DIFFERENTIAL EQUATIONS: AN APPLIED APPROACH

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Contents

1. Preliminaries 1
2. Mathematical Models 13

Chapter 1. First Order Equations 25
1. The Fundamental Existence Theorem 25
2. Approximation of Solutions 33
3. Chapter Summary & Exercises 51
4. Applications 55

Chapter 2. Linear First Order Equations 65
1. The Solution of Linear Equations 67
2. Properties of Solutions 80
3. The Method of Undetermined Coefficients 86
4. Autonomous Linear Equations 93
5. Chapter Summary & Exercises 98
6. Applications 103

Chapter 3. Nonlinear First Order Equations 125
1. Autonomous Equations 126
2. Separable Equations 160
3. Change of Variables 170
4. Approximation Formulas 175
5. Chapter Summary & Exercises 192
6. Applications 199

Chapter 4. Systems and Higher Order Equations 233
1. Systems and Higher Order Equations 235
2. Approximating Solutions of Systems 241
3. Linear Systems of Equations 256
4. Matrix Notation* 271
5. Chapter Summary & Exercises 278
6. Applications 283

Chapter 5. Homogeneous Linear Systems and Higher Order Equations 299
1. Introduction 299
2. Homogeneous Systems with Constant Coefficients 304
3. Homogeneous Second Order Equations 317
4. Phase Plane Portraits 322
5. Matrices and Eigenvalues 341
6. Systems of Three or More Equations 348
7. Chapter Summary & Exercises 354

8. Applications 358

Chapter 6. Nonhomogeneous Linear Systems 379
1. The Method of Undetermined Coefficients 380
2. The Variation of Constants Formula 390
3. Matrix Notation 402
4. Chapter Summary & Exercises 411
5. Applications 415

Chapter 7. Nonlinear Systems 429
1. Introduction 429
2. Phase Plane Portraits and Equilibria 431
3. The Linearization Principle 437
4. Local Phase Plane Portraits 445
5. Global Phase Plane Portraits 451
6. Bifurcations 461
7. Higher Dimensional Systems 471
8. Chapter Summary & Exercises 481
9. Applications 484

Index 511

Appendix A. ANSWERS TO SELECTED **EXERCISES** 515

Introduction

1. Preliminaries

Mathematical applications typically involve one or more equations to be solved for unknown quantities. Often applications involve rates of change, and therefore lead to equations containing derivatives. Such equations are called *differential* equations.

A student's first encounter with differential equations is usually in a calculus course where anti-derivatives (or indefinite integrals) are studied. For example, consider the problem of finding the anti-derivative of t^2. This problem can be formulated as follows: find a function $x = x(t)$ whose derivative is t^2, or in other words find a function $x = x(t)$ that satisfies the equation

$$x' = t^2. \tag{1.1}$$

(Here we have used the notation x' for the derivative of x with respect to t. We will also occasionally use the notation dx/dt.) Equation (1.1) is a differential equation for the unknown function $x = x(t)$. Notice what it means to "solve" this equation: find a function $x = x(t)$ that, when substituted into both sides of the equation, makes the left hand side *identically* equal to the right hand side. That is to say, a solution is a function which upon substitution into the equation reduces the equation to a mathematical identity in t. Also notice it is not accurate to speak of *the* solution of this differential equation. This is because it has many solutions, namely $x(t) = t^3/3 + c$ where c is any constant (the so-called "constant of integration").

It is not always as easy to find formulas for solutions of a differential equation as it is for the equation (1.1). For example, consider the differential equation

$$x' = x. \tag{1.2}$$

This equation is fundamentally different from (1.1) because the unknown function x appears on the right hand side. This equation cannot be solved by an anti-differentiation of the right hand side, because the right hand side is not a known function of t. Later we will learn how to solve this equation, but for now notice that $x(t) = e^t$ is a solution, i.e., a substitution of e^t for x into the left and the right hand sides of the equation yields the same result (namely e^t). Similarly, $x(t) = ce^t$ is a solution of this equation for *any* constant c (including $c = 0$). Notice, however, that $x(t) = e^t + c$ is *not* a solution (unless $c = 0$). To see this, we calculate $x'(t) = e^t$ and note that it is not equal to $x(t) = e^t + c$ (unless $c = 0$). This shows that constants of integration do not always appear additively in formulas for solutions of differential equations.

As another example consider the differential equation

$$x' = x^2. \tag{1.3}$$

The function $x(t) = 1/(1-t)$ is a solution of this equation, so long as $t \neq 1$, because the derivative $x' = 1/(1-t)^2$ is identical to x^2 for $t \neq 1$. We say this function is a solution on the interval $-\infty < t < 1$ or on the interval $1 < t < +\infty$ (or on any interval not containing $t = 1$). Similarly, for a constant c, the function $x(t) = 1/(c-t)$ is a solution on any interval that does not contain $t = c$. Notice each solution obtained by assigning a numerical value to c has a different singular point $t = c$ and hence is associated with a different interval of existence. (Incidentally, the constant function $x \equiv 0$ is also a solution which is not included in the formula $x(t) = 1/(c-t)$.)

A solution of a differential equation is associated with an *interval of existence*. The solutions $x(t) = 1/(c-t)$ of equation (1.3) show there is not necessarily a common interval of existence for all solutions of a differential equation. This example also illustrates that the differential equation itself might give little or no clue about the intervals of existence of its solutions.

For differential equations (1.1), (1.2), and (1.3) it is possible, as we have seen, to write down formulas for solutions. For other equations, it is not possible to calculate solution formulas. In the latter case, we must use other methods to study equations and their solutions. In this book we will study some types of equations for which we can derive solution formulas, but we will also study many methods of analysis that do not require solution formulas. These methods are of particular importance since it is not possible to calculate solution formulas for the differential equations that arise in many, if not most, scientific and engineering applications.

The equations (1.1), (1.2), and (1.3) are examples of a general class of ordinary differential equations of the form

$$x' = f(t, x).$$

Here all terms in the equation not involving the derivative have been placed on the right hand side. In general both the independent variable t and the dependent variable x can appear on the right hand side. Letters or symbols representing unspecified numerical constants called "coefficients" or "parameters" might also appear. Here are some further examples:

$$\begin{aligned}
&x' = x^2 + t^2 \\
&x' = -2x \\
&x' = px, \quad \text{where } p \text{ is a constant} \\
&x' = r\left(1 - \frac{x}{K}\right)x, \quad \text{where } r > 0,\ K > 0 \text{ are constants.}
\end{aligned}$$

It is important to recognize those letters and symbols that represent independent variables, those that represent dependent variables, and those that represent coefficients or parameters. The independent variable is, of course, the variable with respect to which the derivative is being taken. In the above equations we use the letter t for the independent variable; this will be done throughout the book. This choice is motivated by the many applications in which the independent variable represents time. (Other letters can, of course, be used.) On the other hand, throughout the book we use a variety of letters for the *dependent* variable (sometimes referred to as the "state variable"). In applications, a letter suggestive of the meaning of the variable in that application is usually chosen. For example, we will encounter differential equations involving symbols such as x', y', N', and P' for the derivatives of the dependent variables x, y, N, and P with respect to t. If it is necessary to emphasize the role of the independent variable t we sometimes write derivatives as

$$\frac{dx}{dt}, \ \frac{dy}{dt}, \ \frac{dN}{dt}, \ \frac{dP}{dt}.$$

Applications often involve several differential equations for several unknown functions, i.e. a *system of differential equations*. Some examples are

$$\begin{aligned} x' &= y \\ y' &= -\sin x \end{aligned}$$

$$\begin{aligned} x' &= -r_1 x - r_2 y \\ y' &= r_1 x - (r_1 + r_2)\, y \end{aligned}$$

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m}x - \frac{c}{m}y \end{aligned}$$

$$\begin{aligned} x' &= r\left(1 - \frac{x}{K}\right)x - cxy \\ y' &= -dy + xy \end{aligned}$$

$$\begin{aligned} x' &= y \\ y' &= -x - \alpha\left(x^2 - 1\right) y\ . \end{aligned}$$

In each of these examples there are two differential equations for two unknown functions x and y. All other letters represent coefficients (or parameters).

A solution of a system of two equations is a *pair* of functions $x = x(t)$, $y = y(t)$. For example, the pair $x(t) = 2e^{2t}$, $y(t) = -e^{2t}$ is a solution of the system

$$\begin{aligned} x' &= 5x + 6y \\ y' &= x + 4y. \end{aligned}$$

To see this, we note that $x' = 4e^{2t}$ is identical to

$$5x + 6y = 5\left(2e^{2t}\right) + 6\left(-e^{2t}\right) = 4e^{2t}$$

(i.e., the *first* equation is satisfied for all t) and *also* that $y' = -2e^{2t}$ is identical to

$$x + 4y = \left(2e^{2t}\right) + 4\left(-e^{2t}\right) = -2e^{2t}$$

(i.e., the *second* equation is also satisfied for all t). The reader can check that $x(t) = 3e^{7t},\ y(t) = e^{7t}$ is another solution pair of this same system.

Applications also arise in which higher order derivatives appear in the equation. Here are some examples of higher order differential equations:

$$x'' + x = 0$$

$$mx'' + cx' + kx = a\sin\beta t$$

$$x''' + 3x'' + 3x' + 2x = 0$$

$$\begin{aligned} m_1 x'' + (k_1 + k_2)x - k_2 y &= 0 \\ m_2 y'' - k_2 x + k_2 y &= 0. \end{aligned}$$

The *order* of a differential equation is that of the highest order derivative appearing in the equation. Thus, the equation $x' = x$ is a *first order* equation. The first two equations above are *second order* and the third equation is *third order*. The last pair of equations constitute a *second order system* of equations.

Solutions of higher order equations must reduce the equation(s) to identities upon substitution. For example, $x(t) = \sin t$ is a solution of the second order equation $x'' + x = 0$ for all t (as is $x(t) = \cos t$). The exponential function $x(t) = e^{-2t}$ is a solution of the third order equation $x''' + 3x'' + 3x' + 2x = 0$ for all t.

Any higher order equation (or system of higher order equations) can be associated with an equivalent system of first order equations. The following example illustrates the most common way to convert a higher order equation to an equivalent first order system. The function $x(t) = \sin t$ is a solution (for all t) of the second order equation

$$x'' + x = 0. \tag{1.4}$$

Define y to be the derivative of x, i.e., $y = x'$. Then $y(t) = \cos t$ and the pair $x(t) = \sin t$, $y(t) = \cos t$ solves the first order system

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \tag{1.5}$$

This shows how a particular solution of the second order equation (1.4) can be used to construct a solution of the first order system (1.5).

More generally, suppose $x = x(t)$ is *any* solution of the second order equation (1.4), i.e., $x''(t) + x(t) = 0$. Define $y = x'(t)$. The calculations

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) = x''(t) &= -x(t) \end{aligned}$$

show the pair $x(t)$, $x'(t)$ solves the system (1.5). This shows that *any* solution of the second order equation (1.4) gives rise to a solution pair for the first order system (1.5). Is the converse true? Can a solution of the first order system (1.5) be used to obtain a solution of the second order equation (1.4)? If so, then we could say that the second order equation (1.4) is "equivalent" to the first order system (1.5) in the sense that solving one is the same as solving the other.

Suppose $x = x(t)$, $y = y(t)$ is a solution pair of the first order system (1.5). Then

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{1.6}$$

We need to show how we can obtain a solution of the second order equation (1.4) from the solution pair of the system. The way to do this is simply to choose the first component x of the solution pair. We can show that the first component $x = x(t)$ satisfies the second order equation by differentiating both sides of the first equation in the system (1.6), to obtain $x''(t) = y'(t)$, and then use the second equation in the system to obtain $x''(t) = -\,x(t)$, or in other words $x'' + x = 0$.

The procedure we used to derive the system (1.5) equivalent to the equation (1.4) is not peculiar to that second order equation. For example, by the same method, we can show that the second order equation

$$x'' + \sin x = 0$$

is equivalent to the first order system

$$\begin{aligned} x' &= y \\ y' &= -\sin x. \end{aligned}$$

In general, we can show (by a similar procedure) that *any* second order differential equation of the general form

$$x'' = f(t, x, x')$$

is equivalent to the first order system

$$\begin{aligned} x' &= y \\ y' &= f(t, x, y). \end{aligned}$$

An extension of the method also applies to equations of order higher than two. For example, we can obtain an equivalent first order system for the third order equation

$$x''' + 3x'' + 3x' + 2x = 0$$

by defining *two* new dependent variables

$$y = x', \quad z = x''.$$

As above, we can show solutions of this equation give rise to solutions of the system

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -2x - 3y - 3z \end{aligned}$$

and vice versa.

A further extension of the method can be used for higher order systems as well. For example, consider the second order system

$$\begin{aligned} x'' + 2x - z &= 0 \\ 2z'' - x' + z &= 0 \end{aligned}$$

for two unknowns x and z. We apply the procedure twice, once on each equation, by defining two new dependent variables

$$y = x', \quad w = z'$$

and obtaining the equivalent first order system of four equations

$$\begin{aligned} x' &= y \\ y' &= -2x + z \\ z' &= w \\ w' &= \tfrac{1}{2}y - \tfrac{1}{2}z. \end{aligned}$$

The ability to convert higher order equations to a first order system is required by many (if not most) computer programs available for the study of differential equations.

One way to classify differential equations is by their order. Another way to classify equations is based on the notion of "linearity". *A differential equation is linear if the dependent variable and all of its derivatives appear linearly.* Thus, in a linear first order equation, both x and x' appear linearly. This means

$$\begin{aligned} x' &= 3x + 1 \\ 2x' - x &= 2 + \sin t \\ x' &= tx + a \\ e^t x' &= \frac{x}{t} + \ln t \end{aligned}$$

are all linear (first order) differential equations. Note that the independent variable plays no role in the definition of linearity. For example, the second equation is linear even though the independent variable t appears in a nonlinear way (in the $\sin t$ term). We can write each of these equations in the form

$$x' = p(t)x + q(t)$$

for appropriate coefficients $p(t)$ and $q(t)$. By definition, an equation is linear if it has this form (or can be rewritten in this form).

The equations

$$\begin{aligned} x' &= x^2 - 1 \\ xx' &= x + t \\ (x')^2 &= tx - 4 \\ x' &= r\left(1 - \frac{x}{K}\right)x \end{aligned}$$

are nonlinear. The first and fourth equations are nonlinear because of the term x^2. The second equation is nonlinear because of the term xx' and the third equation is nonlinear because of the term $(x')^2$ (not because of the term tx).

A second or higher order equation is linear if the dependent variable and all of its derivatives appear linearly in the equation. The second order equations

$$\begin{aligned} x'' + x &= 0 \\ x'' + x' + x &= \sin t \\ x'' + (\sin t)x &= 0 \end{aligned}$$

are linear because x, x' and x'' appear linearly. The equations

$$\begin{aligned} x'' + \alpha(1 - x)x' + x &= 0 \\ x'' + \sin x &= 0 \end{aligned}$$

are nonlinear (the first because of the term xx' and the second because of the term $\sin x$).

Systems of equations are linear if (and only if) *all* of the equations are linear in *all* of the *dependent* variables and their derivatives. Thus,

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

$$\begin{aligned} x' &= -rx + ry \\ y' &= rx - 2ry \end{aligned}$$

are linear systems and

$$\begin{aligned} x' &= \left(1 - x - \tfrac{1}{2}y\right)x \\ y' &= \left(1 - \tfrac{1}{2}y - x\right)y \end{aligned}$$

$$\begin{aligned} x' &= (x_{in} - x)\,d - \tfrac{1}{\gamma}\tfrac{mx}{a+x}y \\ y' &= \left(\tfrac{mx}{a+x} - d\right)y \end{aligned}$$

are nonlinear systems (because of the terms x^2, xy, and y^2 in the first system and the term $mxy\,(a+x)$ in the second).

EXERCISES

What are the orders of the following equations? Explain your answers.

Exercise 1.1. $t^2x' + x^3 = 0$

Exercise 1.2. $3x' - 2x^2 = 0$

Exercise 1.3. $e^t\,(x')^2 + x^3 = 0$

Exercise 1.4. $3\,(x'')^3 - 2x^5\,(x')^2 = 0$

Exercise 1.5. $x'x^3x'' - t^7x^{1/2} = 0$

Exercise 1.6. $x' + t^2x^2 + x''' = 2$

Exercise 1.7. $x' + t^{1/2}x = \ln t$

Exercise 1.8. $x'(x'')^2 - 5t^{1/2}x^3 = 2$

Exercise 1.9. $tx = e^x + (x')^2$

Exercise 1.10. $x'' + a\sin x = 0$

Exercise 1.11. $t^2x' + x^3 = t\cos t$

Exercise 1.12. $x' = p(t)x + q(t)$

Exercise 1.13. $2xx' + x''' + (x'')^3 - x^4 = 0$

Exercise 1.14. $(x''')^2 + (x'')^5 + 3(x')^7 - \sin x = 0$

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Which of the following are solutions and which are not solutions of the equation $x' + 3x = 0$? Explain your answers.

Exercise 1.15. e^{-3t}

Exercise 1.16. e^{3t}

Exercise 1.17. $-e^{-3t}$

Exercise 1.18. $3e^{-t}$

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Which of the following are solutions and which are not solutions of the equation $x' - 2tx = 0$? Explain your answers.

Exercise 1.19. e^{2t}

Exercise 1.20. $2e^{-2t}$

Exercise 1.21. $-7e^{t^2}$

Exercise 1.22. $1 + e^{t^2}$

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Which of the following are solutions and which are not solutions of the equation $2x' + 3x^{5/3} = 0$? Explain your answers.

Exercise 1.23. $t^{-3/2}$

Exercise 1.24. $-t$

Exercise 1.25. $(t-1)^{-3/2}$

Exercise 1.26. $(1-t)^{-3/2}$

Exercise 1.27. $t^{3/2}$

Exercise 1.28. $-(t+3)^{-3/2}$

Exercise 1.29. $(t-2)^{-2/3}$

Exercise 1.30. $(t-c)^{-3/2}$ *(where c is any constant)*

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Which of the following are solutions and which are not solutions of the equation $x'' - 5x' + 6x = 0$? Explain your answers.

Exercise 1.31. e^{-2t}

Exercise 1.32. e^{2t}

Exercise 1.33. e^{3t}

Exercise 1.34. e^{-3t}

Exercise 1.35. $5e^{2t}$

Exercise 1.36. $-7e^{3t}$

Exercise 1.37. $e^{2t} + e^{3t}$

Exercise 1.38. $c_1e^{2t} + c_2e^{3t}$ *for constants c_1 and c_2*

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In the Exercises 1.39-1.43 determine which of the functions are solutions of the given differential equation and which are not.

Exercise 1.39. *For the equation $x' + 5x = 0$:*
(a) $x = e^{-5t}$ *(b)* $x = 3e^{-5t}$ *(c)* $x = 5e^{-3t}$

Exercise 1.40. *For the equation $x' = 2x$:*
(a) $x = e^{3t}$ *(b)* $x = -3e^{2t}$ *(c)* $x = e^{2t}$

Exercise 1.41. *For the equation $x' + x^2 = 0$:*
(a) $x = \frac{1}{t}$ *(b)* $x = \frac{2}{t}$ *(c)* $x = \frac{1}{t-2}$

Exercise 1.42. *For the equation $x' = x + e^t$:*
(a) $x = e^t$ *(b)* $x = te^t$ *(c)* $x = e^t + te^t$

Exercise 1.43. *For the equation $tx'' + x' = 0$.*
(a) $x = \ln t$ *(b)* $x = 1$ *(c)* $x = t$

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Which of the following are solutions and which are not solutions of the equation $x''' - 4x'' - 4x' + 16x = 0$? Explain your answers.

EXERCISE 1.44. $x = e^{4t}$

EXERCISE 1.45. $x = -2e^{4t}$

EXERCISE 1.46. $x = ce^{4t}$ *where c is any constant*

EXERCISE 1.47. $x = e^{2t}$

EXERCISE 1.48. $x = \frac{1}{2}e^{-2t}$

EXERCISE 1.49. $x = c_1e^{4t} + c_2e^{2t} + c_3e^{-2t}$ *for any constants c_1, c_2, c_3*

EXERCISE 1.50. $x = e^{4t}e^{2t}$

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Which of the following are solutions and which are not solutions of the equation $x'' + x' - 2x = 0$? Explain your answers.

EXERCISE 1.51. $x = e^t$

EXERCISE 1.52. $x = e^{-2t}$

EXERCISE 1.53. $x = e^te^{-2t}$

EXERCISE 1.54. $x = e^t + 2e^{-2t}$

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EXERCISE 1.55. *Do $x = e^{4t}$ and $y = -2e^{4t}$ form a solution pair for the two equations $x' = 2x - y$, $y' = -6x + y$?*

EXERCISE 1.56. *Do $x = e^{3t}\sin 5t$ and $y = e^{3t}\cos 5t$ form a solution pair for the equations $x' = 3x + 5y$, $y' = -5x + 3y$?*

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Which of the following are solution pairs of the system below? Which are not solution pairs? Explain your answers.

$$\begin{aligned} x' &= 4x + 3y \\ y' &= -2x - y. \end{aligned}$$

EXERCISE 1.57. $x = e^t$, $y = -e^t$

EXERCISE 1.58. $x = -e^t$, $y = e^t$

EXERCISE 1.59. $x = e^t$, $y = e^t$

EXERCISE 1.60. $x = -e^t$, $y = -e^t$

EXERCISE 1.61. $x = 3e^{2t}$, $y = -2e^{2t}$

EXERCISE 1.62. $x = e^{2t}$, $y = -e^{2t}$

EXERCISE 1.63. $x = e^t + 3e^{2t}$, $y = -e^t - 2e^{2t}$

EXERCISE 1.64. $x = -2e^t + 6e^{2t}$, $y = 2e^t - 4e^{2t}$

EXERCISE 1.65. $x = c_1 e^t + 3c_2 e^{2t}$, $y = -c_1 e^t - 2c_2 e^{2t}$ *for constants* c_1 *and* c_2

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EXERCISE 1.66. *For each function that is a solution in Exercise 1.15-1.18 identify the interval on which it is a solution.*

EXERCISE 1.67. *For each function that is a solution in Exercise 1.23-1.30 identify the interval on which it is a solution.*

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Convert the equations below to equivalent first order systems.

EXERCISE 1.68. $x'' + x' - 3x = 0$

EXERCISE 1.69. $x'' - 6x' + 4x = 0$

EXERCISE 1.70. $3x'' - 6xx' + 12x^2 = 1$

EXERCISE 1.71. $5x'' + 10x'x = 5e^t$

EXERCISE 1.72. $2x''' - 6x'' + 4x' + x = -3$

EXERCISE 1.73. $x''' + 2x'' - x' + x = 1$

EXERCISE 1.74. $x'' + 2x' + 4x = \cos t$

EXERCISE 1.75. $2x'' + 3x' + 9x = 0$

EXERCISE 1.76. $t^2 x'' + (x')^2 + \cos x = 0$

EXERCISE 1.77. $xx'' + (x')^2 + x^{1/2} = e^t$

EXERCISE 1.78. $x'' = -2x' - x + z$, $z'' = -z' + 2x - z$

EXERCISE 1.79. $x''' + x'' - 2x' + 7x = t$

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EXERCISE 1.80. *Convert the second order system*

$$\begin{aligned} 2x'' - x' + 2z' + 4x - 8z &= 0 \\ z'' + 2x' - z' - x + 3z &= \sin t \end{aligned}$$

to an equivalent first order system.

EXERCISE 1.81. *Convert the second order system*

$$\begin{aligned} x'' - 5x' - 6z' + x - z &= 0 \\ 3z'' - 6x' - z' + 12x + 3z &= 21e^{-3t} \end{aligned}$$

to an equivalent first order system.

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Which of the following first order equations are linear? If an equation is nonlinear, explain why.

EXERCISE 1.82. $x' = 2x + 1$

EXERCISE 1.83. $3x' + 4x = \frac{1}{2}$

EXERCISE 1.84. $x' = tx^2 - 1$

EXERCISE 1.85. $x' = t^2 x - 1$

EXERCISE 1.86. $t^2 x' = x$

EXERCISE 1.87. $x' = x \sin t$

EXERCISE 1.88. $x' = t \sin x$

EXERCISE 1.89. $x' = e^x$

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Which of the following second order equations are linear? If an equation is nonlinear, explain why. (a is a constant.)

EXERCISE 1.90. $x'' + xx' + x = 0$

EXERCISE 1.91. $x'' + tx' + x = 0$

EXERCISE 1.92. $t^2 x'' + tx' + x = 1$

EXERCISE 1.93. $x^2 x'' + tx' + x = 1$

EXERCISE 1.94. $x'' + a(1 - x)x = 0$

EXERCISE 1.95. $x'' + a(1 - t)x = t$

EXERCISE 1.96. $x'' + e^{-x} t = \sin t$

EXERCISE 1.97. $x'' + e^{-t} x = \sin t$

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Which of the following systems are linear? If a system is nonlinear, explain why.

EXERCISE 1.98. $\begin{cases} x' = x + y \\ y' = x - y \end{cases}$

EXERCISE 1.99. $\begin{cases} x' = (1 - x)x - xy \\ y' = -y + xy \end{cases}$

EXERCISE 1.100. $\begin{cases} x' = x - y \\ y' = xy \end{cases}$

EXERCISE 1.101. $\begin{cases} x' = ax + by \\ y' = cx + dy \\ \textit{where } a, b, c \textit{ are constants.} \end{cases}$

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Which of the following equations (or systems of equations) are linear?

Exercise 1.102. $x' = a(r - x)$ *where a and r are constants*

Exercise 1.103. $x' = a(r - x)$ *where $a = a(x)$ is a decreasing function of x*

Exercise 1.104. $x'' + f(x)x = 0$ *where $f = f(x)$ is a function of x satisfying $\frac{df(x)}{dx} > 0$ (for all x).*

Exercise 1.105. $mx'' + cx' + kx = a \sin \beta t$ *where m, c, k, a and β are positive constants*

Exercise 1.106. $mx'' + c \sin x = 0$ *where m and c are positive constants*

Exercise 1.107. $\begin{cases} x' + y' = x + y \\ x' - y' = 2x + y \end{cases}$

Exercise 1.108. $\begin{cases} x' = \ln(ty) \\ y' = x \end{cases}$

Exercise 1.109. $\begin{cases} x' = y \sin t \\ y'x' = x + y \end{cases}$

Exercise 1.110. $\begin{cases} x' - 2x = y + \cos t \\ y - e^{2t}x = y' - 1 \end{cases}$

Determine whether the following equations can be rewritten as linear equations or not.

Exercise 1.111. $x' = \ln(2^x)$

Exercise 1.112. $x' = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

2. Mathematical Models

Mathematical models are descriptions of phenomena that involve mathematical equations, symbols, and concepts. These descriptions express laws, assumptions and/or hypotheses relevant to questions arising within some scientific discipline. One wishes to obtain answers to these questions by using the model. This involves "solving" the equations appearing in the model, where "solving" might mean the usual process of calculating a formula for the solution, or it might instead mean obtaining an approximation to the solution or even applying some other means of analysis that derives information about the solution. Even after the solution step is completed, further work might be needed to answer the scientific questions. Information about the mathematical solution needs to be interpreted and applied in the original scientific context. Thus, a modeling exercise involves three major steps that we can term the derivation ("setup") step, the solution step, and the interpretation step. Often the interpretation step reveals deficiencies in the model (e.g., implications and predictions of the model solution might not compare well with data). Such shortcomings can provide feedback to the derivation step by means of which a modified (presumably improved) model is constructed. Fig. 2.1 shows a schematic representation of these stages of a modeling effort. While these are idealized procedural steps, they can often help to orient and guide one's self while embedded in the details of an elaborate model.

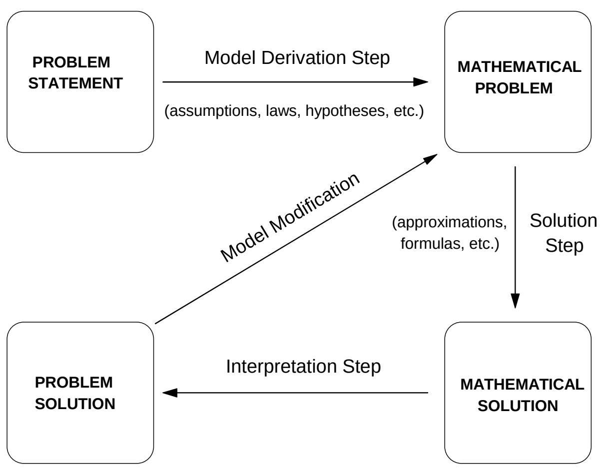

Figure 2.1. The Modeling Cycle.

The Model Derivation Step in the Modeling Cycle involves translating the statement of a problem from the language and jargon of a particular discipline (e.g., physics, chemistry, biology, engineering, economics, etc.) into mathematical terminology, symbols, and equations. The statement of the problem involves laws, principles, and/or assumptions that are to be used in the application. The first task is to identify the relevant unknown quantity or quantities and assign symbols to them. It may be necessary to choose symbols for other quantities as well (time, length, mass, growth and decay rates, coefficients of friction, etc.). These symbols must then be related to each other according to the statement of the problem, utilizing the stated laws and assumptions. The result will be a mathematical problem, usually one or more mathematical equations, to be solved for the unknown quantity (or quantities).

A “law" (or assumption) often used in mathematical modeling states that one quantity “is proportional to” another. This means one quantity is a constant multiple of the other. The mathematical expression of the assumption requires the designation of a symbol for the constant multiple, called the “constant of proportionality”. For example, if the force F exerted by a spring is proportional to its elongation s, then we write $F = ks$ where k is a constant of proportionality.

If rates of change are involved in an application, then the mathematical model usually involves a differential equation. For example, suppose the problem is to determine the velocity of an object with mass m subject to “Newton’s Law of Motion” $F = ma$. This law states that the force F exerted on the object equals its mass m times its acceleration a. Denoting time by t and velocity by $v = v(t)$ and recalling that $a = v'$, we obtain the differential equation $mv' = F$ for the velocity v. For this differential equation to be fully specified we need more information (assumptions, laws, etc.) about the force*s* acting on the object so that we can write a mathematical expression for F.

In many applications the time rate of change of a quantity x is proportional to the quantity itself, in which case $x' = rx$ for a constant of proportionality r. This case is often described in another way, namely that the “per unit rate of change” x'/x is constant. In some applications, the rate of change of a quantity x is proportional to other functions of the quantity. For example, if the rate of change of x is proportional to the square of x, then $x' = px^2$ for a constant of proportionality p.

Another modeling assumption is that a quantity “is jointly proportional to” other quantities. This means the quantity is a constant multiple of the product of the other quantities. Thus, if a is jointly proportional to b and c then we write $a = kbc$, where k is the constant of proportionality. For example, suppose the reaction rate of a chemical substrate is jointly proportional to its own concentration and to the concentration of an enzyme that catalyzes the reaction. If c and e denote the concentrations of the substrate and the enzyme respectively, then $c' = kce$ for a constant of proportionality k. In this case, the constant k is negative in this case, since the reaction decreases the concentration of the substrate c. To emphasize this, a more convenient notation is to write the constant of proportionality as $k = -m$ where $m > 0$. Thus $c' = -mce$. As a general rule, when defining symbols in an application it is useful to let letters stand for positive quantities.

In many applications involving rates of change a useful model derivation procedure is called *compartmental modeling*. In compartmental models the unknown quantities move into and out of designated “compartments” at certain rates. A “compartment” may be a physically well defined entity, such as a reaction tank in which chemical reactions occur. In other applications a “compartment” may be more loosely defined, such as the soil bank in a forest or the body tissues in a physiological problem involving the absorption of a medicinal drug. The *balance law (or balance equation)*

$$x' = \text{ inflow rate } - \text{ outflow rate} \tag{2.1}$$

applies to the amount of the quantity $x = x(t)$ contained in a compartment at time t. The inflow and outflow rates are those of the quantity into and out of the designated compartment. Coupled with information about these rates that allows us to write mathematical expressions for each of them, the balance law (2.1) becomes a differential equation for x. For example, if the quantity flows into the compartment at a constant rate r and flows out of the compartment at a rate proportional to the amount present, then the inflow rate is r and the outflow rate is px (where p is a constant of proportionality p). Then the balance law (2.1) yields the differential equation $x' = r - px$ for x.

The Solution Step of the Modeling Cycle in Fig. 2.1 focuses on the mathematical problem of "solving" the equation(s) in the model. A basic mathematical question is whether the equations even have a solution or not. If not, the problem is "ill-posed", and one must reassess the original statement of the problem and/or the derivation step. Another fundamental question concerns the number of solutions and, if there is more than one, which solution is relevant to the application. A problem is usually called "well-posed" if it has a solution and only one solution. Assuming the mathematical problem in the model is well-posed, one would then like to "solve" the equation(s). One way to do this is to find a formula for the solution. However, it is not possible to find solution formulas for most differential equations. In such cases, one can seek a formula that approximates the solution, or use a computer to calculate numerical approximations to the solution $x(t)$ at selected values of t. From these approximations we can draw approximate graphs of the solution. Another approach is to approximate the differential equation by a "simpler" equation, where by "simpler" we mean one for which we are able to calculate a solution formula. Yet another approach is to obtain the desired information about the solution directly from the differential equation itself, without the aid of solution formulas or approximations.

Finally, in the Interpretation Step of the Modeling Cycle the mathematical results from the Solution Step are utilized and interpreted so as to provide an answer to the original question. The mathematical solution may not immediately provide the answer and further use and manipulations of the solution, as well as additional information, might be necessary.

The following examples illustrate the Modeling Cycle.

EXAMPLE 2.1. *Assume the number of bacteria in a culture grows at a constant per capita rate. If an initial population of one thousand bacteria doubles in thirty minutes in how many minutes will there be one million bacteria present?*

To derive the model equations (Model Derivation Step) we let t be time measured in minutes, $t = 0$ be the initial time, and $x = x(t)$ be the number of bacteria at time t. With these symbols the given initial condition becomes $x(0) = 10^3$ and the growth rate assumption yields the differential equation $x' = px$, where p is a constant of proportionality. A formula for the solution of these equations is $x = 10^3 e^{pt}$ (Model Solution Step). To answer the question (Model Interpretation Step) we determine the time t at which $x(t) = 10^6$, i.e., we solve the equation $10^3 e^{pt} = 10^6$ for

$$t = \frac{1}{p} \ln 10^3 = \frac{3}{p} \ln 10.$$

To obtain a numerical answer in minutes we need a numerical value for p. This value is found from the stated fact that the population doubles in thirty minutes, i.e., in symbols, from $x(30) = 2 \times 10^3$. Solving the resulting equation

$$10^3 e^{30p} = 2 \times 10^3$$

for

$$p = \frac{1}{30} \ln 2$$

we obtain our final answer of

$$t = \frac{90}{\ln 2} \ln 10$$

or approximately 298.97 minutes.

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Example 2.2. *A chemical pesticide is applied to a stand of trees. This pesticide is absorbed into the tissues of the trees and, because of the natural exchange of material between the trees and the soil, the pesticide is transferred from the soil to the trees and vice versa. Assume these transitions take place at a per unit (pesticide) rate of* 2 *per year. In addition, the pesticide decomposes in the soil at a per unit rate of* 3 *per year. No amount of pesticide is initially present in the soil at which time a pesticide dosage of* $d > 0$ *units is applied to the trees. Determine the maximum amount of pesticide that will occur in the soil and the time at which this maximum occurs. In the long run what fraction of the pesticide is in the soil?*

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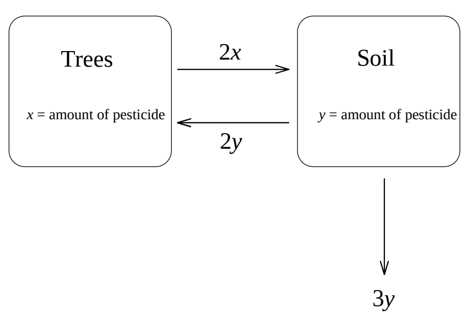

Figure 2.2

As part of the Model Derivation Step we consider two compartments: trees and soil. Let $x = x(t)$ denote the amount of pesticide in the trees at time t and let $y = y(t)$ denote the amount in the soil. The "compartmental diagram" in Fig. 2.2 shows how the pesticide moves between the two compartments. According to the stated assumptions, the outflow rate from the trees to the soil is $2x$ (units of pesticide per year) and the inflow rate is $2y$. The balance equation (2.1) yields the differential equation

$$x' = 2y - 2x. \tag{2.2}$$

The flow-rate into the soil is $2x$. There are two outflow rates from the soil, the absorption rate $2y$ into the trees and the decomposition rate $3y$, yielding a total outflow rate from the tree compartment of $5y$. The balance equation (2.1) yields the differential equation

$$y' = 2x - 5y. \tag{2.3}$$

Together these equations form a system of two first order linear differential equations for the two unknowns x and y. These unknowns are subject to the given initial conditions

$$x(0) = d, \qquad y(0) = 0. \tag{2.4}$$

As part of the Model Solution Step, we begin with some approximate solutions of these equations, which we use to form some tentative answers to the questions. Fig. 2.3 shows some computer drawn plots of the $y = y(t)$ component of the solution for a selection of initial doses d. In Fig. 2.4 the fraction of pesticide in the soil, $y/(x + y)$, is plotted for each of these cases.

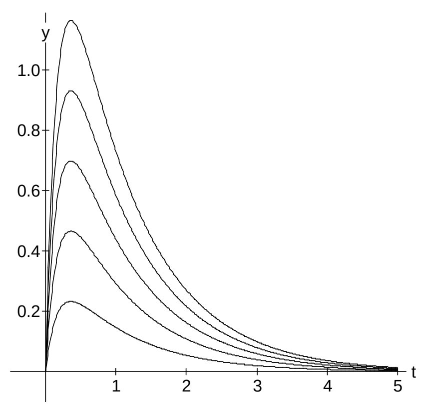

FIGURE 2.3 The amount of pesticide in the soil, $y(t)$, is plotted as a function of time t for initial doses $d = 1, 2, 3, 4, 5$. The maximum of y occurs at approximately $t = 1/3$ year and appears to be independent of the initial dose.

A visual inspection of these graphs suggests the following conclusions. The maximum amount of pesticide occurring in the soil is proportion to the initial dosage (i.e. if d is doubled, tripled, etc. the maximum amount is doubled, tripled, etc.). On the other hand, the time at which the maximum occurs is independent of the initial dosage d and is approximately equal to 1/3 year. The graph in Fig. 2.3 suggests the fraction of pesticide in the soil approaches approximately 1/3 as time goes on, i.e.,

$$\lim_{t\to+\infty} \frac{y(t)}{x(t) + y(t)} = \frac{1}{3}.$$

These conjectures are formulated from the plots shown in Fig. 2.3 and 2.4 (which were obtained from computer generated approximations to the solution of the differential equations).

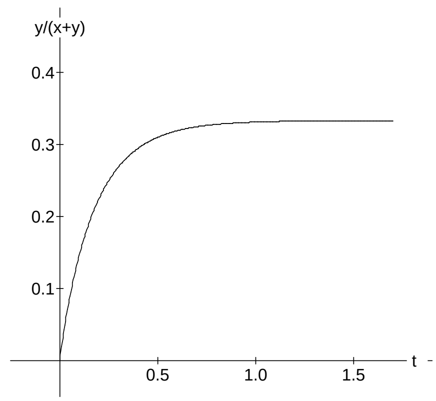

FIGURE 2.4 The fraction of pesticide in the soil $y/(x+y)$ is plotted as a function of time for initial doses $d = 1, 2, 3, 4, 5$. The result, which is the same for all doses, shows an increase from 0 to approximately 1/3.

It turns out there are formulas for the solution of the system of differential equations (2.2)-(2.3) and the initial conditions (2.4), namely

$$x(t) = d\left(\frac{1}{5}e^{-6t} + \frac{4}{5}e^{-t}\right)$$
$$y(t) = d\left(\frac{2}{5}e^{-t} - \frac{2}{5}e^{-6t}\right).$$

(See Exercise 2.19 in Chapter 5.) From these formulas we can obtain more accurate and general answers to our questions than we can get from computer experiments.

For example, using calculus methods we find the maximum of $y(t)$ to occur at $t = (\ln 6)/5$, the root of the derivative

$$y'(t) = -\frac{2}{5}e^{-t} + \frac{12}{5}e^{-6t}.$$

Our conjecture was accurate, since $(\ln 6)/5 = 0.35835 \approx 1/3$. Our conjecture that the maximum is proportional to d was also correct, as we see from the calculation

$$y\left(\frac{1}{5}\ln 6\right) = \frac{1}{3}6^{-1/5}d \approx 0.2329d.$$

Finally, we calculate the long term fraction of pesticide in the soil by taking the limit

$$\lim_{t\to+\infty} \frac{y(t)}{x(t)+y(t)} = \lim_{t\to+\infty} 2\frac{1-e^{-5t}}{6-e^{-5t}} = \frac{1}{3}.$$

This also agrees with our conjecture.

It is important to remember that mathematical models are built from assumptions. Thus, models take the form of "what if" questions: what are the logical conclusions if the assumptions are valid? The assumptions state not only which laws and principles are used in the model, but also (by exclusion) what phenomena and mechanisms are ignored. Some effects are small (relative to those included in the model) and can presumably be safely excluded while still obtaining useful and accurate answers. For example, in some circumstances friction may be negligible compared to other forces acting on an object in motion. However, if the predictions of the model turn out to be unacceptably inaccurate and if this is due to frictional forces, then one must return to the derivation step, include frictional forces in the model and derive new mathematical equations. Usually this cycle results in more complicated equations to solve, and thus there is a relentless trade-off between the accuracy or "realism" of models and their mathematical tractability.

Here is an illustration. In a laboratory experiment an object was dropped and left to fall vertically to the ground under the influence of gravity. We would like a model that predicts the distance fallen $x = x(t)$ at each instant of time $t > 0$. In the Model Derivation Step we assume Newton's Law of Motion $F = ma$. We also assume that near the surface of the earth objects fall with constant acceleration g (this is an approximation to another gravitational law of Newton). This constant is known to be approximately $g = 9.8$ m/sec^2 (32 ft/sec^2). Under these assumptions we have $F = mg = mv'$, where $v = x'$ is the object's velocity, and the mathematical model becomes, after the cancellation of a factor m,

$$v' = 9.8, \quad v(0) = 0.$$

The initial condition $v(0) = 0$ results from the object being dropped (i.e., it is not given any initial velocity). We can carry out the Model Solution Step in this example by performing a straightforward integration of $v' = 9.8$. The result, when $v(0) = 0$ is taken into account, is the solution formula $v(t) = 9.8t$. Since $x' = v$ this formula for v implies $x(t) = 4.9t^2$. (The constant of integration is 0 because at the initial instant the distance fallen equals 0, i.e., $x(0) = 0$). This formula predicts the distance fallen at each time $t > 0$. For example, it predicts the object falls 4.9 meters in $t = 1$ second and $4(2)^2 = 19.6$ meters in $t = 2$ seconds.

To test the accuracy of the model, its predictions can be compared to observational data obtained from the experiment. In the experiment it turned out that the object fell 0.61 meter in $t = 0.347$ second and 7.0 meters in $t = 1.501$ seconds. The model prediction $x(0.347) = 4.9(0.347)^2 = 0.59$ for $t = 0.347$ is fairly accurate, making an error of approximately 3%. However, at $t = 1.501$ seconds the model predicts $x(1.501) = 4.9(1.501)^2 = 11.04$ meters whereas the object actually fell only 7 meters, an error of over 50%. The Model Interpretation Step reveals a deficiency. To obtain more accurate predictions we reconsider the model and its derivation (Model Modification Step).

The falling object in the experiment, it turns out, was a shuttlecock used in the game of badminton. A shuttlecock is a designed to experience considerable air resistance (it is essentially a small ball with a comet-like tail of feathers). Therefore, in addition to gravity, the force of friction should be included in the total force F acting on the shuttlecock. To do this requires a modeling assumption about how friction acts on the shuttlecock.

Since friction is related to the motion of the shuttlecock, the force due to friction is a function of the velocity v. This function equals 0 when $v = 0$ (there is no friction when the shuttlecock is at rest) and increases as v increases (friction increases as velocity increases). A simple relationship of this kind is direct proportionality, i.e., the force due to friction equals $-kv$ for a constant k of proportionality. (We assume $k > 0$ and the minus sign occurs because friction works against the motion of the object). The constant k is called the "coefficient of friction". Under this assumption we have $F = mg - kv$. and Newton's Law $F = ma$ lead to the modified model and differential equation $mg - kv = mv'$ for the object's velocity. After dividing by m and letting $g = 9.8$, we obtain the differential equation

$$v' = 9.8 - k_0 v.$$

Here k_0 stands for k/m (the per unit mass coefficient of friction).

In order to make numerical predictions, k_0 needs to be assigned a numerical value. Data yields an estimated value of $k_0 = 1.128$ (see Sec. 6.2, Chapter 3). This yields the equations

$$v' = 9.8 - 1.128v, \quad v(0) = 0 \tag{2.5}$$

for the velocity v of the falling shuttlecock. After these equations are solved (either by formula or approximation), the distance fallen is found by integrating v, i.e., $x = \int_0^t v dt$.

It turns out that this modified model, with friction included, makes more accurate predictions at later times than the original model without the frictional force does. For example, $x(1.501) = 6.75$ meters, an error of only 3.5%.

For more details about this application see Sec. 6.2, Chapter 3.

Mathematical books naturally focus on the Solution Step of the Modeling Cycle. In this book we will study methods for approximating solutions of differential equations (graphically, numerically and analytically), methods for obtaining solution formulas, and methods for analyzing properties of solution. Throughout the book we will, however, use model equations arising from applications to illustrate these methods.

EXERCISES

In the exercises below you are asked to derive differential equations for unknown functions and, in some cases, initial conditions for the equations. Except when explicitly asked to do so, *do not solve* the equations.

Exercise 2.1. *Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called "continuous compounding".) Suppose an initial deposit of d dollars is made.*

(a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t.

(b) Suppose withdrawals are made at a constant rate w. Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t.

Exercise 2.2. *Samples of radioactive isotopes decay at a rate proportional to the amount present. Suppose an initial amount x_0 is present.*

(a) Write a differential equation and initial condition for the amount of radioactive isotope $x = x(t)$ present at time t.

(b) Suppose an amount of radioactive material is added at a constant rate a. Write a differential equation and initial condition for the amount of radioactive isotope $x = x(t)$ present at time t.

Exercise 2.3. *A container holds a volume v of fluid. Suppose a fluid is pumped into the container at a rate $d > 0$ (volume per unit time) and is rapidly mixed. To keep the volume constant at v, the well mixed fluid is pumped out at a rate d. Suppose the incoming fluid contains a concentration c_{in} of a chemical substrate, which is initially absent.*

(a) Write a differential equation and initial condition for the concentration $c = c(t)$ of the substrate in the container at time t.

(b) An enzyme is added to the container in such a way that its concentration e is held constant. Suppose this enzyme reacts with the substrate at a rate (per unit volume) jointly proportional to both concentrations. Write a differential equation and initial condition for the concentration $c = c(t)$.

EXERCISE 2.4. *(a) Write a differential equation and initial condition for the velocity of a dropped shuttlecock under the assumption that the force of friction is proportional to the square of its velocity.*

(b) Show

$$v(t) = \sqrt{\frac{9.8}{k_0}} \frac{1 - \exp\left(-2t\sqrt{9.8k_0}\right)}{1 + \exp\left(-2t\sqrt{9.8k_0}\right)}$$

solves the equations in (a) where k_0 is the per unit mass coefficient of friction.

(c) Show $v(t)$ has a limit as $t \to +\infty$. What does this mean about the motion of the falling shuttlecock?

EXERCISE 2.5. *Suppose a population has a constant per unit death rate $d > 0$ and a constant per unit birth rate $b > 0$.*

(a) Using the inflow-outflow rule (2.1), write a differential equation for the population concentration $x(t)$.

(b) Suppose the per unit death rate d is not constant, but is instead proportional to population concentration. Write a differential equation for the population concentration $x(t)$.

EXERCISE 2.6. *Suppose a population has a constant per unit death rate $d > 0$ and a per unit birth rate that is proportional to population concentration x (with constant of proportionality denoted by $a > 0$). Using the inflow-outflow rule (2.1), write a differential equation for the population concentration $x(t)$.*

EXERCISE 2.7. *A basic model of the growth of a tumor is based upon the assumption that the per unit volume growth rate of the tumor decreases exponentially with time (i.e. proportionally to an exponential of the form e^{-bt} for a constant $b > 0$). Assume the tumor initially has volume $v_0 > 0$. Write a differential equation and initial condition for the tumor volume $v = v(t)$ as a function of time.*

EXERCISE 2.8. *Assume a population with constant per unit birth and death rates b and d is subjected to an influx of immigrants at a constant rate I (not a per unit rate, but a constant rate!). Write a differential equation for the size of the population.*

EXERCISE 2.9. *Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and its surrounding environment. Write a differential equation for the temperature of the body.*

EXERCISE 2.10. *Modify the system of equations for the tree and soil compartmental Example 2.2 to account for the application of pesticide to the trees continuously at a constant rate p. Assume initially there is no pesticide in either the trees or the soil.*

EXERCISE 2.11. *Suppose two distinct cultural groups live in the same city. Each grows at a rate proportional to its numbers. However, each group loses population* *numbers at a rate proportional to the size of the other (since the two groups do not get along). Write a compartmental model (two differential equations) for the rate of change of each group's size. Is the system of differential equations linear or nonlinear?*

EXERCISE 2.12. *Consider two populations that grow exponentially in the absence of the other (i.e. their per unit birth and death rates are constants). When placed together in a common habitat the two populations compete for a vital resource. Because of this competition each population's per unit death rate is increased, but the per unit birth rates remain constant. Specifically, assume each population's per capita death rate is proportional to the other population's density. Write down a system of differential equations for the population densities x and y of each population. Is the system linear or nonlinear? (This model is a famous system in theoretical ecology called the Lotka-Volterra Competition Model.)*

EXERCISE 2.13. *A salt water concentration of* 2 *lbs per gallon is added to a* 150 *gallon tank full of initially pure water at a rate of* 5 *gallons per minute. Suppose the mixture is well-stirred so that the tank always has a uniform concentration of salt throughout. The well-stirred mixture is drained out the bottom of the tank at a rate of* 5 *gallons per minute (so that the tank remains full at* 150 *gallons). Write a differential equation for the number of lbs of salt in the tank. Is the equation linear or nonlinear?*

EXERCISE 2.14. *Suppose a container initially contains an amount $x_0 > 0$ of a chemical substance. Suppose the chemical flows into the container at a constant rate $r > 0$ and flows out at a rate proportional to the elapsed time (with proportionality constant $k > 0$).*

(a) Using the inflow-outflow rule (2.1) write a differential equation for the amount of the chemical $x = x(t)$ as a function of time t. Is the equation linear or nonlinear?

(b) Obtain a formula for the solution x.

(c) Show the amount of the chemical in the container initially increases to a maximum level x_m at a time $t_m > 0$ before decreasing to 0 *in a finite amount of time t_f. Find formulas for x_m, t_m, and t_f.*

EXERCISE 2.15. *Write a differential equation and initial condition for the concentration $x = x(t)$ of a population under the following assumptions: the initial population concentration is x_0; the birth rate is proportional to population concentration; the death rate is proportional to the square of population concentration; and the population is harvested at a constant rate $h > 0$. Is the differential equation linear or nonlinear?*

EXERCISE 2.16. *Suppose a drug is to be injected into the blood of a patient. Consider the circulatory system as one compartment and all of the tissues that serve to eliminate the drug from the circulatory system (such as the kidney) as another compartment. Let x and y be the mass of the drug in these two compartments respectively. Suppose the rate at which the drug flows out of a compartment (and into the other) is proportional to the mass present in the compartment. Also assume the drug is removed from a compartment, through elimination or degradation, at a rate that is proportional to the amount of mass in the compartment. Finally, assume no drug is present in the patient until an initial injection of amount $x_0 > 0$*

is given into the blood stream (after which no further drug is added to the system). Using the inflow-outflow rule (2.1) write differential equations and initial conditions for $x = x(t)$ and $y = y(t)$.

EXERCISE 2.17. *A manufactured item is given an initial set price of p_0. However, as supply and demand for the item changes with time t, so does the price $p = p(t)$. To complicate matters, the supply and demand in turn depend on the price. Write a differential equation for the price p under the following assumptions: the rate of change of the price p is proportional to the difference between the supply s and demand d (price increases if demand exceeds supply); the supply s is proportional to the price (if the price goes up the supply goes up proportionally); and the demand d is inversely proportional to the price. Is the equation linear or nonlinear?*

EXERCISE 2.18. *Consider two armies engaged in a battle. Let x and y represent the strengths of the two armies (measured, for example, as the number of troops and/or armaments). Assume the strength of each army decreases at a rate proportional to the strength of the other army. In addition, each army receives reinforcements at a constant rate. Write differential equations for x and y. Are these equations linear or nonlinear?*

EXERCISE 2.19. *Pure water is pumped into a tank of volume V (liters) initially filled with salt water. Suppose the pure water is pumped in at a rate of r liters per hour and the well-stirred salt water mixture is pumped out at the same rate (so that the volume in the tank remains constant at V).*

(a) Write a differential equation for the salt concentration $x = x(t)$ in the tank as a function of time t.

(b) If the water pumped into the tank is not pure, but instead has a salt concentration of s grams/liter, modify your equation in (a) accordingly.

EXERCISE 2.20. *In biology, allometry is the study of the relative size and growth of different parts of an organism. Suppose $x = x(t)$ and $y = y(t)$ are measures of the sizes of two different parts of a particular organism as functions of time. A simple model of allometry is based on the assumption that the per unit growth rates of the different parts are proportional to each other. Treating y as a function of x, derive a differential equation for y. Is this equation linear or nonlinear?*

EXERCISE 2.21. *The birth rate of a population is proportional to its size $x = x(t)$ with constant of proportionality $b > 0$. The death rate of the population is proportional the population size of a deadly virus $y = y(t)$ with constant of proportionality $c > 0$. The virus population has a negative per unit growth rate $-r$.*

(a) Write a differential equation and initial condition for the population sizes $x = x(t)$ and $y = y(t)$, assuming initial sizes of x_0 and y_0.

(b) Show

$$x = \left(x_0 - \frac{c}{r+b}y_0\right)e^{bt} + \frac{c}{r+b}y_0e^{-rt}, \qquad y = y_0e^{-rt}$$

solve the differential equations you obtained in (a).

(c) Only positive values of x and y are of relevance. Keeping this in mind describe what happens to x as $t \to +\infty$ and explain the implications with regard to the survival of the population x.

CHAPTER 1

First Order Equations

In this Chapter we consider first order differential equations of the form

$$x' = f(t, x).$$

A fundamental question concerns the existence of solutions to such an equation. Under what conditions (i.e., for what kind of expressions $f(t, x)$) can we be assured that solutions exist? Another question concerns the number of solutions. We know from calculus that integration problems have infinitely many solutions and, therefore, we anticipate that this is also true for a first order differential equation. On the other hand, in applications there are often requirements (in addition to the differential equation) that serve to select exactly one solution. For a first order differential equation the most common requirement is that the solution $x(t)$ equal a specified value x_0 for a specified value of t, that is to say, that $x(t_0) = x_0$ for a given t_0 and x_0. A fundamental mathematical question is whether the resulting *initial value problem*

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a solution. In this chapter we learn conditions which, when placed on $f(t, x)$, guarantee that this initial value problem has one and only one solution (i.e., has a "unique" solution).

For specialized equations (i.e., for $f(t, x)$ with special properties) one can calculate formulas for solutions. We study some examples in Chapters 2 and 3. However, for most differential equations it is not possible to find solution formulas. Nonetheless, it is possible to obtain useful approximations to solutions of any first order equation, especially with the aid of a computer. In this chapter we study some basic methods for approximating solutions, both graphically and quantitatively. In applications these methods are often sufficient to obtain the desired answers. Other approximation methods appear in Chapter 3.

1. The Fundamental Existence Theorem

We begin with a definition.

DEFINITION 1.1. *A **solution** of a differential equation $x' = f(t, x)$ on an interval $a < t < b$ is a differentiable function $x = x(t)$ that reduces the equation to an identity on the interval, i.e., $x'(t) = f(t, x(t))$ for all values of t from the interval.*[1] *The interval $a < t < b$ may be the whole real line, in which case we say the function is a solution for all t.*

[1] As a mathematical function $f(t, x)$ has a domain of t and x values. It is assumed, in this definition, that all values of t taken from the interval $a < t < b$ and the corresponding values of $x(t)$ (i.e., the range of the function $x(t)$) lie in the domain of f. Otherwise $f(t, x(t))$ makes no sense.

For the differential equation

$$x' = t^2 \tag{1.1}$$

we have $f(t,x) = t^2$. The function $x(t) = t^3/3 + 1$ is a solution of this equation for all t because $x'(t) = t^2$ equals $f(t, x(t)) = t^2$ for all t.

More generally, the unknown x might appear in $f(t,x)$. For example, for the equation $x' = tx$ we have $f(t,x) = tx$. The function $x(t) = e^{t^2/2}$ is a solution of this equation for all t because $x'(t) = te^{t^2/2}$ and $f(t, x(t)) = tx(t) = te^{t^2/2}$ are identical for all t.

From calculus we know the differential equation (1.1) has infinitely many solutions and the set of all solutions is given by the formula

$$x = \frac{1}{3}t^3 + c \tag{1.2}$$

where c is an arbitrary constant. This is an example of a "general solution" of a differential equation.

Definition 1.2. *The collection of all solutions of the differential equation $x' = f(t,x)$ is called the **general solution** (or the **solution set**).*

An initial condition $x(t_0) = x_0$ selects a particular solution from the general solution. For example, suppose we require that a solution of the equation (1.1) satisfy the initial condition $x(0) = 1$. From the general solution (1.2) we obtain $x(0) = c$ and therefore this initial condition is satisfied by choosing (and only by choosing) $c = 1$. That is to say, there is a unique solution of the initial value problem

$$x' = t^2, \quad x(0) = 1$$

namely, $x = t^3/3 + 1$.

In an initial value problem the "initial" time need not be $t_0 = 0$. For example, we can use the general solution (1.2) to find the unique solution

$$x(t) = \frac{1}{3}t^3 - \frac{11}{3}$$

of the initial value problem

$$x' = t^2, \quad x(2) = -1.$$

In fact, we can solve the general initial value problem

$$x' = t^2, \quad x(t_0) = x_0.$$

using the general solution (1.2) by setting

$$x(t_0) = \frac{1}{3}t_0^3 + c$$

equal to the desired initial value x_0 and solving for

$$c = x_0 - \frac{1}{3}t_0^3.$$

This results in the unique solution

$$x(t) = \frac{1}{3}t^3 + x_0 - \frac{1}{3}t_0^3.$$

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EXAMPLE 1.1. *A differential equation for the velocity $v = v(t)$ of a falling object subject to the force of gravity and air resistance is $v' = f(t, v)$ where $f(t, v) = g - k_0 v$. Here g and k_0 are constants (the acceleration due to gravity and the per unit mass coefficient of friction respectively). The function*

$$x(t) = e^{-k_o t} + \frac{g}{k_0}$$

is a solution for all t. To see this note

$$x'(t) = -k_0 e^{-k_0 t}$$

is equal to

$$f(t, x(t)) = g - k_0 \left(e^{-k_o t} + g/k_0 \right)$$

for all t.

For any constant c the function

$$x(t) = ce^{-k_o t} + \frac{g}{k_0}$$

is also solution for all t since

$$x'(t) = -k_0 c e^{-k_0 t}$$

is equal to

$$f(t, x(t)) = g - k_0 \left(ce^{-k_o t} + \frac{g}{k_0} \right)$$

for all t. In Chapter 2 it is shown that this formula is in fact the general solution.

The solution satisfying the initial condition $v(0) = 0$ (which describes an object that is initially dropped) is found from the general solution by solving

$$x(0) = c + \frac{g}{k_0} = 0$$

for

$$c = -\frac{g}{k_0}.$$

This yields the solution

$$x(t) = -\frac{g}{k_0} e^{-k_o t} + \frac{g}{k_0}.$$

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In applications solutions are not always defined for all t. Here is an example.

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EXAMPLE 1.2. *An equation describing the growth of the world's human population $x(t)$ in billions as a function of time t (in years) is*

$$x' = kx^{p+1}$$

where $k > 0$ and $p > 0$ are positive constants estimated from data (see Chapter 3, Sec. 6.) The function

$$x(t) = \frac{1}{(1 - pkt)^{\frac{1}{p}}}$$

is defined on the interval $t < 1/pk$. *(The denominator vanishes at* $t = 1/pk$.*) This function is a solution for* $t < 1/pk$ *since*

$$x'(t) = k\frac{1}{(1-pkt)^{\frac{p+1}{p}}}$$

and

$$f(t,x(t)) = k\left(\frac{1}{(1-pkt)^{\frac{1}{p}}}\right)^{p+1} = k\frac{1}{(1-pkt)^{\frac{p+1}{p}}}$$

are identically equal for all $t < 1/pk$.

Similar calculations show the function

$$x(t) = \frac{x_0}{(1-pkx_0^p t)^{\frac{1}{p}}}$$

is a solution on the interval $t < 1/pkx_0^p$ *for any constant* $x_0 > 0$. *This solution satisfies the initial condition* $x(0) = x_0$.

A formula for the general solution of an equation $x' = f(t,x)$ cannot always be found. The right hand side of the equation $f(t,x)$ involves the unknown solution x and therefore is not a known function of t that we can integrate. Nonetheless, the initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{1.3}$$

has one and only one solution under appropriate conditions placed on $f(t,x)$ as a function of t and x. The derivative of $f(t,x)$ with respect to x is denoted by $\partial f(t,x)/\partial x$ (and called the "partial derivative" of f with respect to x).

THEOREM 1.1. ***(Fundamental Existence and Uniqueness Theorem)*** *Suppose* $f(t,x)$ *and its derivative* $\partial f(t,x)/\partial x$ *with respect to* x *are continuous for* x *near* x_0 *and* t *near* t_0[2]. *Then the initial value problem (1.3) has a solution on an interval containing* t_0. *Moreover, there is no other solution of the initial value problem on this interval.*

For example, consider the initial value problem

$$x' = tx, \quad x(0) = \frac{1}{2}.$$

The function

$$f(t,x) = tx$$

and its derivative

$$\frac{\partial f(t,x)}{\partial x} = t,$$

are continuous for all x and t (and therefore, certainly for x near $x_0 = 1/2$ and t near $t_0 = 0$). Therefore, by Theorem 1.1 this initial value problem has a unique solution on an interval containing $t_0 = 0$. (From the formula $x(t) = e^{t^2/2}/2$ for the

[2]By "continuous for x near x_0" we mean continuous on an interval $a < x_0 < b$ containing x_0. Similarly, by "continuous for t near t_0" we mean continuous on an interval $c < t_0 < d$ containing t_0.

solution it is seen that the solution is defined for all t, a fact not obtainable from Theorem 1.1.)

EXAMPLE 1.3. *An initial value problem describing the growth of a population in a periodically fluctuating environment is*

$$x' = rx\left(1 - \frac{x}{K + a\sin t}\right), \quad x(0) = x_0$$

where x_0 is the initial population size and r, K and $a < K$ are positive constants. Since the denominator never vanishes the function

$$f(t,x) = r\left(x - \frac{x^2}{K + a\sin t}\right)$$

and its derivative with respect to x

$$\frac{\partial f(t,x)}{\partial x} = r\left(1 - 2\frac{x}{K + a\sin t}\right)$$

are continuous for all x and t. Therefore, the initial value problem has a unique solution on an interval containing $t_0 = 0$. No algebraic formula is available for the general solution of this equation, nor for the solution of initial value problems.

If one or both of the conditions on $f(t,x)$ in the existence and uniqueness Theorem 1.1 fail to hold, then one can draw no conclusions from this theorem. In particular, one *cannot* conclude in this case that there is not a solution. For example, for the initial value problem

$$x' = x^{1/3}, \quad x(0) = 0 \tag{1.4}$$

the function

$$f(t,x) = x^{1/3}$$

fails to satisfy the conditions in Theorem 1.1 because the derivative

$$\frac{\partial f(t,x)}{\partial x} = \frac{1}{3}x^{-2/3},$$

is not continuous at $x_0 = 0$ (it is not even defined there). Yet this initial value problem does have a solution: $x(t) = 0$. For an example of an initial value problem that has no solution see Exercise 1.25.

The initial value problem (1.4) also provides an example of non-uniqueness since $x(t) = 0$ and

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}$$

are two different solutions. This does not contradict Theorem 1.1 because the theorem does not apply to this initial value problem.

The Fundamental Existence and Uniqueness Theorem 1.1 provides criteria under which an initial value problem has a solution on an interval containing the initial point $t = t_0$. The *maximal interval* of the solution is the largest interval containing t_0 on which it solves the differential equation. Theorem 1.1 gives no information about the maximal interval of a solution. In fact, without a solution formula it is usually difficult to determine the maximal interval. The function $f(t, x)$ may satisfy the criteria of Theorem 1.1 for all values of t and x and yet solutions may not be defined for all t.

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EXAMPLE 1.4. *Consider the initial value problem*

$$x' = 2tx^2, \quad x(0) = 1.$$

The function

$$f(t, x) = 2tx^2$$

and its derivative

$$\frac{\partial f(t, x)}{\partial x} = 4tx$$

are continuous for all x and t. Theorem 1.1 implies there exists a unique solution on an interval containing $t_0 = 0$. The solution formula

$$x(t) = \frac{1}{1 - t^2}$$

shows the maximal interval is $-1 < t < 1$. See Fig. 1.1.

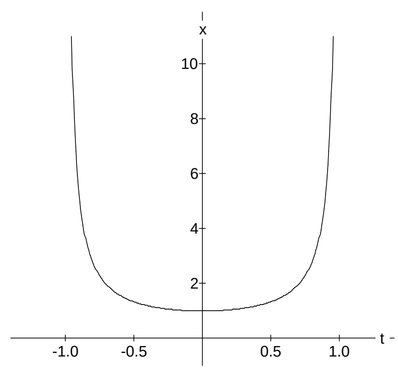

FIGURE 1.1 The solution of the initial value problem $x' = 2tx^2$, $x(0) = 1$ has vertical asymptotes at $t = \pm 1$.

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The importance of the interval of existence of a solution can sometimes be overlooked. Here is an example.

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EXAMPLE 1.5. *A popular computer program gives the formula $x(t) = \sin t$ for the solution of the initial value problem*

$$x' = \sqrt{1 - x^2}, \quad x(0) = 0$$

without indicating the solution interval. Since $\sin t$ is defined for all t, the implication is that $\sin t$ is a solution for all t. This is false, however, since

$$x'(t) = \cos t$$

and

$$f(t, x(t)) = \sqrt{1 - \sin^2 t}$$

are equal only on intervals where $\cos t$ *is positive. Thus, the formula* $x(t) = \sin t$ *defines a solution on the interval* $-\pi/2 < t < \pi/2$*, but on no larger interval containing* $t_0 = 0$*. (However, this interval is not the maximal interval of the solution of the initial value problem! See Exercise 1.26.)*

EXERCISES

Find the general solution of the following differential equations.

Exercise 1.1. $x' = 1 + t^2$

Exercise 1.2. $x' = \cos \pi t$

Exercise 1.3. $x' = e^{2t}$

Exercise 1.4. $x' = te^{-t}$

Find the unique solution of the following initial value problems.

Exercise 1.5. $x' = t^2$, $x(1) = 2$

Exercise 1.6. $x' = e^{-3t}$, $x(0) = 1$

Exercise 1.7. $x' = te^{-t}$, $x(0) = 1$

Exercise 1.8. $x' = \sin 3t$, $x\left(\frac{\pi}{6}\right) = 0$

For which initial value problems can the Fundamental Existence and Uniqueness Theorem 1.1 be applied? Explain your answer. In each case, what do you conclude from this theorem?

Exercise 1.9. $x' = t^2 + x^2$, $x(0) = 0$

Exercise 1.10. $x' = \frac{t^2}{x^2}$, $x(0) = 0$

Exercise 1.11. $x' = \tan x$, $x\left(\frac{\pi}{2}\right) = 0$

Exercise 1.12. $x' = \tan x$, $x(0) = 0$

Exercise 1.13. $x' = \tan x$, $x(0) = \frac{\pi}{2}$

Exercise 1.14. $x' = \ln(tx)$, $x(1) = 2$

Exercise 1.15. $x' = \frac{1}{\sin x}$, $x(0) = \frac{\pi}{2}$

Exercise 1.16. $x' = \frac{1}{t-x}$, $x(-1) = 2$

For what values of the constant a can the Fundamental Existence and Uniqueness Theorem 1.1 be applied to the initial value problems below? Explain your answer. What do you conclude from this theorem for such values of a? What do you conclude from this theorem for other values of a?

EXERCISE 1.17. $x' = \ln(a - x)$, $x(0) = 0$

EXERCISE 1.18. $x' = \tan ax$, $x(0) = \frac{\pi}{2}$

EXERCISE 1.19. $x' = \sqrt{a^2 - x^2}$, $x(1) = 2$

EXERCISE 1.20. $x' = \frac{1}{a-x}$, $x(1) = 2$

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For which t_0 and x_0 does the Fundamental Existence and Uniqueness Theorem 1.1 apply to the initial value problem $x' = f(t,x)$, $x(t_0) = x_0$, with the functions $f(t,x)$ below? Explain your answer. What do you conclude from this theorem for such initial points? What do you conclude from this theorem for other initial points?

EXERCISE 1.21. $f = \ln\left(t^2 + x^2\right)$

EXERCISE 1.22. $f = \frac{t^2}{x}$

EXERCISE 1.23. $f = \tan bx$, $b =$ *constant*

EXERCISE 1.24. $f = \sqrt{t^2 + x^2 - b^2}$, $0 < b =$ *constant*

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EXERCISE 1.25. *Consider the initial value problem $x' = f(t,x)$, $x(0) = 0$ where*

$$f(t,x) = \begin{cases} 1, & \text{for } t \geq 0 \text{ and all } x \\ -1, & \text{for } t < 0 \text{ and all } x. \end{cases}$$

(a) Show the existence and uniqueness Theorem 1.1 does not apply. What do you conclude?

(b) Show this initial value problem does not have a solution on any interval containing $t_0 = 0$.

EXERCISE 1.26. *Consider the equation $x' = \sqrt{1 - x^2}$.*

(a) Show the constant functions $x(t) = 1$ and $x(t) = -1$ are solutions for all t.

(b) Show the function

$$x(t) = \begin{cases} 1 & \text{for } t \geq \frac{\pi}{2} \\ \sin t & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ -1 & \text{for } t \leq -\frac{\pi}{2} \end{cases}$$

is a solution for all t. Thus, the maximal interval for the solution of the initial value problem $x(0) = 0$ is the whole real line.

(c) The solution $x(t) = 1$ and the solution in (b) both satisfy the same initial value problem $x(\pi/2) = 1$ for all t. Why does this not contradict Theorem 1.1?

2. Approximation of Solutions

Formulas for solutions of differential equations are not in general available. For this reason we need other methods for studying equations and their solutions. For some applications it is sufficient to obtain approximations to solutions. For example, roughly sketched graphs of solutions are sometimes adequate. In other applications, more accurate graphs or even numerical approximations are necessary. One can also obtain algebraic formulas for approximations to solutions. In this section we study some graphical and numerical approximation methods. Analytic approximation methods are studied in Chapter 3. We begin with a procedure for making sketches of solution graphs.

2.1. Slope Fields. From algebra and calculus we learn that graphs are a useful way to study functions. The derivative of a function is the slope of its graph. A differential equation therefore tells us something about the slopes of the graphs of its solutions.

Specifically, if the graph of a solution $x = x(t)$ of

$$x' = f(t, x) \tag{2.1}$$

passes through a point (t, x), then the slope $x'(t)$ of its graph at this point equals $f(t, x(t))$. In other words, each point (t, x) in the domain of f is associated with a slope equal to the number $f(t, x)$.

For example, the graph of a solution of $x' = t^2 + x^2$ that passes through the point $(t, x) = (1, 1)$ necessarily has slope $1^2 + 1^2 = 2$ at this point. Similarly, the solution whose graph passes through the point $(-2, 1/3)$ must have slope $(-2)^2 + (1/3)^2 = 37/9$ at this point.

The association of a slope $f(t, x)$ with each point (t, x) defines the *slope field* of the differential equation (2.1). Solutions of differential equation must "fit" its slope field. This means at each point on a solution's graph the slope (of the tangent) must equal the slope associated with that point.

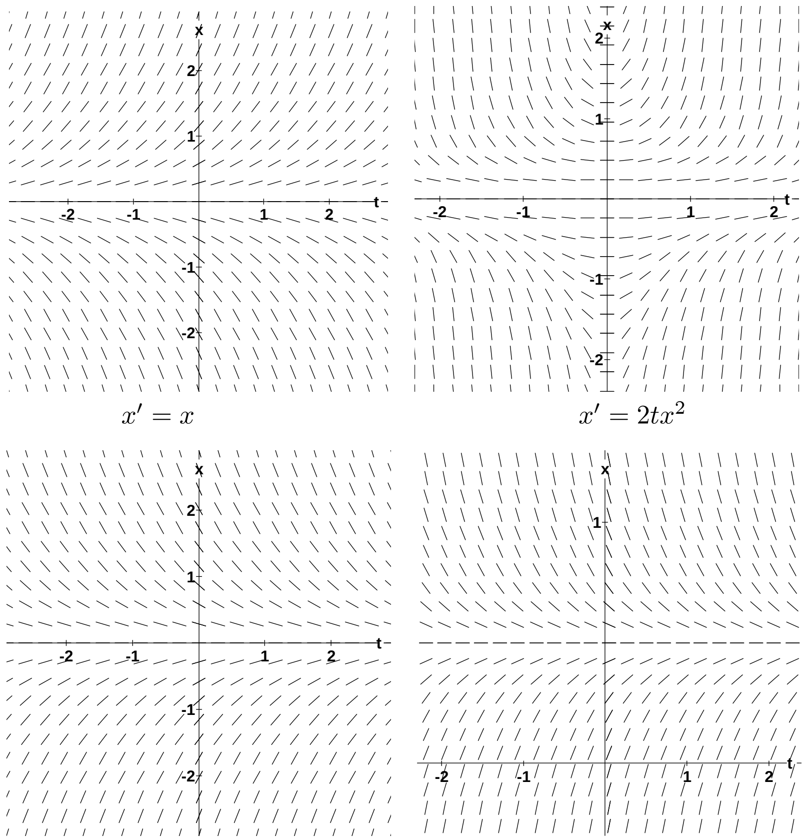
One way to obtain a picture of a slope field is to draw, through each of several points in the (t, x)-plane, a short straight line segment that has the slope associated with that point. By drawing such line segments through a sufficient number of points in the plane, we can get a good approximation to the overall slope field and hence the graphs of solutions.

Rather than randomly choosing points in the plane, it is better to proceed in a systematic manner. We discuss two ways to do this: the "grid" and the "isocline" methods. The grid method is particularly well suited for computer use. The isocline method is sometimes a convenient way to obtain a sketch of the slope field by hand.

THE GRID METHOD

One way to approximate a slope field is to draw a short line segment with the appropriate slope at points lying on a rectangular grid in the (t, x)-plane. This *grid method* can be done by hand; however, most computer programs that "solve" differential equations will also draw slope fields using this "grid" method and display the results graphically.

When sketching a slope field by the grid method, one must chose a grid fine enough so that the essential features of the slope field are apparent, but coarse enough so as not to be visually cluttered. It usually takes a several attempts to find a suitable gird size. Sample slope fields for several differential equations, drawn using the grid method, appear in Fig. 2.1.



$x' = x$ $\qquad$ $x' = 2tx^2$

$x' = -x$ $\qquad$ $x' = 1 - 2x$

FIGURE 2.1. Slope fields are shown for four different differential equations.

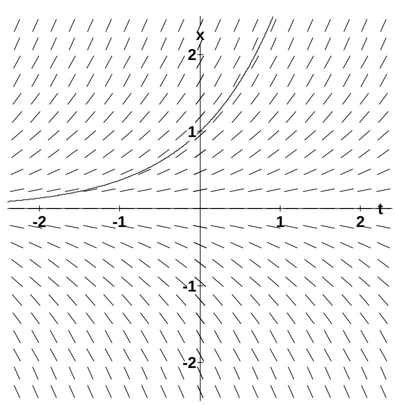


FIGURE 2.2 The slope field for $x' = x$ and the solution satisfying the initial condition $x(0) = 1$.

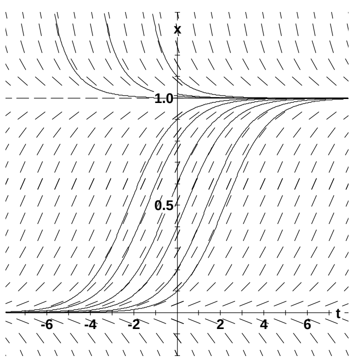
One can sketch the solution graph of an initial value problem $x(t_0) = x_0$ by drawing a curve that both fits the slope field and passes through the point (t_0, x_0). Such a sketch can often suggest important properties of solutions. For example, the slope field and solution sketched in Fig. 2.2 suggest that the solution is monotonically increasing without bound as $t \to +\infty$ and that the x-axis is a horizontal asymptote as $t \to -\infty$.

The next example shows how a slope field can yield important general properties of solutions.

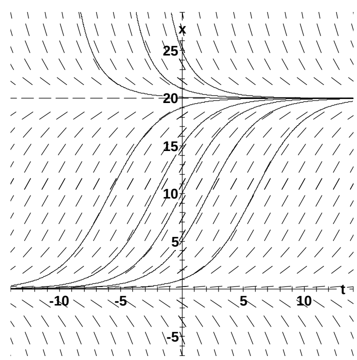
EXAMPLE 2.1. *Fig. 2.3 shows the slope fields of the logistic equation*

$$x' = rx\left(1 - \frac{x}{K}\right)$$

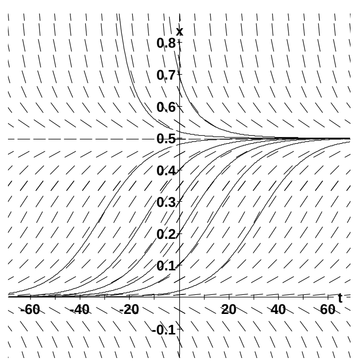
for several choices of the parameters r and K. These slope fields, together with the sample solution graphs, suggest that solutions with positive initial conditions $x(0) = x_0 < K$ tend monotonically to a horizontal asymptote at $x = K$ as $t \to +\infty$. This important fact about the logistic equation will be proved in Chapter 3. Note that $x(t) = K$ is a solution.



(a) $r = 1$, $K = 1$



(b) $r = 0.5$, $K = 20$



(c) $r = 0.1$, $K = 0.5$

FIGURE 2.3. Selected slope fields and solutions for the logistic equation $x' = r\,(1 - x/K)\,x$.

THE ISOCLINE METHOD

In Fig. 2.3 it is interesting to note that the points lying on a horizontal straight line appear to be associated with the same slope. The reason for this is that $f(t,x) = rx(1-x/K)$, and hence the slope at a point (t,x), does not depend on t.This observation in fact applies to any equation whose right hand side f does not depend on the independent variable t, i.e. to any so-called "autonomous" equation (Chapter 3).

A curve all of whose points are associated with the same slopes in the slope field of a differential equation is called an *isocline*. ("iso" means "same" and "cline" means "slope" .) The isoclines of an autonomous equation $x' = f(x)$ are horizontal straight lines. Points on a horizontal line $x = a$ are associated with slope $f(a)$. This fact can be a useful aid in sketching the slope field of an autonomous equation. Fig. 2.4 shows a sketch of the slope field for the equation $x' = x(1-x)$ obtained using this isocline method.

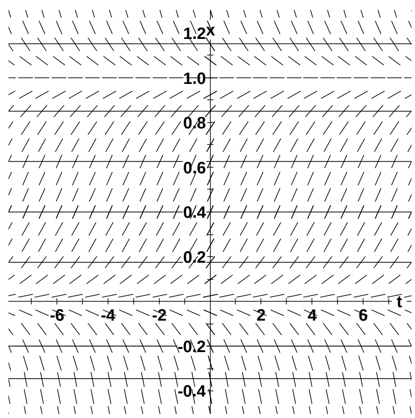


FIGURE 2.4. Some isoclines for $x' = x\,(1-x)$ are shown.

The concept of an isocline is not restricted to autonomous equations. For any equation $x' = f(t,x)$ we can find isoclines by determining those points in the plane that are associated with a common slope m. These points satisfy the *isocline equation*

$$f(t,x) = m.$$

The graph of this equation is, in general, a curve in the plane called the *isocline* associated with slope m.

For nonautonomous equations isoclines are not necessarily horizontal lines. If they can be conveniently graphed, isoclines can be used to sketch slope fields for nonautonomous equations in the same way they were used for autonomous equations. On an isocline we draw several short line segments each having the slope associated with that isocline. Doing this for a collection of isoclines we obtain a sketch of the slope field. The following example illustrates the method.

EXAMPLE 2.2. *What are the isoclines associated with the equation*

$$x' = t^2 + x^2 \ ?$$

Suppose we find the isocline associated with slope $m = 1$. *The equation for this isocline is* $t^2 + x^2 = 1$ *which we recognize as the equation the circle with radius* 1 *and center at the origin* $(0, 0)$. *Drawing this circle and placing on it several short line segments with slope* 1, *we obtain part of the slope field. This procedure can be repeated using other slopes* m. *Points associated with slope* $m = 2$ *lie on the circle of radius* $\sqrt{2}$ *while points associated with slope* $m = 0.25$ *lie on the circle of radius* $\sqrt{0.25}$ *and so on. The typical isocline equation* $t^2 + x^2 = m$ *yields the circle of radius* $\sqrt{m}$, *provided* $m > 0$. *A "degenerate" isocline is obtained for slope* $m = 0$, *namely the single point* $(0, 0)$. *There are no isoclines associated with negative slopes* $m < 0$. *See Fig. 2.5(a).*

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Isoclines are not necessarily easy to identify or graph. Their usefulness for slope field sketching depends on the right hand side $f(t, x)$ of the differential equation. If we can easily identify and graph isoclines, then this method for drawing slope fields is convenient. Otherwise it is not.

Caution: A common mistake is to confuse isoclines with the solution graphs. Isoclines are *not* graphs of solutions. For example, compare the solution graph in Fig. 2.5(b) to the circular isoclines in Fig. 2.5(a).

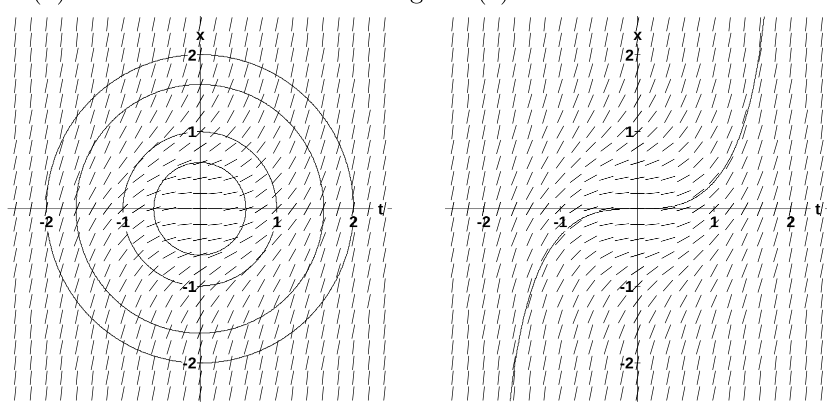

FIGURE 2.6. (a) Selected circular isoclines of $x' = t^2 + x^2$. (b) The solution satisfying the initial condition $x(0) = 0$.

EXERCISES

Use a computer to obtain sketches of the slope fields for the differential equations in the exercises below. Using the slope field, sketch (by hand) the graph of the solutions satisfying each of the given initial conditions.

EXERCISE 2.1. $x' = 1 - x$
$x(0) = 3, \quad x(0) = 0, \quad x(-1) = 2$

EXERCISE 2.2. $x' = 2 - 3x$
$x(0) = 1, \quad x(0) = \frac{2}{3}, \quad x(0) = -1$

EXERCISE 2.3. $x' = 1 - x^2$
$x(0) = -1, \quad x(-2) = 1, \quad x(0) = 0, \quad x(1) = 1.2$

EXERCISE 2.4. $x' = x\left(1 - \frac{x}{2+\cos t}\right)$
$x(0) = 0, \quad x(0) = 1, \quad x(0) = -0.1, \quad x(-2) = 2$

EXERCISE 2.5. $x' = x\cos t$
$x(0) = 0, \quad x(1) = 4, \quad x(0) = 1, \quad x(0) = -1$

EXERCISE 2.6. $x' = -\frac{1}{2}x + \sin t$
$x(0) = 0, \quad x(0) = 1, \quad x(-2) = -1, \quad x(0) = -1$

EXERCISE 2.7. $x' = x\sin x$
$x(0) = 0, \quad x(0) = \frac{\pi}{2}, \quad x(0) = -3, \quad x(0) = 4$

EXERCISE 2.8. $x' = \left(1 + t^2 + x^2\right)^{-1/2}$
$x(0) = 0, \quad x(-1) = -1.5$

EXERCISE 2.9. $x' = (1 - x)\, x \sin^2 t$
$x(0) = -0.25, \quad x(0) = 2, \quad x(-2) = 0.5$

EXERCISE 2.10. $x' = (1 - x^2)\,(\sin t - x)$
$x(0) = 0, \quad x(0) = -0.5, \quad x(1) = 1.5, \quad x(0) = 1$

EXERCISE 2.11. $x' = x(1 - x)(x + 1)$
$x(0) = 0.5, \quad x(0) = -0.5, \quad x(0) = 1.5, \quad x(0) = -1.5$

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EXERCISE 2.12. *Consider the differential equation in Example 1.3 :*

$$x' = rx\left(1 - \frac{x}{K + a\sin t}\right).$$

(a) Use a computer to sketch the slope fields of the equation in the window $0 \le t \le 20$, $0 \le x \le 10$ *for the cases below.*

$$\begin{array}{ll} \textit{(i)} & r = 1, \quad K = 2, \quad a = 1 \\ \textit{(ii)} & r = 1, \quad K = 5, \quad a = 1 \\ \textit{(iii)} & r = 0.5, \quad K = 5, \quad a = 2 \\ \textit{(iv)} & r = 0.5, \quad K = 5, \quad a = 4 \end{array}$$

(b) For each case in (a), use the slope field to sketch (by hand) the graphs of the solutions satisfying the initial condition $x(0) = 1$.

(c) What do all the solutions graphed in (b) seem to have in common?

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Use a computer to obtain a sketch of the slope field for the equations below. Do this for a selection of values for the constant a. How are the slope fields for $a > 1$ different from those for $a < 1$?

EXERCISE 2.13. $x' = -a + 2x - x^2$

EXERCISE 2.14. $x' = x(x - a)(1 - x)$

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Describe (geometrically) and sketch the isoclines for the differential equations below and use them to obtain a sketch of the slope fields.

EXERCISE 2.15. $x' = 1 - x$

EXERCISE 2.16. $x' = 4 - 2x$

EXERCISE 2.17. $x' = \left(1 + t^2 + x^2\right)^{-1/2}$

EXERCISE 2.18. $x' = -x + \sin t$

Find first order differential equations whose isoclines are as described below, if possible. Here m denotes the slope in the field slope. If there is no such equation, explain why.

EXERCISE 2.19. *The family of lines $x = t + m$ where m allowed to be any constant.*

EXERCISE 2.20. *The family of parabolas $x = t^2 + m$ where m is allowed to be any constant.*

EXERCISE 2.21. *The family of lines $x = t + \frac{1}{m}$ where m is allowed to be any nonzero constant.*

EXERCISE 2.22. *The family of parabolas $x = t^2 + \frac{1}{m}$ where m is allowed to be any nonzero constant.*

EXERCISE 2.23. *The family of ellipses $2x^2 + 3t^2 = m^{1/3}$ where m is allowed to be any positive constant.*

EXERCISE 2.24. *The family of circles $x^2 + t^2 = 1 - 2m^2$ where m is allowed to be any positive constant satisfying $0 < c < 1/\sqrt{2}$.*

2.2. Numerical and Graphical Approximations. Slope fields provide approximate graphs of solutions of differential equations. However, it is often desirable to have a more accurate approximation to a solution and its graph than can be obtained from a slope field. Another way to obtain an approximate graph of a solution on an interval $t_0 \le t \le T$ is to calculate numerical approximations x_i to the solution $x(t_i)$ at $t = t_i$ where

$$t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T.$$

and, in the (t, x)-plane, connect the points (t_0, x_0), (t_1, x_1),..., (t_n, x_n) by straight line segments. See Fig. 2.6.

We want to obtain the approximations $x_i \approx x(t_i)$ in such a way that if the number of points t_i increases (and the distances between them tend to zero) then the approximations x_i become more accurate and the approximate ("broken line") graphs approach the (smooth) graph of the solution $x = x(t)$.

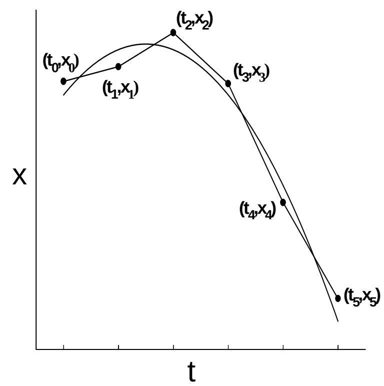

FIGURE 2.6. A "broken line" approximate graph obtained from approximations x_i to the solution at t_i.

In this section we study a basic method for approximating the solution of the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0 \tag{2.2}$$

at specified values of $t > t_0$. The method, called the *Euler Algorithm*, is a fundamental method that serves as an introduction to the numerical approximation of solutions of differential equations. It is, however, rarely used for other than pedagogical reasons because it "converges" too slowly. Sec. 2.3 gives some methods that converge more quickly (and hence are more commonly used). Nonetheless Euler's Algorithm, by providing a basis for understanding how solutions are numerically approximated, is a good starting point for the study of more efficient (and hence complicated) algorithms.

Consider the problem of approximating the solution $x = x(t)$ of (2.2) at $t = t_1 > t_0$. Since $x(t)$ is a solution, we can integrate both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_0$ to $t = t_1$ to obtain

$$x(t_1) - x(t_0) = \int_{t_0}^{t_1} f(t, x(t))dt$$

or, using the initial condition,

$$x(t_1) = x_0 + \int_{t_0}^{t_1} f(t, x(t))dt. \tag{2.3}$$

The right hand side of this equation does not give a formula for $x(t_1)$ because it involves the unknown solution $x(t)$. However, we can use (2.3) to approximate $x(t_1)$ by making an approximation to the integral on the right hand side. For example, we can use integration approximation methods studied in calculus, such as the rectangle rule, the trapezoid rule, or Simpson's rule.

The Euler Algorithm is obtained by using the (left hand) rectangle rule to approximate the integral :

$$\int_{t_0}^{t_1} f(t, x(t))dt \approx (t_1 - t_0)f(t_0, x(t_0)).$$

Defining the first step size by $s_0 = t_1 - t_0$ and recalling the initial condition $x(t_0) = x_0$ we have

$$\int_{t_0}^{t_1} f(t, x(t))dt \approx s_0 f(t_0, x_0)$$

and consequently from (2.3) we have the approximation

$$x(t_1) \approx x_0 + s_0 f(t_0, x_0).$$

Denote this approximation by x_1; that is, we define x_1 by

$$x_1 = x_0 + s_0 f(t_0, x_0).$$

To obtain an approximation x_2 to the solution value $x(t_2)$ at the next point t_2 we proceed in a similar manner. Integrate both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_1$ to $t = t_2$. Using the Fundamental Theorem of Calculus, the (left hand) rectangle rule to approximate the integral and the approximation $x_1 \approx x(t_1)$, we obtain

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} f(t, x(t))dt \approx x_1 + (t_2 - t_1)f(t_1, x_1)$$

We denote this approximation to the solution at $t = t_2$ by

$$x_2 = x_1 + s_1 f(t_1, x_1), \quad s_1 = t_2 - t_1.$$

In calculating the approximation x_2 we introduced *two* sources of error. First, there is the error made in using the rectangle rule to approximate the integral (called the "truncation error") and, secondly, there is the error in using the approximation x_1 to $x(t_1)$. Together these errors account for the "accumulation error" at the point $t = t_2$.

If this procedure is repeated we obtain the following formulas

$$\begin{aligned} x_0 &= x_0 \\ x_{i+1} &= x_i + s_i f(t_i, x_i), \quad s_i = t_{i+1} - t_i, \quad i = 0, 1, 2, \cdots \end{aligned}$$

of the Euler Algorithm. The number x_i is an approximation to the solution $x = x(t)$ of the initial value problem (2.2) at the point $t = t_i$. Usually equally spaced points are chosen, in which case $s_i = s$ for all i and the algorithm reduces to

$$x_0 = x_0 \qquad (2.4)$$
$$x_{i+1} = x_i + sf(t_i, x_i) \quad \text{for} \quad i = 0, 1, 2, \cdots, n.$$

The common distance s is called the *step size* of the algorithm.

The formulas (2.4) are recursive. That is to say, one utilizes the same formula sequentially to calculate the approximations at each of the points t_1, $t_2, \ldots t_n$, using at each step the approximation made at the previous step. This makes the method ideally suited for programming on a computer or calculator.

The accuracy of the integral approximation obtained by the rectangle rule increases if the step size s decreases. For this reason we expect the accuracy of the approximations obtained from the Euler Algorithm (2.4) to increase if the step size s decreases. There is a cost for this increased accuracy, however, because decreasing the step size s will increase the number n of steps necessary to get from the initial condition t_0 to the end point T. This means more repetitions of the algorithm (2.4) are required, and consequently more arithmetic work is necessary to reach the end point $t_n = T$. (This also means more round off errors!)

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EXAMPLE 2.3. *In this example we use the Euler Algorithm (2.4) to approximate the solution $x = x(t)$ of the initial value problem*

$$x' = x, \quad x(0) = 1$$

at $T = 1$ using step size $s = 0.2$. The Euler algorithm (2.4) for this problem is

$$x_{i+1} = x_i + sx_i \quad \textit{for} \quad i = 0, 1, 2, \cdots$$

with $x_0 = 1$. Using step size $s = 0.2$ we need to calculate approximations at the five points $t = 0.2$, 0.4, 0.6, 0.8, 1.0. The calculations are

$$\begin{aligned}
x_1 &= x_0 + sx_0 = 1 + 0.2 \times 1 = 1.2 \\
x_2 &= x_1 + sx_1 = 1.2 + 0.2 \times 1.2 = 1.44 \\
x_3 &= x_2 + sx_2 = 1.44 + 0.2 \times 1.44 = 1.728 \\
x_4 &= x_3 + sx_3 = 1.728 + 0.2 \times 1.728 = 2.0736 \\
x_5 &= x_4 + sx_4 = 2.0736 + 0.2 \times 2.0736 = 2.48832.
\end{aligned}$$

The Euler Algorithm with step size $s = 0.2$ yields the approximation $x(1) \approx x_5 =$ 2.48832.

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How good is the approximation x_5 in the previous example? More generally, how accurate are the approximations (2.4) of the Euler Algorithm? Can we estimate the size of the error and if not how can we have any confidence in the numerical approximations obtained from the formulas (2.4)?

An accurate estimate of the error resulting from approximation methods such as the Euler Algorithm is usually not possible. However, we expect the numerical approximations will get more accurate as the step size s decreases and that they will tend to the exact solution in the limit as $s \to 0$. This turns out to be true for the Euler Algorithm, on the solution's interval of existence, under the assumptions of the Fundamental Existence and Uniqueness Theorem 1.1.

One useful way to study the accuracy of the Euler Algorithm (and of other algorithms as well) is to consider the *rate* at which the approximations converge to the exact solution. The Euler Algorithm is said to be "first order" or of "order 1". What this means is that the magnitude of the error at $t = T$ is no larger than constant multiple of the first power of s. That is to say, there exists a constant $c > 0$ such that $\mid x(T) - x_n \mid \leq cs$. This inequality guarantees the Euler approximations converge to the value of the solution at least as fast as s decreases to 0. Thus, roughly speaking, if the step size s is halved, then in general we expect the error to be (at least) halved. If the step size is decreased by a factor of 1/10, then in general we expect the error to decrease by a factor of 1/10 and so on. (For an example, see Table 2.2 below.) We summarize this by saying that the Euler Algorithm is "$O(s)$" (pronounced "Oh of s").

We can gain confidence in the accuracy of numerical approximations by observing their changes as the step size s decreases. This is commonly done by decreasing s by a fixed fraction. For example, if s is decreased by one half several times, we expect the error to be cut in half each time. Since the approximations at a fixed t approach the solution value $x(t)$, the leading digits in the resulting sequence of approximations should eventually "stabilize" (i.e., remain unchanged as s decreases further). As a practical matter we accept these digits as correct. However, none of these digits may be accurate, since we cannot be sure that they will remain unchanged if the step size s decreases further.

EXAMPLE 2.4. *In this example we repeat Example 2.3 by halving the step size s six consecutive times and observe the resulting change in the approximation to* $x(1)$. *The number of calculations necessary to perform the approximation increases as s decreases. For example, the algorithm (2.4) must be used* 320 *times for the step size* $s = 0.003125$.

We use a computer to perform the calculations and the results appear in Table 2.1. We expect the approximation $x(1) \approx 2.714047$ *obtained from the smallest step size* $s = 0.003125$ *to be the most accurate, but how many of these digits are correct? We know the sequence of approximations converges to the exact value of the solution at* $T = 1$. *Since only two digits appear to have stabilized in Table 2.1, we accept only the two digit approximation* 2.7 *as accurate.*

Step size s	Approximation to $x(1)$
0.200000	2.488320
0.100000	2.593742
0.050000	2.653298
0.025000	2.685064
0.012500	2.701485
0.006250	2.709836
0.003125	2.714047

TABLE 2.1. The Euler Algorithm approximations to the solution at $t = 1$ of the initial value problem $x' = x$, $x(0) = 1$ obtained by repeatedly halving the step size.

There is a formula for the solution of the initial value problem in Examples 2.3 and 2.4, namely $x(t) = e^t$. Therefore, the exact value of the solution at $t = 1$ is

$x(1) = e$ (recall $e \approx 2.718282$). Using this formula we can investigate how accurate the approximations in Table 2.1 really are.

Step size s	Approximation to $x(1)$	% Error
0.200000	2.488320	8.4598
0.100000	2.593742	4.5816
0.050000	2.653298	2.3906
0.025000	2.685064	1.2220
0.012500	2.701485	0.6179
0.006250	2.709836	0.3107
0.003125	2.714047	0.1558

TABLE 2.2. The percent errors of the approximations in Table 2.1.

The percent error of each approximation is given in Table 2.2. Notice the percent error decreases by a factor of (approximately) 1/2 at each consecutive step. This is what we expect, since the step size s decreases by a factor of 1/2 at each step and the Euler Algorithm is $O(s)$.

We approximate the graph of solution of the initial value problem $x' = x$, $x(0) = 1$ by connecting the points (t_i, x_i) with straight line segments. This is done in Fig. 2.7 for decreasing step sizes on the interval $0 \le t \le 5$. The convergence, as s decreases, of these approximate graphs to the graph of the solution $x = e^t$ is apparent.

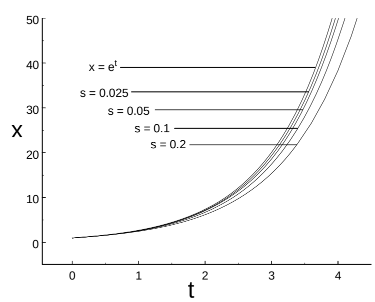

FIGURE 2.7. The broken line graphs calculated from approximations using Euler's Algorithm converge to the solution of the initial value problem $x' = x$, $x(0) = 1$ as the step size s decreases.

One should not accept a graphical approximation to a solution obtained from a single step size s alone (e.g., the default step size in a computer program). Instead, before accepting a graphical approximation, one should decrease the step size until little change occurs in two consecutive graphical approximations.

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EXAMPLE 2.5. *The equation* $x' = ae^{-bt}x$ *describes the growth of a tumor where* $x = x(t)$ *is a measure of its size (e.g., weight or number of cells) and* t *is time. Fig. 2.8 shows approximate graphs of the solution of the initial value problem with* $x_0 = 5$ *and parameter values* $a = 20$ *and* $b = 15$. *These graphs result from the Euler Algorithm using a decreasing sequence of step sizes starting with* $s = 0.1$. *Little change occurs in the graphs for the last two steps sizes* $s = 0.003125$ *and* 0.0015625 *and therefore we accept the final graph as an accurate approximation. All* *of the graphs indicate that the tumor size x approaches a maximal size as $t \to +\infty$. However, the inaccurate graphs obtained from the larger steps sizes considerably over estimate the maximal size of the tumor.*

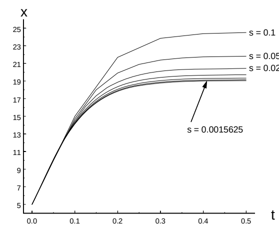

FIGURE 2.8 The Euler Algorithm with a decreasing sequence of steps sizes yields converging approximate graphs for the solution of the initial value problem $x' = 20e^{-15t}x$, $x(0) = 5$.

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The convergence rate $O(s)$ of the Euler Algorithm is sometimes too slow for practical purposes. In Table 2.2 only two digits of accuracy for $x(1)$ are obtained with a step size $s = 0.003125$. To obtain more accuracy a smaller step size is needed. However, there are more intermediate steps with each decrease in step size and it takes longer to perform all of the necessary calculations. Furthermore, other sources of error, such as round-off errors at each step, might eventually prevent increased accuracy if the number of steps (and hence calculations) becomes too large.

Table 2.3 shows an example that dramatically illustrates the slow convergence of the Euler Algorithm. In this example no accurate digits are found with a step size as small as $s = 0.000391$.

Step size s	Approximation to $x(1)$
0.100000	5.862897
0.050000	8.905711
0.025000	13.766320
0.012500	21.242856
0.006250	31.967263
0.003125	45.709606
0.001563	60.736659
0.000781	74.330963
0.000391	84.517375

TABLE 2.3. Euler Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.[3]

Fortunately, practical algorithms with faster rates of convergent are available. In the following section we discuss algorithms of orders two and four. An algorithm

[3]The interval of existence for the solution is $-\infty < t < 100/99 \approx 1.0101$. It is interesting to note that the Euler Algorithm will calculate "approximations" at t values outside of this interval. For example, with step size $s = 0.1$, eleven repetitions of the algorithm produce the number $x_{11} = 9.30025$. However, this number cannot be taken as an approximation to the solution at $t = 1.1$ because the solution is not defined at this value of $t > 100/99$.

has *order of convergence* p (or more succinctly is of *order* p**),** written $O(s^p)$, if the accumulative error is bounded in magnitude by a constant multiple of s^p, i.e., if $\mid x(T) - x_n \mid \leq cs^p$.

To see the advantage of a convergence rate of order greater than $p = 1$ consider an algorithm of order $p = 2$, for which the error satisfies $\mid x(T) - x_n \mid \leq cs^2$. We can expect the error to decrease by a factor of $(1/2)^2 = 1/4$ if the step size s is decreased by a factor of $1/2$, or by a factor of $(1/10)^2 = 1/100$ if the step size s is decreased by a factor of $1/10$, and so on. For an algorithm of order 4 the error decreases even faster, e.g., by a factor of $(1/10)^4 = 1/10000$ if the step size is decreased by a factor of $1/10$.

2.3. Another Numerical Algorithm. In deriving the Euler Algorithm we used the Rectangle Rule to approximate the integral

$$\int_{t_i}^{t_{i+1}} f(t, x(t))dt.$$

More accurate approximations to this integral lead to algorithms that converge faster than the Euler Algorithm. For example, we could use the Trapezoid Rule. (Another choice is Simpson's Rule; see Exercise 2.34.) Integrating both sides of the equation $x'(t) = f(t, x(t))$ from $t = t_i$ to $t = t_{i+1}$ we obtain

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(t, x(t))dt.$$

From the Trapezoid Rule approximation

$$\int_{t_i}^{t_{i+1}} f(t, x(t))dt \approx \frac{s_i}{2}\left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x(t_i))\right]$$

we get

$$x(t_{i+1}) \approx x(t_i) + \frac{s_i}{2}\left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x(t_i))\right].$$

Assuming that we already have an approximation $x(t_i) \approx x_i$ to the solution at the point $t = t_i$, we can write

$$x(t_{i+1}) \approx x_i + \frac{s_i}{2}\left[f(t_{i+1}, x(t_{i+1})) + f(t_i, x_i)\right].$$

Unfortunately we cannot use the right hand side to calculate an approximation x_{i+1} to $x(t_{i+1})$ because it involves $x(t_{i+1})$. This is an example of what is called an *implicit* algorithm because the equation

$$x_{i+1} = x_i + \frac{s_i}{2}\left[f(t_{i+1}, x_{i+1}) + f(t_i, x_i)\right]$$

is not explicitly solved for the approximation x_{i+1}. (The Euler Algorithm is an example of an *explicit* algorithm.) To find the approximation x_{i+1}, we have to solve this equation. To do this at each step results in a highly complicated algorithm. One way to deal with this difficulty is to perform another approximation. For example, we can use the Euler approximation for the x_{i+1} on the right hand side. Thus, at each step we use the formulas

$$\begin{aligned} x^*_{i+1} &= x_i + s_i f(t_i, x_i) \\ x_{i+1} &= x_i + \frac{s_i}{2}\left[f(t_{i+1}, x^*_{i+1}) + f(t_i, x_i)\right], \quad i = 0, 1, 2, \cdots \end{aligned}$$

to calculate the approximation x_{i+1}. This algorithm is called *Heun's Algorithm* (sometimes the *Improved Euler Algorithm* or the *Modified Euler Algorithm*). It is an example of a "predictor-corrector" algorithm. At each step the Euler approximation x^*_{i+1} is the prediction and x_{i+1} is the correction.

If equal step sizes $s_i = s$ are used Heun's Algorithm is

$$\begin{aligned} x^*_{i+1} &= x_i + sf(t_i, x_i) \\ x_{i+1} &= x_i + \frac{s}{2}\left[f(t_{i+1}, x^*_{i+1}) + f(t_i, x_i)\right], \quad i = 0, 1, 2, \cdots . \end{aligned} \tag{2.5}$$

The initial condition x_0 starts the algorithm. It turns out that Heun's Algorithm of order $O(s^2)$.

Compare the results in Table 2.4 with those in Table 2.2. Note that the error in Table 2.4 decreases approximately by a factor of 1/4 as the steps size is decreased by a factor of 1/2. Heun's Algorithm is a popular procedure; for example, it is often used with programmable hand calculators.

Step size s	Approximation to $x(1)$	% Error
0.200000	2.702708	0.5729
0.100000	2.714081	0.1545
0.050000	2.717191	0.0401
0.025000	2.718004	0.0102
0.012500	2.718212	0.0026
0.006250	2.718264	0.0007

TABLE 2.4. The Heun Algorithm approximations to the solution of the initial value problem $x' = x$, $x(0) = 1$, at $t = 1$ obtained by repeatedly halving the step size.

We saw in Table 2.3 an example of an initial value problem for which the Euler Algorithm converges too slowly to be practical. Table 2.5 shows the results of applying Heun's Method to the same initial value problem. The estimates obtained from the two numerical algorithms differ considerably. At each step size Heun's Algorithm provides a more accurate approximation to $x(1) = 99$ than does the Euler Algorithm.

Step size s	Approximation to $x(1)$
0.100000	19.346653
0.050000	33.073325
0.025000	52.217973
0.012500	72.662362
0.006250	87.787581
0.003125	95.273334
0.001563	97.719807
0.000781	98.719804
0.000391	98.928245

TABLE 2.5. Heun's Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.

Even higher order algorithms are available, although they involve more complicated formulas at each step. A widely used class of algorithms are called Runge-Kutta algorithms. These algorithms are available for any order of convergence. A popular algorithm is the fourth order Runge-Kutta algorithm. You can see the complicated formulas for this algorithm in Exercise 2.25. Table 2.6 shows the results applying this algorithm to the same initial value problem in Table 2.3 and 2.5. This faster converging algorithm provides an accurate approximation to $x(1) = 99$.

Step size s	Approximation to $x(1)$
0.100000	53.355933
0.050000	75.881773
0.025000	91.639594
0.012500	97.671604
0.006250	98.856123
0.003125	98.988718
0.001563	98.999238
0.000781	98.999951
0.000391	98.999997

TABLE 2.6. Fourth order Runge-Kutta Algorithm estimates to the solution of the initial value problem $x' = x^2$, $x(0) = 0.99$, at $t = 1$ for a decreasing sequence of step sizes. The solution formula $x(t) = 99/(100 - 99t)$ for this initial value problem gives the exact value $x(1) = 99$.

EXERCISES

EXERCISE 2.25. *The following formulas constitute the fourth order Runge-Kutta Algorithm :*

$$x_0 = x_0, \qquad x_{i+1} = x_i + s\tfrac{L_1+2L_2+2L_3+L_4}{6} \quad \text{for} \quad i = 0, 1, 2, \cdots$$

where

$$\begin{aligned} L_1 &= f(t_i, x_i) \\ L_2 &= f\left(t_i + \frac{s}{2}, x_i + \frac{s}{2}L_1\right) \\ L_3 &= f\left(t_i + \frac{s}{2}, x_i + \frac{s}{2}L_2\right) \\ L_4 &= f(t_i + s, x_i + sL_3) \end{aligned}$$

At each step one must calculate, in order, the four numbers L_1, L_2, L_3, and L_4 before calculating x_{i+1}.

(a) Use the fourth order Runge-Kutta method to approximate the solution of $x' = x$, $x(0) = 1$ at $t = 1$. Start with step size $s = 0.2$ and calculate a sequence of approximations by repeated step size halving.

(b) Use the solution formula $x = e^t$ to calculate percent errors. Do the errors decrease at the expected rate?

(c) Compare the results in (a) and (b) with those of the Euler and Heun's Algorithms in Tables 1.2 and 1.4.

EXERCISE 2.26. *Let $x = x(t)$ denote the solution of the initial value problem $x' = x^3$, $x(0) = 0.6$. It turns out that $x(1) \approx 1.1338934190$.*

(a) Use the Euler Algorithm to obtain an approximation to $x(1)$ with step size $s = 0.1$. How many correct significant digits does this approximation have?

(b) Obtain Euler approximations by repeatedly halving the step size (starting at $s = 0.1$*). At which step size* s *is the Euler approximation first correct to* 2 *decimal places? To* 3 *decimal places?*

(c) Compute the absolute error at each step size, starting from $s = 0.1$ *and halving four times. Is the fractional decrease in the error correct for the Euler Algorithm?*

EXERCISE 2.27. *Repeat Exercise 2.26 using Heun's Algorithm.*

EXERCISE 2.28. *Repeat Exercise 2.26 using the Runge-Kutta Algorithm.*

EXERCISE 2.29. *Repeat Exercise 2.26 using any other algorithm available on your computer.*

EXERCISE 2.30. *Let* $x = x(t)$ *denote the solution of the initial value problem* $x' = e^x$, $x(0) = 0$. *It turns out that* $x(0.8) \approx 1.6094379124$.

(a) Use the Euler Algorithm to obtain an approximation to $x(0.8)$ *with step size* $s = 0.1$. *How many correct significant digits does this approximation have?*

(b) Obtain Euler approximations by repeatedly halving the step size. At which step size s *is the Euler approximation first correct to* 2 *decimal places? To* 3 *decimal places?*

(c) Compute the absolute error at each step size, starting from $s = 0.1$ *and halving four times. Is the fractional decrease in the error correct for the Euler Algorithm?*

EXERCISE 2.31. *Repeat Exercise 2.30 using Heun's Algorithm.*

EXERCISE 2.32. *Repeat Exercise 2.30 using the Runge-Kutta Algorithm.*

EXERCISE 2.33. *Repeat Exercise 2.30 using any other algorithm available on your computer.*

EXERCISE 2.34. *Euler's Algorithm was derived by using the Rectangle Rule to approximate the integral* $\int_{t_i}^{t_{i+1}} f(t, x(t))dt$ *and Heun's Algorithm was derived by using the Trapezoid Rule. In this exercise you derive an algorithm by using Simpson's rule to approximate this integral. Simpson's rule for approximating an integral* $\int_a^b g(t)dt$ *is*

$$\int_a^b g(t)dt \approx \frac{1}{3}\left[g(b) + 4g\left(\frac{a+b}{2}\right) + g(a)\right].$$

(a) Use Simpson's rule to obtain a predictor-corrector algorithm for an approximation x_{i+1} *of the solution* $x = x(t)$ *of the initial value problem* $x' = f(t, x)$, $x(t_0) = x_0$ *at* $t = t_{i+1}$. *Use equal step sizes of length* s.

(b) Why is the algorithm derived in (a) called a "two step" algorithm? What problem does this cause at the start (i.e., at t_1*) and how might this be solved?*

(c) Use you answers from (a) and (b) to obtain an approximation to the solution of $x' = x$, $x(0) = 1$ *at* $T = 0.2$ *using a step size of* $s = 0.1$. *(The exact solution is* $e^{0.2} = 1.2214027582$*).*

(d) Compare your answer in (c) to the approximations obtained by the Euler and Heun's. If your computer has the fourth order Runge-Kutta Algorithm, compare its approximations also. Which algorithm gives the best approximation at $T = 0.2$*?*

EXERCISE 2.35. *Approximate the solution of the initial value problem* $x' = t^2 + x^2$, $x(0) = 0$ *at* $T = 0.5$ *using the Euler Algorithm, Heun's Algorithm, and the*

Runge-Kutta Algorithm. Start with step size $s = 0.1$ and repeat by halving the step size four times. What are the accurate digits obtained from each algorithm? What is the best approximation obtained from all methods?

EXERCISE 2.36. *Use a computer obtain an accurate graphical solution of the initial value problem $x' = t^2 + x^2$, $x(0) = 0$ on the interval from $t = 0$ to $T = 1$ using the Euler Algorithm. Repeatedly halve the step size s starting with $s = 0.1$. What step size did you stop with and why?*

EXERCISE 2.37. *Repeat Exercise 2.36 using Heun's Algorithm.*

EXERCISE 2.38. *Repeat Exercise 2.36 using the Runge-Kutta Algorithm.*

EXERCISE 2.39. *Repeat Exercise 2.36 using any other algorithm available on your computer.*

EXERCISE 2.40. *Use a computer obtain an accurate graphical solution of the initial value problem $x' = \frac{x^3}{x-t}$, $x(0) = 1$ on the interval from $t = 0$ to $T = 1$ using the Euler Algorithm. Use a window size of $-20 < x < 20$. Repeatedly decrease the step size s by a factor of one tenth, starting with $s = 0.1$. What step size did you stop with and why?*

EXERCISE 2.41. *Repeat Exercise 2.40 using Heun's Algorithm.*

EXERCISE 2.42. *Repeat Exercise 2.40 using the Runge-Kutta Algorithm.*

EXERCISE 2.43. *Repeat Exercise 2.40 using any other algorithm available on your computer.*

EXERCISE 2.44. *(a) Use any algorithm you wish to obtain a graphical solution of the initial value problem $x' = 500\cos(200t)$, $x(0) = 0$. Start with step size $s = 0.1$ and decrease until the graph has stabilized. What do you conclude about the solution?*

(b) Obtain a formula for the solution and use it to explain the graphical solution.

3. Chapter Summary & Exercises

A solution $x = x(t)$ of the differential equation $x' = f(t, x)$ is a differentiable function for which $x'(t) = f(t, x(t))$ holds for all t on an interval. In general a differential equation has infinitely many solutions. The general solution is the set of all solutions. We need an additional requirement in order to specify a unique solution. For a given point (t_0, x_0), the initial condition $x(t_0) = x_0$ is such a requirement. Theorem 1.1 gives conditions under which an initial value problem $x' = f(t, x)$, $x(t_0) = x_0$ has one and only one solution. Specifically, if $f(t, x)$ and its derivative $\partial f(t, x)/\partial x$ with respect to x are both continuous for t near t_0 and x near x_0, then there is one and only one solution. Although formulas for the solution cannot always be calculated, many kinds of approximation methods are available. The slope field associated with the differential equation helps in to sketching a graph of the solutions. A computers is useful for plotting the slope fields by the grid method; this method associates the slope $f(t, x)$ with each point (t, x) on from a chosen grid of points in the (t, x)-plane. Also useful for sketching slope fields are isoclines, which are curves in the (t, x)-plane made up of those points associated with a common slope. Numerical approximations to solution values $x(t)$ yield more accurate graphs of the solution. If x_1, $x_2, ...,$ x_n approximate the solution values $x(t_1)$, $x(t_2), ..., x(t_n)$ for $t_1 < t_2 < \cdots < t_n$, then by connecting the points (t_i, x_i) with straight line segments we construct an approximate (broken line segment) solution graph. Usually equally spaced points t_i are chosen and the common distance between them is the step size s of the method. If the approximations converge to the solution values as s tends to 0, then the broken line graph tends to the solution graph as s tends to 0. The Euler Algorithm is one method for calculating such approximations. It is based on the left hand rectangle rule for approximating an integral. Under the conditions on $f(t, x)$ in Theorem 1.1 the Euler approximations converge to the solution values as the step size s decreases to 0. The Euler Algorithm is of order 1, which means the errors tend to 0 at the same rate that s tends to 0. Faster converging algorithms are available. Heun's Algorithm is of order 2, which means the error tends to 0 at the same rate that s^2 tends to 0. A fourth order method called the Runge-Kutta Algorithm is commonly used.

EXERCISES

Find formulas for the general solutions of the differential equations below.

Exercise 3.1. $x' = \frac{1}{(1-t)t}$

Exercise 3.2. $x' = \frac{2t}{1+t^2}$

Find solution formulas for the following initial value problems.

Exercise 3.3. $x' = \frac{1}{1+t^2}$, $x(1) = \frac{\pi}{2}$

Exercise 3.4. $x' = \frac{1+t+t^2}{(1+t^2)t}$, $x(1) = 1$

Exercise 3.5. *Does existence and uniqueness Theorem 1.1 apply to the initial value problem $x' = \sqrt{1-x}$, $x(1) = 0$? Explain your answer. What do you conclude?*

EXERCISE 3.6. *Does the existence and uniqueness Theorem 1.1 apply to the initial value problem* $x' = \left(4 - x^2\right)^{-1}$, $x(2) = 0$. *Explain your answer. What do you conclude?*

For which initial values t_0 and x_0 does the existence and uniqueness Theorem 1.1 apply to the problems below? Explain your answer. What do you conclude? What can you conclude about initial value problems for other t_0 and x_0?

EXERCISE 3.7. $x' = \ln|x - t|$, $x(t_0) = x_0$

EXERCISE 3.8. $x' = \sqrt{9 - x^2 - t^2}$, $x(t_0) = x_0$

EXERCISE 3.9. $x' = |x|$, $x(t_0) = x_0$

EXERCISE 3.10. $x' = t^{\frac{1}{3}}x$, $x(t_0) = x_0$

—————————————————————

Explain why Theorem 1.1 does not apply to the initial value problems below. What do you conclude?

EXERCISE 3.11. $x' = \sqrt{x^2 + t^2}$, $x(0) = 0$

EXERCISE 3.12. $x' = \sqrt{\sin(x^2 + t^2)}$, $x(0) = 0$

—————————————————————

EXERCISE 3.13. *Apply the existence and uniqueness Theorem 1.1 to the initial value problem* $x' = \sqrt{1 - x^2}$, $x(0) = 0$ *in Example 1.5. What do you conclude?*

EXERCISE 3.14. *Let* $f(t, x)$ *be a polynomial in* t *and* x. *Prove that any initial value problem* $x' = f(t, x)$, $x(t_0) = x_0$ *has a unique solution on an interval containing* t_0.

EXERCISE 3.15. *Let* $p(z, w)$ *be a polynomial in* z *and* w *and let* $f(t, x) = p(\sin t, \sin x)$. *Prove that any initial value problem* $x' = f(t, x)$, $x(t_0) = x_0$, *has a unique solution on an interval containing* t_0.

—————————————————————

Use a computer to obtain sketches of the slope fields associated with the following differential equations. By hand, sketch graphs of the solutions satisfying each of the given initial conditions.

EXERCISE 3.16. $x' = t^2 + 4x^2$
$x(0) = 0, \quad x(0.5) = 0.5$

EXERCISE 3.17. $x' = -\frac{t}{x}$
$x(0) = 1, \quad x(1) = -1$

EXERCISE 3.18. $x' = \frac{t^2 - x^2}{t^2 + x^2}$
$x(0) = 1, \quad x(-1) = -1$

EXERCISE 3.19. $x' = \ln\left(t^2 + x^2\right)$
$x(1) = 0, \quad x(0) = 0.1$

———————————————————————————————

EXERCISE 3.20. *Find the isocline equation for the differential equations in Exercises 3.16-3.19 and graph several typical isoclines. Use your results to sketch the slope field of the equation.*

———————————————————————————————

Use a computer to obtain sketches the slope fields associated with the equations in the initial value problems below. Hand sketch a graph of the solution satisfying each of the given initial conditions.

EXERCISE 3.21. $x' = 1 - x$
$x(0) = 0, \qquad x(0) = 1.5$

EXERCISE 3.22. $x' = x - 1$
$x(0) = 0, \qquad x(0) = 1.5$

EXERCISE 3.23. $x' = 1 - x^2$
$x(0) = 0, \qquad x(0) = 1.5$

EXERCISE 3.24. $x' = \sin(x^2 + t^2)$
$x(0) = 0, \qquad x(0) = -0.5$

———————————————————————————————

EXERCISE 3.25. *Consider the initial value problem $x' = x^3e^{-t}$, $x(0) = 1$. Apply the Euler Algorithm to approximate the solution at $T = 0.6$.*

(a) Start with step size $s = 0.1$ and halve it four times. Which digits in the resulting approximations do you think are accurate? Explain your answer.

(b) Halve the step size four more times. Now which digits in the resulting approximations do you think are accurate? Explain your answer.

EXERCISE 3.26. *Consider the initial value problem $x' = x^3e^{-t}$, $x(0) = 1$. Apply Heun's Algorithm to approximate the solution at $T = 0.6$.*

(a) Start with step size $s = 0.1$ and halve it four times. Which digits in the resulting four approximations do you think are accurate? Explain your answer.

(b) Halve the step size four more times. Now which digits in the resulting four approximations do you think are accurate? Explain your answer.

EXERCISE 3.27. *Consider the initial value problem $x' = x^3e^{-t}$, $x(0) = 1$. Apply the Runge-Kutta algorithm to approximate the solution at $T = 0.6$. (See Exercise 2.25.)*

(a) Start with step size $s = 0.1$ and halve it four times. Which digits in the resulting four approximations do you think are accurate? Explain your answer.

(b) Halve the step size four more times. Now which digits in the resulting four approximations do you think are accurate? Explain your answer.

EXERCISE 3.28. *Use the formula $x(t) = \left(2e^{-t} - 1\right)^{-1/2}$ for the solution of the initial value problem in Exercises 3.25, 3.26, and 3.27 to calculate the error and the per cent error of the approximations in these exercises for step size $s = 0.00625$. Round all numbers to 6 significant digits.*

EXERCISE 3.29. *Use the Euler Algorithm and a computer program to obtain an accurate graph of the solution of the initial value problem* $x' = 1.5x^3 \sin 10t$, $x(0) = 1$ *on the interval from* $t = 0$ *to* $T = 1$. *Use a window size of* $-2 < x < 2$. *Repeatedly halve the step size* s *starting with* $s = 0.2$. *At what step size did you stop and why?*

EXERCISE 3.30. *Repeat Exercise 3.29 using Heun's Algorithm.*

EXERCISE 3.31. *Repeat Exercise 3.29 using the Runge-Kutta Algorithm.*

EXERCISE 3.32. *Suppose the decay rate of a radioactive isotope is* $r = -0.35$ *per year. The differential equation for the amount* $x(t)$ *at time* t *is* $x' = -0.35x$.

(a) Use a computer to study the graphs of solutions with many different initial conditions $x_0 > 0$ *and formulate a conjecture about the length of time it takes a sample amount of the isotope to decay to one half of its initial amount.*

(b) Use the solution formula $x(t) = x_0 e^{-0.35t}$ *to verify or disprove your conjecture.*

EXERCISE 3.33. *Let* $x = x(t)$ *be the dollars in an investment account which is compounded continuously at a rate of* 4.5%.

(a) Perform numerical experiments on the model equations $x' = 0.045x$, $x(0) = s$ *to formulate a conjecture about how long will it take for the initial investment of* s *dollars to triple.*

(b) Use the solution formula $x(t) = te^{0.045t}$ *to prove or disprove your conjecture.*

EXERCISE 3.34. *Suppose a population has a per capita death rate* $d > 0$ *and a per capita birth rate that is proportional to population size* x *(with constant of proportionality denoted by* $a > 0$*).*

(a) Use the inflow-outflow rule (2.1) to write down a model differential equation for the population size $x = x(t)$.

(b) Perform numerical experiments and formulate a conjecture about the fate of the population. (Hint: choose a pair of model parameter values, such as $a = 1$ *and* $d = 1$*, and compute solution graphs for many initial population sizes* $x(0) = x_0$. *Then repeat for other values for* a *and* d.*)*

(c) Use the solution formula

$$x(t) = \frac{dx_0}{x_0 a + e^{dt}(d - x_0 a)}$$

to verify or disprove your conjectures in (b).

4. Applications

4.1. Bacterial Cell Growth. When placed in an environment of abundant resources (nutrients, space, etc.) cell cultures typically grow in such a way that their per capita rate of change is constant. Mathematically, this means the number of cells $x = x(t)$ at time t satisfies the differential equation

$$x' = rx$$

where the constant $r > 0$ is the "per capita growth rate". Often a particular microorganism's growth rate is described by the time it takes the number of cells in the culture to double. This time δ is called the "doubling time" (or "generation time") and it is related to the growth rate according to the formula

$$r = \frac{\ln 2}{\delta}.$$

For more detailed discussion of these topics and of population growth models see Sec.6, Chapter 3.

As an example, the doubling time of the bacterium *Staphylococcus aureus* is approximately $\delta = 30$ minutes, which corresponds to a per capita growth rate of

$$r = \frac{\ln 2}{30} = 0.02310 \text{ (per minute)}.$$

The growth of a culture of *S. aureus* initially consisting of 10^6 cells is described by the initial value problem

$$\begin{aligned} x' &= 0.02310x \\ x(0) &= 1. \end{aligned} \tag{4.1}$$

Here x is measured in units of 10^6 cells.

According to Theorem 1.1, this initial value problem has a unique solution $x = x(t)$. A slope field and a solution graph (drawn using Heun's Algorithm with step size $s = 0.05$) appear in Fig. 2.9. Notice the number of cells grows rapidly, following a seemingly exponential-like curve. Indeed, the solution formula for the initial value problem

$$x = e^{0.02310t}$$

shows the growth is indeed exponential.

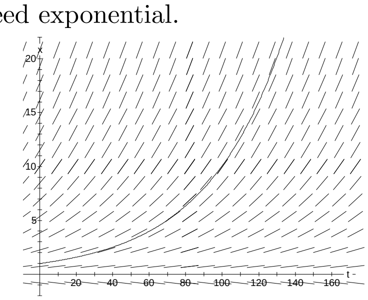

FIGURE 2.9. The slope field of the differential equation $x' = 0.02310x$ and the solution of the initial value problem (4.1) drawn using Heun's Algorithm with step size $s = 0.05$.

S. aureus is a common cause of bacterial skin infection (particularly in patients with HIV). The rapid exponential growth of a staph infection can be a serious problem if left untreated. Our modeling application involves determining the effect of a medical treatment that removes staph cells from the patient at a certain rate $h > 0$ (cells/minute). We set up our mathematical model (i.e., perform the Model Derivation Step in Fig. 2.1 of the Introduction) by applying the inflow-outflow rule (2.1) to the staph cell population numbers. This leads to the differential equation

$$x' = 0.02310x - h \tag{4.2}$$

More specifically, suppose a milligram (mg) of antibiotic in a particular patients kills staph cells at a rate of 10^4 per minute. Then a dosage of d mg kills a total of $10^4 d$ staph cells per minute. In units of 10^6 cells, we have

$$h = \frac{10^4}{10^6} d = 0.01d. \text{(per minute)} \tag{4.3}$$

Suppose, for the moment, that this removal rate h remains constant in time, as might be the case for example if the antibiotic were continuously administered intravenously. We want to know what dosages d, if any, will eliminate the staph infection from the patient, and if so in what amount of time.

The antibiotic kill rate h in (4.3) leads to the initial value problem

$$\begin{aligned} x' &= 0.02310x - 0.01d \\ x(0) &= 1. \end{aligned} \tag{4.4}$$

for the number of staph cell $x = x(t)$. Our next goal is to perform the Model Solution Step in the Modeling Cycle. What we want to learn from the solution $x = x(t)$ is whether or not it continues to increase or whether it decreases and eventually equals 0. The answer will presumably depend on the dosage d.

One way to obtain answers to our questions would be from a formula for the solution $x(t)$. We will learn how to find such a formula in Chapter 2. Here, however, we will investigate the solution by means of the methods developed in Sec. 2 and 2.2.

Fig. 2.10 shows slope fields and solution graphs, for a selection of dosages d, obtained by a computer. These graphs indicate the existence of a critical dosage level d_{cr} above which the staph infection is eliminated and below which it is not. From Fig. 2.10 this critical dose lies between 1.5 gm and 3.0 gm. Further computer explorations, using other values of d, suggest this critical value is approximately $d_{cr} = 2.31$ gm.

Another way to determine the critical value is to reason as follows. For $d < d_{cr}$, the staph infection increases ($x' > 0$) and for $d > d_{cr}$ it decreases ($x' < 0$). Therefore, at the critical dose $d = d_{cr}$ the infection should do neither, but instead remain constant. From the initial value problem (4.4), we see that x remains at $x(0) = 1$, and hence $x' = 0$, means

$$0.02310 - 0.01d_{cr} = 0$$

or

$$d_{cr} = 2.31.$$

At the critical dose d_{cr} the staph infection remains constant, but at a higher dose $d > d_{cr}$ our computer studies indicate that $x(t) = 0$ at some finite time t_c. This ("cured") time $t_c = t_c(d)$ when the infection is eliminated depends on d, as Fig. 2.10 shows. The higher the dose, the quicker the staph is eliminated; that is, $t_c(d)$ is a decreasing function.

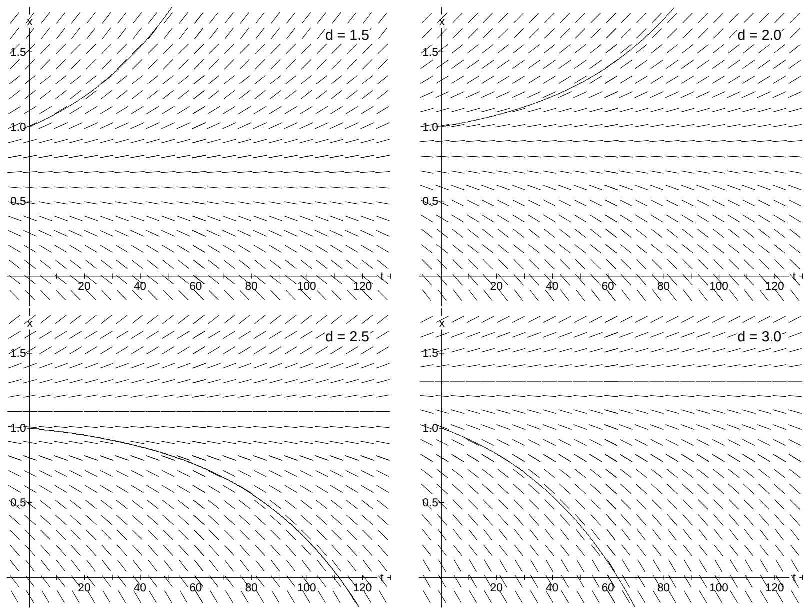

FIGURE 2.10. The slope field of the differential equation $x' = 0.02310x - 0.01d$ and the solution of the initial value problem (4.1) for selected values of the antibiotic dose d.

We emphasize that computer explorations do not "prove" our conclusions about the existence of a critical dosage and the dependence of t_c on d. This is because, when doing computer studies, we can calculate only a finite number of solutions for only a finite selection of dosages d. An advantage of a solution formula, if available (or, if not, other methods of analysis) is that these conclusions can be rigorously established. (See Exercise 6.36 in Chapter 2).

Often antibiotics are not continuously administered to a patient, but a dose is applied by pill or injection. In this case, the effect of the antibiotic is not constant, but decreases over time. To account for this change we return to the model equation (4.2) to see what adjustments must be made (this is the Model Modification Step of the Modeling Cycle). To proceed we need information concerning how the effectiveness of the antibiotic changes over time, so that we can derive a formula for the staph removal rate h.

Suppose, for example, the effectiveness of the antibiotic decreases exponentially so that

$$h = 0.01de^{-at}$$

Under this model assumption, the initial effectiveness of the antibiotic is $0.01d$ (cells/minute), but the effectiveness decreases over time with an exponential decay rate of $a > 0$. Suppose it is observed that the effectiveness decreases by 50% every hour. This allows us to calculate a. In 60 minutes, h is decreased by a fraction of $1/2$ and therefore

$$e^{-a60} = 0.5.$$

or

$$a = 0.01155.$$

These assumptions lead us to a new initial problem for a staph infection starting with 10^6 cells:

$$x' = 0.02310x - 0.01de^{-0.01155t} \tag{4.5}$$
$$x(0) = 1.$$

(Recall x is measured in units of 10^6.)

Again we ask: what dosages d, if any, will eliminate the staph infection?

Fig. 2.11 shows the slope field and the solution of the initial value problem (4.5) for some selected values of the dose d. These samples suggest that this initial value problem also has a critical dosage d_{cr} below which the treatment does not eliminate the staph infection. The particular examples in Fig. 2.11 indicate that d_{cr} lies between 2.5 gm and 4.0 gm. (See Exercise 6.37 in Chapter 2.)

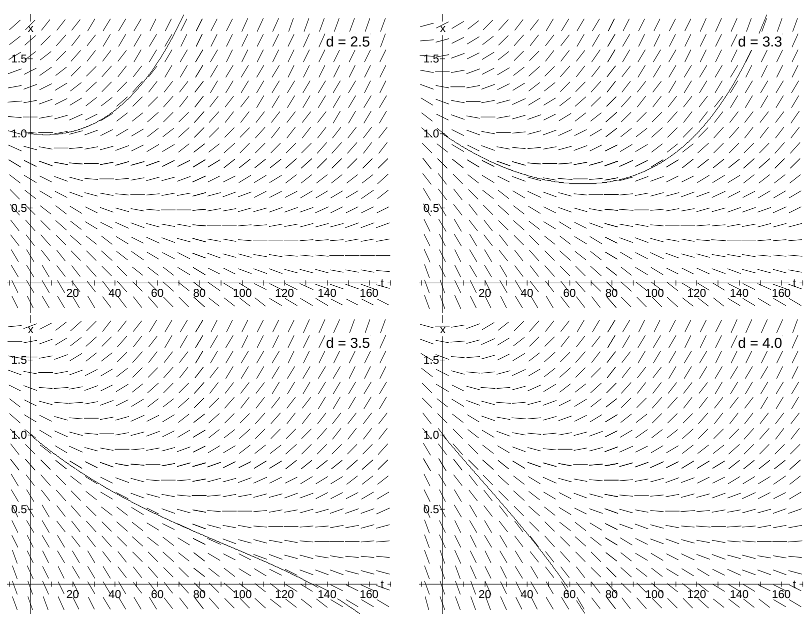

FIGURE 2.11. The slope field of the differential equation $x' = 0.02310x - 0.01de^{-0.01155t}$ and the solution of the initial value problem (4.5) for selected values of the antibiotic dose d.

An interesting difference between the intravenous treatment modeled by (4.4) and the pill or injection treatment modeled by (4.5) occurs for doses below the critical level d_{cr}. Unlike the intravenous treatment, the pill or injection treatment can show an initial improvement (x initially decreases in Fig. 2.11 for $d = 2.5$ and 3.3) even though the infection ultimately "bounces back" and grows unabated. Thus, one must guard against a mistaken conclusion, based on its early effectiveness, that the treatment will result in a cure.

4.2. Running a Curve. One of the most famous laws of physics is Newton's law of motion given by the equation $F = ma$. Here m is the mass of a moving object and a is its acceleration. The letter F represents the force (or a collection of many forces $F = F_1 + F_2 + \cdots$) acting on the object. Since

$$a = \frac{dv}{dt} = \frac{d^2x}{dt},$$

where v is the object's velocity and x is its position (measured from some reference point), the application of Newton's law usually results in a differential equation that describes the motion of the object when subject to the force F. The Modeling Derivation step of the Modeling Cycle involves describing the relevant forces acting on the object so as to obtain a mathematical expression for F. We will utilize this law in a variety of applications throughout the book. In this section, we will use Newton's law to study a sprinter running a race of fixed distance.

One model of a sprinter running in a straight line assumes two forces are involved: the propulsive force exerted by the runner and a resistive force (due mainly to air resistance).[4] Thus, $F = F_p + F_r$ in Newton's law. In this model it is assumed that the runner exerts a constant propulsive force throughout the race, an assumption that seems reasonable for sprint of short distance. Thus,

$$F_p = mf$$

were $f > 0$ is the per unit mass force characteristic of a particular individual runner. The resistive force, on the other hand, depends on the runner's velocity. It is absence when the runner is not moving and it increases with the runner's speed. The simplest law assumes the resistive force is proportional to velocity v, i.e.,

$$F_r = -cv$$

where the "coefficient of friction" $c > 0$ is another characteristic of each particular runner. The reason for the negative sign is that the resistive force works against the runner.

In the absence of other forces, Newton's law yields the differential equation

$$m\frac{dv}{dt} = mf - cv$$

for the runner's velocity v. If we divide both sides by m and denote the per unit mass coefficient of friction c/m by σ, this equation becomes

$$\frac{dv}{dt} = f - \sigma v. \tag{4.6}$$

The model parameters f and σ can be approximated from performance records. For example, the parameters for 1968 Mexico City Olympics gold medalist Tommie Smith have been estimated to be $f = 13.46$ (Newtons,/kg) and $\sigma = 1.252$ (per second).[5]

In races one is usually interested in the time it takes to run a fixed distance x_d from a starting line at $x = 0$ from which the runner's begin from a standing start ($v(0) = 0$). To determine this time from the initial value problem

$$\frac{dv}{dt} = f - \sigma v, \quad v(0) = 0$$

[4]J. B. Keller, 1973. *Physics Today*, 26(9), p.42

[5]A. Armenti, Jr., 1993. *The Physics of Sports*, American Institute of Physics, New York, pp. 105-108

we calculate the runner's position from $v = dx/dt$, i.e.,

$$x(t) = \int_0^t v(s)ds,$$

and then solve the equation $x(t) = x_d$ for t.

For example, consider gold medalist Tommie Smith running a 100m race. We can use a computer to approximate the solution of the initial value problem

$$\begin{aligned} \frac{dv}{dt} &= 13.46 - 1.252v \\ v(0) &= 0. \end{aligned} \tag{4.7}$$

The algorithms studied in Sec. 2.2 produce approximations to the velocity $v(t)$ at points t_i (depending on the chosen step size s) lying in, say, the interval $0 \leq t \leq 12$. The resulting table of approximations for $v(t_i)$ permits us to approximate the distance

$$x(t_i) = \int_0^{t_i} v(s)dx$$

run at each point in time t_i by using an numerical integration procedure (for example, the trapezoid rule). Table 2.7 shows some of the results.

We make two observations from the numerical solution of the initial value problem (4.7). First, the model predicts that Tommie Smith could run 100m in approximately 10.1 seconds (Table 2.7). Secondly, the model predicts that after about 6 seconds (55 m), Smith's velocity $v(t)$ is very nearly constant at 10.75 (m/sec) for the rest of the race.

t	$x(t)$	$v(t)$
10.00	98.92	10.75
10.01	99.03	10.75
10.02	99.14	10.75
10.03	99.24	10.75
10.04	99.35	10.75
10.05	99.46	10.75
10.06	99.57	10.75
10.07	99.67	10.75
10.08	99.78	10.75
10.09	99.89	10.75
10.10	99.99	10.75
10.11	100.10	10.75
10.12	100.21	10.75

TABLE 2.7. Some results of applying Heun's Algorithm (Sec. 2.3) with step size $s = 0.01$ to the initial value problem (4.7).

Some sprints are not run in a straight line, but involve running a curve at the beginning of the race, with staggered starting positions for the racers. For example, this is the case for most 200m races in which the course lies on a (circular) curve for 100m before straightening out for the last 100m.

When running along the curve, the sprinter's propulsive force must supply an additional centripetal acceleration which depends on the radius of curvature of the curve. Furthermore, the radius of curvature is different for each lane. Lanes are typically 1.22 meters wide and the inner radius of the n^{th} lane is given by the formula

$$R(n) = \frac{100}{\pi} + 1.22\,(n-1)\,.$$

We will not delve into the physics of the derivation here, but leave it to say that the new equation motion that results when the additional force due to the centripetal acceleration is taken into account leads to the initial value problem[6]

$$\begin{aligned} \frac{dv}{dt} &= \left(f^2 - \left(\frac{v^2}{R(n)}\right)^2\right)^{1/2} - \sigma v \\ v(0) &= 0. \end{aligned} \tag{4.8}$$

These are the equations of motion during the first 100m of the 200m race.

During the second 100m of the race equation (4.6) is applicable. The initial condition associated with (4.6) would be the runner's velocity v_c at the end of the first 100m (along the curve portion) of the race.

Thus, the Model Solution Step of the Modeling Cycle involves, in this application, the numerical solution of the initial value problem (4.6) until the time t_c is reached at which $x(t_c) = 100$. At this time the runner's velocity is $v_c = v(t_c)$, which constitutes the initial condition for equation (4.8). This initial value problem is solved until the finishing time for the runner is reached, i.e., the time t_f at which $x\,(t_f) = 100$ (the second 100m of the race).

We can, however, simply the second step of the solution procedure as follows. It will turn out (as in the 100m sprint example above) that the runner's velocity will reach a constant by the time the final 100m portion of the race is reached. Therefore, rather than solve a second initial value problem using equation (4.8) we can obtain a good approximation to the time for the last 100m by assuming a constant velocity v_f is maintained, in which case the final 100m time is given by the formula $100/v_c$. The model predicted sprint time for the 200m race is then

$$t_f = t_c + \frac{100}{v_c}. \tag{4.9}$$

Notice all that is needed by the model to make a prediction for a sprinter's time in a 200m race are the parameter values f and σ (obtained from the sprinter's performance data on straight courses) and the lane assignment n.

As an example we consider gold medalist Tommie Smith's performance in the 1968 Mexico City Olympics. In the 200m finals Smith was assigned lane $n = 3$. Using $f = 13.46$, $\sigma = 1.252$ and $R(3) = 34.27$ we approximate the solution of (4.8) using Heun's Algorithm.

From Table 2.8 we see that the model predicts Smith will run the first 100 meters along the curve in approximately $t_c = 10.33$ seconds and at the end of the curve his velocity will be approximately $v_c = 10.45$. From (4.9) we calculate the predicted time for Smith's 200m sprint in lane $n = 3$ to be approximately

$$t_f = 10.33 + \frac{100}{10.45} = 19.90.$$

In fact Smith ran the race in 19.83 seconds (at that time a world record).

[6]A. Armenti, Jr., 1993. *The Physics of Sports*, American Institute of Physics, New York, pp. 105-108

t	$x(t)$	$v(t)$
10.25	99.20	10.45
10.26	99.31	10.45
10.27	99.41	10.45
10.28	99.51	10.45
10.29	99.62	10.45
10.30	99.72	10.45
10.31	99.83	10.45
10.32	99.93	10.45
10.33	100.0	10.45
10.34	100.1	10.45
10.35	100.2	10.45
10.36	100.4	10.45
10.37	100.6	10.45

TABLE 2.8. Some results of applying Heun's Algorithm with step size $s = 0.01$ (Sec. 2.3) with step size $s = 0.01$ to the initial value problem (4.8) with $f = 13.46$, $\sigma = 1.252$ and $R(3) = 34.27$.

We can use the model (4.8) to predict what might have been the result if Smith been given a different lane assignment. For example, the results in Table 2.9 for lane $n = 1$ show a slower predicted time of

$$t_f = 10.36 + \frac{100}{10.40} = 19.98.$$

Runners dislike lane 1 as being "too tight". The slower time predicted by the model for $n = 1$ bears out this opinion. Had Smith run in lane $n = 8$, however, his world record, according to the model, would have been even lower than 19.83 seconds. See Exercise 4.10.

t	$x(t)$	$v(t)$
10.25	98.85	10.40
10.26	98.96	10.40
10.27	99.06	10.40
10.28	99.17	10.40
10.29	99.27	10.40
10.30	99.37	10.40
10.31	99.48	10.40
10.32	99.58	10.40
10.33	99.69	10.40
10.34	99.79	10.40
10.35	99.89	10.40
10.36	100.0	10.40
10.37	100.1	10.40

TABLE 2.9. Some results of applying Heun's Algorithm with step size $s = 0.01$ (Sec. 2.3) with step size $s = 0.01$ to the initial value problem (4.8) with $f = 13.46$, $\sigma = 1.252$ and $R(1) = 31.83$.

EXERCISES

When we talk of a population's doubling time in the exercises below we imply that the population grows, in the absence of any limiting facts, with a constant per unit growth rate: $x' = rx$.

EXERCISE 4.1. *E. coli has a doubling time of approximately 20 minutes. Assume h cells per minute are removed from a culture initially at 10^8 cells. Use a computer to solve the initial value problem for the number of cells $x = x(t)$ at time t. Determine the critical value h_{cr} of h above which the culture will die out.*

EXERCISE 4.2. *E. coli has a doubling time of approximately 20 minutes. Assume he^{-at} cells per minute are removed from a culture initially at 10^8 cells. Use a computer to solve the initial value problem for the number of cells $x = x(t)$ at time t. Explore those values of a and h for which the culture goes extinct. Specifically, for selected values of a, calculate the critical value h_{cr} for h above which the culture goes extinct. Determine a relationship between a and h_{cr}.*

EXERCISE 4.3. *The bacterium M. Tuberculosis has a doubling time of approximately 13 hours. Assume h cells per hour are removed from a culture initially at 10^7 cells. Use a computer to solve the initial value problem for the number of cells $x = x(t)$ at time t. Determine the critical value h_{cr} of h above which the culture will die out.*

EXERCISE 4.4. *The bacterium M. Tuberculosis has a doubling time of approximately 13 hours. Assume he^{-at} cells per hour are removed from a culture initially at 10^7 cells. Use a computer to solve the initial value problem for the number of cells $x = x(t)$ at time t. Explore those values of a and h for which the culture goes extinct. Specifically, for selected values of a, calculate the critical value h_{cr} for h above which the culture goes extinct. Determine a relationship between a and h_{cr}.*

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EXERCISE 4.5. *In the model (4.5) suppose the effectiveness of the antibiotic decays more slowly than exponentially. Specifically, assume $h = 200d\,(1 + at)^{-1}$ where $a > 0$ is a constant. Assume there is a 50% drop in effectiveness after 60 minutes.*

(a) Modify the initial value problem (4.5) to account for this new assumption.

(b) Using slope fields and computer calculated solution graphs, determine whether or not this new model has a critical dosage value d_{cr} below which the infection is not controlled and above which the infection is eliminated.

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No population can grow exponentially indefinitely. Many populations eventually decrease their rate of growth and level off at a number K appropriate to its environment and available resources. A differential equation often used to model this kind of growth is $x' = rx\,(1 - x/K)$ where r is the exponential growth rate at low population numbers. Suppose such a population is harvested at a constant rate h. Then the equation governing the populations growth is $x' = rx\,(1 - x/K) - h$. As an example, suppose the number of fish in a large lake grows according to this law. It is estimated that the lake can support $K = 10^4$ fish, and it is known that during the exponential grow phase (i.e., low population numbers) the fish population will double in two years. Use a computer to investigate the following question: at what maximal annual rate h_{cr} can the fish be harvested without causing extinction if initially there are the following numbers in the lake?

EXERCISE 4.6. $x(0) = 10^2$

EXERCISE 4.7. $x(0) = 10^3$

EXERCISE 4.8. $x(0) = 10^4$

EXERCISE 4.9. *Investigate many initial conditions* $x(0) > K/2 = 0.5 \times 10^4$. *What do you notice about* h_{cr}*?*

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EXERCISE 4.10. *Calculate the model (4.8) predicted time for gold medalist Tommie Smith had he run in lane* $n = 8$ *of the 200m finals in the 1968 Mexico City Olympics.*

EXERCISE 4.11. *The current world record for 200m of 19.32 seconds, set at the 1996 Atlanta Olympics, is held by Michael Johnson. In order for Tommie Smith to equal this time on a straight course what higher value of the per unit mass propulsive form* f *would he have to attain?*

EXERCISE 4.12. *The current world record for 200m of 19.32 seconds, set at the 1996 Atlanta Olympics, is held by Michael Johnson. In order for Tommie Smith to better this time on a curved course in lane* $n = 3$ *what higher value of the per unit mass propulsive form* f *would he have to attain?*

EXERCISE 4.13. *Estimated parameter values for sprinter Jim Hines are* $f = 7.10$ *(N/kg) and* $\sigma = 0.581$ *(per second). In a 100m (straight line) race who would win, Jim Hines or Tommie Smith?*

EXERCISE 4.14. *Estimated parameter values for sprinter Jim Hines are* $f = 7.10$ *(N/kg) and* $\sigma = 0.581$ *(per second). In a 200m (curve course) race who would win, Jim Hines in lane* $n = 5$ *or Tommie Smith in lane* $n = 4$*?*

EXERCISE 4.15. *Who would win if Hines and Smith switched lanes in Exercise 4.14?*