Mirror couplings of reflecting Brownian motions and applications

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Abstract

Rodrigo Bañuelos, Krzysztof Burdzy et. al. ([BaBu], [Bu], [AtBu1], [AtBu2], [BuKe]) introduced the \textit{mirror coupling} of reflecting Brownian motions in a smooth domain $D \subset \mathbb{R}^d$ and used it in order to derive properties of Neumann eigenvalues/eigenfunctions of the Neumann Laplaceian on $D$. 

In the present talk we will show that the coupling can be extended to the case when the two reflecting Brownian motions live in different domains $D_1, D_2 \subset \mathbb{R}^d$. As an application of the construction, we will derive a unifying proof of the two most important results on Chavel's conjecture on the domain monotonicity of the Neumann heat kernel ([Ch], [Ke]).
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A 1-dimensional Brownian motion starting at \( x \in \mathbb{R} \) is a continuous stochastic process \((B_t)_{t \geq 0}\) with \( B_0 = x \) a.s for which \( B_t - B_s \) is a normal random variable \( \mathcal{N}(0, t-s) \), independent of the \( \sigma \)-algebra \( \mathcal{F}_s = \sigma(B_r : r \leq s) \), for all \( 0 \leq s < t \).
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A \( d \)-dimensional Brownian motion starting at \( x = (x^1, \ldots, x^d) \in \mathbb{R}^d \) is a stochastic process \( B_t = (B^1_t, \ldots, B^d_t) \), where the components \( B^i_t \) are independent 1-dimensional Brownian motions starting at \( x^i \), \( 1 \leq i \leq d \).
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Definition

Reflecting Brownian motion in a smooth domain $D \subset \mathbb{R}^d$ starting at $x_0 \in \overline{D}$ is a solution of the stochastic differential equation

$$X_t = x_0 + B_t + \int_0^t \nu_D(X_s) \, dL^X_s, \quad t \geq 0,$$

(1)

where $B_t$ is a $d$-dimensional BM starting at $B_0 = 0$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, $L^X_t$ is the local time of $X$ on the boundary of $D$, $X_t$ is $\mathcal{F}_t$-adapted and almost surely $X_t \in \overline{D}$ for all $t \geq 0$. 

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Skorokhod map

Remark

It can be shown ([LiSz]) that there exists a unique $\mathcal{F}_t$-semimartingale which solves (1). In fact, there exists a map (Skorokhod map)

$$\Gamma : C \left([0, \infty) : \mathbb{R}^d\right) \rightarrow C \left([0, \infty) : \bar{D}\right)$$

such that $X = \Gamma (x + B)$ a.s.

For each $T > 0$ fixed, $\Gamma|_{[0,T]}$ is Hölder continuous of order 1/2 on compact subsets of $C \left([0, T] : \mathbb{R}^d\right)$. 
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Mirror coupling of Brownian motions

Given a hyperplane $\mathcal{H}$ (the mirror) and a Brownian motion $X_t$, we define the Brownian motion $Y_t$ as the mirror image of $X_t$ with respect to $\mathcal{H}$ until the coupling time

$$\xi = \inf \{ s > 0 : X_s = Y_s \} ,$$

after which the processes $X_t$ and $Y_t$ evolve together.

Figure: The mirror coupling of Brownian motions.
If $m$ is a unit normal to $\mathcal{H}$, then $Y_t$ is given explicitly by

$$Y_t = X_t - 2 (X_t \cdot m) m, \quad t \leq \xi.$$  \hfill (2)
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Introducing the $d \times d$ matrix $H$ by

$$H(m) = I - 2mm^T = (\delta_{ij} - 2m_im_j)_{1 \leq i,j \leq d},$$  \hfill (3)

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since for $t \leq \xi$ we have $m = \frac{Y_t - X_t}{|Y_t - X_t|}$ and for $t \geq \xi$ we have $Y_t = X_t,$
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the above relation can be written in the form

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Y_t = G (Y_t - X_t) X_t, \quad t \geq 0,
\]

(4)

where

\[
G (u) = \begin{cases} 
H \left( \frac{u}{|u|} \right), & u \neq 0 \\
I, & u = 0
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(5)
If $m$ is a unit normal to $\mathcal{H}$, then $Y_t$ is given explicitly by

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Consider $D_{1,2} \subset \mathbb{R}^d$ smooth bounded domains with $\overline{D_2} \subset D_1$ and $D_2$-convex.
Mirror coupling of reflecting Brownian motions

Consider $D_{1,2} \subset \mathbb{R}^d$ smooth bounded domains with $\overline{D_2} \subset D_1$ and $D_2$-convex.
Given a $d$-dimensional BM $(W_t)_{t \geq 0}$ with $W_0 = 0$, consider the following system of SDEs:

\begin{align}
X_t &= x + W_t + \int_0^t \nu_{D_1}(X_s) \, dL_s^X \\
Y_t &= y + Z_t + \int_0^t \nu_{D_2}(Y_s) \, dL_s^Y \\
Z_t &= \int_0^t G(Y_s - X_s) \, dW_s
\end{align}

where $\nu_{D_1}$ and $\nu_{D_2}$ represent the inward unit normal vector fields on $\partial D_1$, respectively $\partial D_2$. 

Remark: In the particular case when $D_1 = D_2$, (6) – (9) above reduces to the case considered by Burdzy et al. (i.e. mirror coupling of reflecting Brownian motions in $D$).
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X_t &= x + W_t + \int_0^t \nu_{D_1}(X_s) \, dL^X_s \quad (6) \\
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where $\nu_{D_1}$ and $\nu_{D_2}$ represent the inward unit normal vector fields on $\partial D_1$, respectively $\partial D_2$. Considering $\Gamma$ and $\tilde{\Gamma}$ the corresponding Skorokhod maps (i.e. $X = \Gamma(x + W)$, $Y = \tilde{\Gamma}(y + Z)$), the above system is equivalent to

\begin{equation}
Z_t = \int_0^t G \left( \tilde{\Gamma}(y + Z)_s - \Gamma(x + W)_s \right) \, dW_s \quad (9)
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\[ X_t = x + W_t + \int_0^t \nu_{D_1}(X_s) \, dL_s^X \]  \hspace{2cm} (6)

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In the particular case when $D_1 = D_2$, (6) – (9) above reduces to the case considered by Burdzy et. al. (i.e. mirror coupling of reflecting Brownian motions in $D$).
Main result

Theorem

Let $D_{1,2} \subset \mathbb{R}^d$ be smooth bounded domains with $\overline{D}_2 \subset D_1$ and $D_2$ convex domain, and let $x \in \overline{D}_1$ and $y \in \overline{D}_2$ be arbitrarily fixed points.

Then there exists a strong solution $X_t, Y_t$ to (6) – (9) above, referred to as a mirror coupling of Reflecting Brownian motions in $D_1$, respectively $D_2$, starting from $(x, y) \in \overline{D}_1 \times \overline{D}_2$ with driving Brownian motion $W_t$. 
Some remarks

Remark

In the case $D_1 = D_2 = D$, the solution to (9) can be essentially constructed by Picard iterations, since outside of the origin $G$ satisfies

$$\| G(u) - G(u') \| \leq c |u - u'|,$$

where $\| (g_{ij})_{i,j} \| = \left( \sum_{i,j} g_{ij}^2 \right)^{1/2}$.
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Remark

In the general case this method cannot be used. The reason is that once the processes $X_t$ and $Y_t$ have coupled, it is possible for them to decouple: for example if $X_t = Y_t \in \partial D_2$, the solutions will split.

The behaviour of $G$ at the origin becomes therefore essential – we have to show the existence of a degenerate SDE ($G$ is discontinuous at the origin).

Surprisingly, the existence of the solution comes from the convexity of the smaller domain!
Idea of the proof

- Reduce the problem to the case $D_1 = \mathbb{R}^d$ (hence $X_t = X_0 + W_t$)
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- Approximate $D_2 = D$ by an increasing sequence of convex polygonal domains $D_n \nearrow D$
- Show the solution $Y^n_t$ for $D_n$ converges to the solution $Y_t$ for $D$, that is

$$Z^n_t = \int_0^t G(Y^n_s - X_s) \, dW_s \xrightarrow{n \to \infty} \int_0^t G(Y_s - X_s) \, dW_s = Z_t, \quad t \geq 0,$$

where $Z^n_t, Z_t$ are the driving Brownian motions for $Y^n_t$, respectively $Y_t$. 
Applications

Consider $D_{1,2} \subset \mathbb{R}^d$. 

The Dirichlet heat kernel $\tilde{p}_{D}(t, x, y)$ is an increasing function of the domain: if $D_1 \subset D_2$ then $\tilde{p}_{D_1}(t, x, y) \leq \tilde{p}_{D_2}(t, x, y)$, $t > 0$ and $x, y \in D_1$ (one feels warmer in bigger rooms with refrigerated walls than in smaller ones).

Isaac Chavel conjectured that the Neumann heat kernel is a decreasing function of the domain:

**Conjecture (Chavel, 1986)**

If $D_1 \subset D_2$ are convex domains then for all $t > 0$ and $x, y \in D_2$ we have $p_{D_1}(t, x, y) \geq p_{D_2}(t, x, y)$.

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Chavel proved the conjecture in the case $D_2$ is a ball centered at $x$ (or $y$) and $D_1$ is convex (integration by parts).

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Using the mirror coupling we can give a unifying proof of Chavel conjecture in the case $D_1 \subset B \subset D_2$ where $B$ is a ball centered at either $x$ or $y$. 

M. N. Pascu (Transilvania Univ)
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where $B$ is a ball centered at either $x$ or $y$. 
Geometry of the mirror coupling

Consider a mirror coupling \((X_t, Y_t)\) of reflecting Brownian motions in \((D_2, D_1)\) starting at \(x \in D_1\).
The proof of Chavel conjecture

If $D_1 \subset B(y, r) \subset D_2$, then the mirror $M_t$ of the coupling cannot separate $Y_t$ and $y$:

$$|Y_t - y| \leq |X_t - y| , \quad t \geq 0.$$
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We obtain

$$P^y (|X_t - x| < \varepsilon) \leq P^y (|Y_t - x| < \varepsilon),$$
The proof of Chavel conjecture

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$$|Y_t - y| \leq |X_t - y|, \quad t \geq 0.$$ 

We obtain

$$P^y (|X_t - x| < \varepsilon) \leq P^y (|Y_t - x| < \varepsilon),$$

hence

$$p_{D_2} (t, x, y) = \lim_{\varepsilon \searrow 0} \frac{1}{|B(y, \varepsilon)|} P^x (X_t \in B(y, \varepsilon)) \leq \lim_{\varepsilon \searrow 0} \frac{1}{|B(y, \varepsilon)|} P^x (Y_t \in B(y, \varepsilon)) = p_{D_1} (t, x, y).$$
Extensions of the mirror coupling

Same arguments can be used in order to construct the mirror coupling in $D_1, D_2 \subset \mathbb{R}^d$ if:

- $D_1$ and $D_2$ have non-tangential boundaries (needed for localization of the construction)
- $D_1 \cap D_2$ is a convex domain (needed for the construction of the solution).

**Figure:** Generic smooth domains $D_1, D_2$ for the mirror coupling
The solution is not unique.
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In the case $D_1 = D_2 = \mathbb{R}$, with the substitution $U_t = -\frac{Y_t - X_t}{2}$, we obtain the singular SDE:

$$U_t = \int_0^t \sigma(U_s) \, dW_s,$$

where

$$\sigma(u) = \begin{cases} 
1, & u \neq 0 \\
0, & u = 0 
\end{cases}.$$
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$$U_t = \int_0^t \sigma (U_s) \, dW_s, \quad (11)$$

where

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1, & u \neq 0 \\
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\end{cases}.$$  

The above has the solutions $U_t \equiv 0$ and $U_t = W_t$, and a whole range of intermediate solutions (sticky Brownian motion).
Question of uniqueness

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The above has the solutions $U_t \equiv 0$ and $U_t = W_t$, and a whole range of intermediate solutions (sticky Brownian motion). The original equation has solutions $Y_t = X_t = W_t$ (sticky mirror coupling), $Y_t = -X_t = -W_t$ (non-sticky mirror coupling), and a whole range of intermediate solutions (weak/mild sticky mirror coupling).
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