direction of Bifurcation

If $m=1$ the bifurcation in Theorem 4 occurs at the solution pair $(r,\mathbf{0})$. If ε is small, then $a(\varepsilon)$ is close to r (and hence $a(\varepsilon)<1$) so the solution

$x(t,\varepsilon)=\varepsilon r^t+\varepsilon w(t,\varepsilon)$ must decay to zero as $t\to\infty$ (since the zero equilibrium is locally asymptotically stable for $a<1$). Thus, $w_0(\varepsilon)$ is identically zero in the expansion (7) of $w(t,\varepsilon)$, and hence we have $w(t,\varepsilon)=w_2(\varepsilon)r^{2t}+w_3(\varepsilon)r^{3t}+\cdots$. If we examine Eq. (10) in its component form

$$w(t+1,\varepsilon)-rw(t,\varepsilon)=\lambda(\varepsilon)[r^t+w(t,\varepsilon)]+\frac{1}{\varepsilon}h(r+\lambda(\varepsilon),\varepsilon r^t+\varepsilon w(t,\varepsilon))$$

for terms containing r^t, we find

$$0=\lambda(\varepsilon)$$

for each ε. Hence, for each ε we have $a(\varepsilon)=r$ and the solution pairs on the bifurcating branch of Theorem 4 may be written $(a(\varepsilon),\mathbf{x}(\varepsilon))=(r,\mathbf{x}(\varepsilon))$. We call such a bifurcation "vertical".

If $m=M>1$, a branch of solutions in $R_+\times H_r$ bifurcates from the point $(r^M,\mathbf{0})$ by Theorem 4. We wish to know the direction of this bifurcation. Replacing r by r^M and m by 1 in Theorem 4 gives a branch of solution pairs in $R_+\times H_{r^M}$ which also bifurcates from the point $(r^M,\mathbf{0})$; and by the argument in the previous paragraph, this bifurcation is vertical. Since $H_{r^M}\subset H_r$, these two branches are identical by uniqueness. Thus, the bifurcation of solutions in $R_+\times H_r$ is also vertical.

During the above discussion $r\in]0,1[$ has been fixed and we have considered vertical bifurcations of solutions in $R_+\times H_r$ at a countable number of critical values $a_{cr}=r,r^2,r^3,\ldots$ of the parameter a. From another point of view, at each fixed value of $a\in]0,1[$ there is a vertical bifurcation of solutions in $R_+\times H_a$ (where a now plays the role of r). These solutions are nonconstant (since the zero equilibrium is stable) and are geometric transients with convergence rate r equal to the eigenvalue a at zero:

THEOREM 5 *Assume (A1). For each fixed* $a\in]0,1[$ *there exists a vertical continuum* $(a,\boldsymbol{x}(\varepsilon))\in R_+\times H_a$ *of nontrivial solution pairs of* $L\boldsymbol{x}=B(a,\boldsymbol{x})$ *bifurcating from the trivial solution pair* $(a,\mathbf{0})$, *i.e., satisfying* $lim_{\varepsilon\to 0}(a,\boldsymbol{x}(\varepsilon))=(a,\mathbf{0})$. *For each fixed* ε *sufficiently small,* $lim_{t\to\infty}x(t,\varepsilon)=0$.

The branch of solution pairs described in Theorem 5 is a branch of solution pairs of the difference Eq. (3). The branch bifurcates vertically from the equilibrium pair $(a, \mathbf{0})$ on the branch of trivial equilibria. Thus, each pair $(a, \mathbf{x}(\varepsilon)) = (a, \{x(t, \varepsilon)\}_{t=0}^{\infty})$ on the branch provides a solution $x(t, \varepsilon)$ of the difference Eq. (3) that tends geometrically to 0.

In the next section we see that if the difference Eq. (3) also satisfies assumption (A2), we can extend the bifurcation result in Theorem 5 to each equilibrium pair $(a, \mathbf{x}_e)$ lying on the branch of nontrivial (positive) equilibrium pairs. This result will yield solutions of (3) that tend geometrically to the equilibrium x_e.

3.2. Bifurcations from the Positive Equilibrium Branch

Assume (A1)–(A2). Fix $a \in]1, a_{cr}[$ and let $x_e(a)$ be the corresponding positive, stable equilibrium of Eq. (3). Then $(a, \mathbf{x}_e(a))$ is a solution pair of $L\mathbf{x} = B(a, \mathbf{x})$. The variation equation for $y(t) = x(t) - x_e(a)$ may be written

$$y(t+1) = \alpha y(t) + \eta(y(t)) \tag{13}$$

where $\alpha = f_x(a, x_e(a)) = a + h_x(a, x_e(a))$.

Equation (13) satisfies (A1), with α replacing a in the statement of (A1). Since $\alpha \in]0, 1[$ by (A2), an application of Theorem 5 to this variation equation gives a vertical continuum $(\alpha, \mathbf{y}(\varepsilon)) \in R_+ \times H_\alpha$ of nontrivial solution pairs of (13) bifurcating from the trivial solution pair $(\alpha, \mathbf{0})$. Each solution pair $(\alpha, \mathbf{y}(\varepsilon)) \in R_+ \times H_\alpha$ on this branch corresponds to a solution pair $(a, \mathbf{x}(\varepsilon)) \in R_+ \times H_\alpha$ of $L\mathbf{x} = B(a, \mathbf{x})$, where $\mathbf{x}(\varepsilon) = \mathbf{y}(\varepsilon) + \mathbf{x}_e$.

THEOREM 6 *Assume (A1)–(A2). Fix $a \in]1, a_{cr}[$ and let $r = a + h_x(a, x_e(a))$. There exists a vertical continuum of nontrivial solution pairs $(a, \boldsymbol{x}(\varepsilon)) \in R_+ \times H_r$ of $L\boldsymbol{x} = B(a, \boldsymbol{x})$ such that $\lim_{\varepsilon \to 0}(a, \boldsymbol{x}(\varepsilon)) = (a, \boldsymbol{x}_e(a))$. For each fixed ε sufficiently small, $\lim_{t \to \infty} x(t, \varepsilon) = x_e(a)$.*

The branch of solution pairs described in Theorem 6 is a branch of solution pairs of the difference Eq. (3). The branch bifurcates vertically from the equilibrium pair $(a, \mathbf{x}_e(a))$ on the branch of positive equilibria. Each pair $(a, \mathbf{x}(\varepsilon)) = (a, \{x(t, \varepsilon)\}_{t=0}^{\infty})$ on the vertical branch provides a solution $x(t, \varepsilon)$ of the difference Eq. (3) that tends geometrically (as $t \to \infty$) to the equilibrium $x_e(a)$, with geometric convergence rate $r = f_x(a, x_e(a)) = a + h_x(a, x_e(a))$.

In the next section we use Theorems 5 and 6 to obtain H_r-stability results for the equilibria of (3).

3.3. H_r-stability

We have studied the maps

$$x(t+1) = f(a, x(t)) = ax(t) + \sum_{i=2}^{\infty} \frac{1}{i!} \frac{\partial^i f}{\partial x^i}(a, 0) x^i$$

under conditions (A1) and (A2) with locally asymptotically stable trivial equilibria for each $a \in]0, 1[$ and locally asymptotically stable positive equilibria $x_e(a)$ for each $a \in]1, a_{cr}[$. Theorems 5 and 6 show that each of these locally stable equilibria has a one-parameter family of geometric transient solutions converging to it. For $a \in]0, 1[$, the geometric transients have the form

$$x(t) = \varepsilon a^t + \varepsilon \sum_{n=2}^{\infty} c_n a^{nt}$$

and decay to zero as $t \to \infty$. For $a \in]1, a_{cr}[$, the geometric transients have the form

$$x(t) = x_e(a) + \varepsilon r^t + \varepsilon \sum_{n=2}^{\infty} c_n r^{nt},$$

where $r = f_x(a, x_e(a))$, and decay to $x_e(a)$ as $t \to \infty$.

Do the geometric transients account for all solutions which converge to the stable equilibria, at least locally?

For $a \in]0, 1[$, the initial conditions for the family of geometric transient solutions converging to zero are given by

$$x(0) = \varepsilon \left(1 + \sum_{n=2}^{\infty} c_n \right)$$

for sufficiently small ε. For $a \in]1, a_{cr}[$, the initial conditions for the family of geometric transients converging to $x_e(a)$ are given by

$$x(0) = x_e(a) + \varepsilon \left(1 + \sum_{n=2}^{\infty} c_n \right)$$

for sufficiently small ε. In both cases, for fixed a these initial conditions account for all possible initial conditions within a sufficiently small neighborhood of the associated equilibrium. Thus we have shown the following result.

THEOREM 7 *Assume* $(A1)-(A2)$ *for* $x(t+1)=f(a,x(t))$. *Then for each* $a\in\,]0,1[$ *the zero equilibrium is* H_a*-stable; and for each* $a\in\,]1,a_{cr}[$ *the positive equilibrium* $x_e(a)$ *is* H_r*-stable, where* $r=f_x(a,x_e(a))$.

4. EXAMPLE: CALCULATING SOLUTIONS OF THE LOGISTIC MAP

Having established the existence of geometric transient solutions, it is in principle straightforward to calculate their coefficients. By way of example, consider the logistic map

$$x(t+1)=ax(t)[1-x(t)]$$

and a solution of the form $x(t)=\sum_{n=0}^{\infty}c_nr^{nt}$. Equating coefficients on like powers of r, we generate the sequence of equations

$$\begin{aligned} c_0 &= ac_0(1-c_0)\\ rc_1 &= ac_1-2ac_0c_1\\ r^2c_2 &= ac_2-2ac_0c_2-ac_1^2\\ &\ \,\vdots \end{aligned}$$

All possible solutions are classified by the decision tree

$$\begin{array}{cccccc}
 & & c_0=0 & c_0=1-\frac{1}{a} & & \\
 & / & | & | & \backslash & \\
 & c_1=0 & \begin{array}{c} r=a\\ c_1=\varepsilon\end{array} & \begin{array}{c} r=2-a\\ c_1=\varepsilon\end{array} & c_1=0 & \\
/ & | & | & | & | & \backslash\\
c_2=0 & \begin{array}{c} r^2=a\\ c_2=\varepsilon\end{array} & c_2=\frac{r}{r-r^2}\varepsilon^2 & c_2=\frac{2-r}{r-r^2}\varepsilon^2 & \begin{array}{c} r^2=2-a\\ c_2=\varepsilon\end{array} & c_2=0\\
/| & | & | & | & | & |\backslash
\end{array}$$

The left half of the tree corresponds to $a < 1$, while the right half corresponds to $a > 1$. The leftmost path down the tree corresponds to the trivial solution branch, while the rightmost path gives the branch of positive equilibria. Every other path down the tree corresponds to the vertical branch of geometric transients.

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