# PREPARE for MICHIGAN CALCULUS

targeted precalculus practice problems and answers
for students planning to take calculus

It may have been a long time since you systematically practiced the complicated manipulative skills that you will be using throughout calculus. This is a good time to look back and rethink the rules and definitions that you learned and consider how they all fit together.

In your calculus class you will be concentrating on reading the textbook and writing explanations for your mathematical ideas. We want you to begin the course with all the skills you need. As you work through the Prep problems, try to make both the vocabulary and the manipulations second nature so that you will be able to use the words and operations of mathematics quickly and appropriately when you are studying calculus.


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# 1. Exponents

When we multiply numbers together, each of the numbers is called a *factor*. In the special situation where some of the numbers are the same, we use *exponent notation*. When we use the form $a^n$, the "$n$" is called the *exponent* and the "$a$" is called the *base*.

## *Integer Exponents*

**Definitions:**

$$a^0 = 1 \quad (\text{if } a \neq 0)$$

$$a^1 = a$$

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots a}_{n \text{ factors}} \qquad (n \text{ is a positive integer})$$

$$a^{-1} = \frac{1}{a}$$

$$a^{-n} = \frac{1}{a^n}$$

**Rules for Exponents:**

| | |
|---|---|
| $a^m \cdot a^n = a^{m+n}$ | multiplying powers |
| $\frac{a^m}{a^n} = a^{m-n}$ | dividing powers |
| $(a^m)^n = a^{mn}$ | raising a power to a power |
| $(ab)^n = a^n b^n$ | power of a product |
| $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ | power of a quotient |

***Be careful using the following:***

$ab^n = a(b)^n$ (Notice that $ab^n \neq (ab)^n$.)

$-b^n = -(b)^n$ (Notice that $-b^n \neq (-b)^n$.)

$-ab^n = (-a)(b^n)$,

***and watch out for this common mistake,***

$(a+b)^n \neq a^n + b^n$ power of a sum $\neq$ sum of powers.

**Example 1:** $3b(2b)^3(b^{-1}) = 3b^1 2^3 b^3 b^{-1} = 3 \cdot 8 \cdot b^{3+(-1)+1} = 24b^3$

**Example 2:** $\frac{y^4(x^3y^{-2})^2}{2x^{-1}} = \frac{y^4x^6y^{-4}}{2x^{-1}} = \frac{y^{4-4}x^{6-(-1)}}{2} = \frac{y^0x^7}{2} = \frac{x^7}{2} \qquad (y \neq 0)$

**Example 3:** $\left(\dfrac{2^{-3}}{L}\right)^{-2} = \dfrac{2^{(-3)(-2)}}{L^{-2}} = \dfrac{2^6}{L^{-2}} = 64L^2$

**Example 4:** $\dfrac{5(2s+1)^4(s+3)^{-2}}{(2s+1)} = \dfrac{5(2s+1)^{4-1}}{(s+3)^2} = \dfrac{5(2s+1)^3}{(s+3)^2}$

## *Radicals (roots) and fractional exponents*

**Definitions of radicals:**

Roots of positive numbers. For $a > 0$

$\sqrt{a}$ is the positive number whose square is $a$. $\quad \sqrt{49} = +7$

$\sqrt[n]{a}$ is the positive number whose $n^{th}$ power is $a$. $\quad \sqrt[3]{125} = +25$

Roots of negative numbers. For $a < 0$

$\sqrt[n]{a}$ is the negative number whose $n^{th}$ power is $a$ if $n$ is odd. $\quad \sqrt[3]{-8} = -2$

$\sqrt[n]{a}$ is not a real number if $a$ is negative and $n$ is even. $\quad \sqrt{-8}$ is not a real number

**Rules for radicals (whenever defined) :**

$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$ roots of products

$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ roots of quotients

**Example 5:** $\sqrt{36} = +6$ (Notice that the answer is the positive root, not ±6.)

**Example 6:** $\sqrt[3]{-8x^6} = \sqrt[3]{-8}\,\sqrt[3]{x^6} = -2x^2$

**Example 7:** $\sqrt{\dfrac{16R^8}{25}} = \dfrac{\sqrt{16R^8}}{\sqrt{25}} = \dfrac{\sqrt{16}\sqrt{R^8}}{\sqrt{25}} = \dfrac{4R^4}{5}$

***Special case:*** If $n$ is even,

$$\sqrt[n]{a^n} = |a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \end{cases}$$

**Example 8:** $\sqrt[4]{(-1)^4} = \sqrt[4]{+1} = +1$ (Notice that in this case $\sqrt[n]{a^n} \neq a$ .)

**Example 9:** $\sqrt{9x^2} = 3|x|$ (because $x$ can be either positive or negative)

**Definition of fractional exponents (wherever defined):**

$a^{\frac{1}{n}} = \sqrt[n]{a}$ $\qquad$ $a^{\frac{1}{3}} = \sqrt[3]{a}$

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ $\qquad$ $a^{\frac{2}{3}} = \sqrt[3]{a^2}$

If $\frac{m}{n}$ is a fraction in lowest terms, we can interchange the order of raising to a power and finding a root. This gives two different ways of calculating $a^{\frac{m}{n}}$. We write $a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m$. The rules for fractional exponents are the same as for integer exponents.

**Example 10:** $(27)^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{729} = 9$

or equivalently

$$(27)^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$$

**Example 11:** $\left(\dfrac{M^{\frac{1}{5}}}{3N^{-\frac{1}{2}}}\right)^2 = \dfrac{\left(M^{\frac{1}{5}}\right)^2}{\left(3N^{-\frac{1}{2}}\right)^2} = \dfrac{M^{\frac{2}{5}}}{3^2 N^{-\frac{2}{2}}} = \dfrac{M^{\frac{2}{5}}N}{9}$

**Example 12:** $\dfrac{3u^2\sqrt{u}\,w}{w^{\frac{1}{3}}} = \dfrac{3u^2u^{\frac{1}{2}}w}{w^{\frac{1}{3}}} = 3u^{\left(2+\frac{1}{2}\right)}w^{\left(1-\frac{1}{3}\right)} = 3u^{\frac{5}{2}}w^{\frac{2}{3}}$

***Calculator note:*** Most calculators will not compute $a^{\frac{m}{n}}$ for $m \neq 1$ when $a$ is negative, even if $n$ is odd. For example, $(-1)^{\frac{2}{3}}$ is clearly well-defined, but a calculator will likely display "error". To evaluate $(-1)^{\frac{2}{3}}$, enter either $\left((-1)^2\right)^{\frac{1}{3}}$ or $\left((-1)^{\frac{1}{3}}\right)^2$. [ Better yet, do <u>this</u> one in your head! ]

**Example 13:** To find $(-2197)^{\frac{2}{3}}$ on a calculator, you may enter

```
(-2197)^(1/3)^2
```

or

```
(-2197)^2^(1/3)
```

.

In either case you should get 169.

**Example 14:** $(-256)^{\frac{3}{4}}$ is <u>not</u> a real number, since $(-256)^{\frac{1}{4}}$ is undefined.

### *Real Number Exponents*

The rules for real number exponents are the same as those for integer exponents and fracti0onal exponents.

**Example 15:** $p^{\sqrt{2}} \cdot p^{\sqrt{8}} = p^{\left(\sqrt{2}+\sqrt{8}\right)} = p^{\left(\sqrt{2}+2\sqrt{2}\right)} = p^{3\sqrt{2}}$

**Example 16:** $\sqrt{\dfrac{x^{3\pi}}{x^{\pi}}} = \left(\dfrac{x^{3\pi}}{x^{\pi}}\right)^{\frac{1}{2}} = \left(x^{3\pi-\pi}\right)^{\frac{1}{2}} = \left(x^{2\pi}\right)^{\frac{1}{2}} = x^{\pi}$

### *Variables as Exponents*

The same rules of exponents apply whether the exponent is a constant as in $x^3$ or a variable as in $5^x$.

**Example 17:** $-3^x \cdot 3^{-x} = -\left(3^x\right)\left(3^{-x}\right) = -\left(3^{x-x}\right) = -\left(3^0\right) = -1$

**Example 18:** $\dfrac{4^x}{2^x} = \left(\dfrac{4}{2}\right)^x = 2^x$

**Example 19:** $\dfrac{a^{\frac{2}{3}}\left(a^x\right)\left(a^x\right)}{a} = a^{\frac{2}{3}}\left(a^x\right)\left(a^x\right)\left(a^{-1}\right) = a^{\left(\frac{2}{3}+x+x-1\right)} = a^{2x-\frac{1}{3}}$

## Problems for section 1

***Evaluate mentally. First memorize the powers of integers in section 17.***

1. $\dfrac{1}{7^{-2}}$
2. $\dfrac{2^7}{2^3}$
3. $(-1)^{445}$
4. $-11^2$
5. $(-2)(3^2)$
6. $(5^0)^3$
7. $2.1(10^3)$
8. $\sqrt[3]{-125}$
9. $\sqrt{(-4)^2}$
10. $(-1)^3\sqrt{36}$
11. $(0.04)^{\frac{1}{2}}$
12. $(-8)^{\frac{2}{3}}$
13. $\left(\dfrac{1}{27}\right)^{-\frac{1}{3}}$
14. $(0.125)^{\frac{1}{3}}$

***Simplify and leave without radicals. A calculator is not necessary.***

15. $(0.1)^2(4xy^2)^2$
16. $(3)\left(3^{\frac{x}{2}}\right)^2$
17. $\left(4L^{\frac{2}{3}}P\right)^{\frac{3}{2}}(P)^{-\frac{3}{2}}$
18. $7\left(5W^{\frac{1}{2}}\right)\left(2W^{\frac{1}{3}}\right)$
19. $\left(s\sqrt{16st^2}\right)^2$
20. $\sqrt{e^{2x}}$
21. $(3AB)^{-1}(A^2B^{-1})^2$
22. $e^{kt}\cdot e^3\cdot e$
23. $\sqrt{M+2}(2+M)^{\frac{3}{2}}$
24. $\left(3x\sqrt{x^3}\right)^2$
25. $x^e(x^e)^2$
26. $(y^{-2}e^y)^2$
27. $\dfrac{4x^{(3\pi+1)}}{x^2}$
28. $\left(\dfrac{35(2b+1)^9}{7(2b+1)^{-1}}\right)^2$
29. $\dfrac{a^{n+1}3^{n+1}}{a^n 3^n}$
30. $\dfrac{(12u^3)}{3(uv^2w^4)^{-1}}$

***Evaluate if possible. Check your answers with a calculator.***

31. $(-32)^{\frac{3}{5}}$
32. $-32^{\frac{3}{5}}$
33. $-625^{\frac{3}{4}}$
34. $(-625)^{\frac{3}{4}}$
35. $(-1728)^{\frac{4}{3}}$
36. $64^{\frac{-3}{2}}$
37. $-64^{\frac{3}{2}}$
38. $(-64)^{\frac{3}{2}}$

## 2. Multiplying

We use the distributive property combined with the rules of exponents to find products where one of the factors is a sum of terms.

$$a(b+c) = ab + ac$$
$$(b+c)a = ba + ca$$

**Example 1:** $3x^2\left(x + \frac{1}{6}x^{-3}\right) = (3x^2)(x) + (3x^2)\left(\frac{1}{6}x^{-3}\right) = 3x^3 + \frac{1}{2}x^{-1}$

**Example 2:** $\left((2t)^2 - 5\right)\sqrt{t} = (2t)^2\left(\sqrt{t}\right) - 5\sqrt{t} = (4t^2)\left(t^{\frac{1}{2}}\right) - 5t^{\frac{1}{2}} = 4t^{\frac{5}{2}} - 5t^{\frac{1}{2}}$

**Example 3:** $2^x\left(3^x + 2^{x-1}\right) = (2^x)(3^x) + (2^x)(2^{x-1}) = (2\cdot 3)^x + 2^{x+x-1} = 6^x + 2^{2x-1}$

If there are two terms in each factor,

$$(a+b)(c+d) = ac + ad + bc + bd$$
$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

**Example 4:** $(5x^2+2)(x-4) = (5x^2)(x) + (5x^2)(-4) + (2)(x) + (2)(-4) = 5x^3 - 20x^2 + 2x - 8$

**Example 5:** $\left(5\sqrt{r} - 2\right)\left(3\sqrt{r} + 2\right) = (5)(3)\left(\sqrt{r}\right)^2 + (5)(2)\sqrt{r} + (-2)(3)\sqrt{r} + (-2)(2) = 15r + 4r - 4$

**Example 6:** $(e^x+1)(2x+e^{-x}) = (e^x)(2x) + (e^x)(e^{-x}) + (1)(2x) + e^{-x} = 2xe^x + e^{(x-x)} + 2x + e^{-x}$
$= 2xe^x + 1 + 2x + e^{-x}$

**Example 7:** $\left(3 - \frac{1}{2}x\right)^2 = (3)^2 + 2(3)\left(-\frac{1}{2}x\right) + \left(-\frac{1}{2}x\right)^2 = 9 - 3x + \frac{1}{4}x^2$

**Problems for section 2** *Multiply and simplify.*

1. $(3x-2x^2)(4) + (5+4x)(3x-4)$
2. $P(p-3q)^2$
3. $(A^2-B^2)^2$
4. $4(x-3)^2 + 7$
5. $-\left(\sqrt{2x}+1\right)^2$
6. $(t^2+1)(50t) - (25t^2+125)(2t)$
7. $u\left(u^{-1}+2^u\right)2^u$
8. $K(R-r)r^2$
9. $(x+3)\left(\frac{24}{x}+2\right)$
10. $\left(\frac{e^x+e^{-x}}{2}\right)^2$

## 3. Factoring

If we want to write an expanded expression in factored form we "*un-multiply*". Always check factoring by remultiplying.

### *Removing a common factor*

$$ab + ac = a(b + c)$$

$$-a - b = -(a + b)$$

$$a - b = -(b - a)$$

**Example 1:** $\frac{2}{3}x^2 y + \frac{4}{3}xy = \frac{2}{3}xy(x + 2)$ Check by multiplying out.

**Example 2:** $2 - x = \quad (-1)(-2 + x) = \quad -(x - 2)$

**Example 3:** $e^{2x} + xe^x = \quad e^x \cdot e^x + xe^x = \quad e^x(e^x + x)$

**Example 4:** $(2p+1)p^3 - 3p(2p+1) = p(2p+1)(p^2 - 3)$

### *Grouping terms*

Even though all the terms may not have a common factor, we can sometimes factor by first grouping the terms and then removing a common factor.

**Example 5:** $x^2 - hx - x + h = \quad (x^2 - hx) - (x - h) = \quad x(x-h) - (x-h) = \quad (x-h)(x-1)$

### *Factoring quadratics*

The easiest way to factor quadratics is to mentally multiply out the possibilities.

**Example 6:** $t^2 - 4t - 12 = (t + \quad)(t + \quad)$

You are looking for two numbers whose product is -12 and whose sum is -4. By trying combinations in the blanks you will find

$$t^2 - 4t - 12 = (t - 6)(t + 2).$$

**Example 7:** $4 - 2M - 6M^2 = (2 - 3M)(2 + 2M)$

### *Perfect squares*

**Example 8:** $x^2 + 2xz + z^2 = (x + z)^2$

**Example 9:** $16y^2 - 24y + 9 = (4y - 3)^2$

***Difference of two squares***

**Example 10:** $x^2 - y^2 = (x-y)(x+y)$

**Example 11:** $25S^2R^4 - T^6 = \left(5SR^2\right)^2 - \left(T^3\right)^2 = \left(5SR^2 - T^3\right)\left(5SR^2 + T^3\right)$

**Example 12:** $x^2(x-2) + 16(2-x) = x^2(x-2) - 16(x-2) = (x-2)\left(x^2-16\right)$
$= (x-2)(x-4)(x+4)$

**Problems for section 3** ***Factor completely.***

1. $2x^2 + 12 - 10x$
2. $\pi r^2 + 2\pi rh$
3. $24 - 10B + B^2$
4. $x \sin x - \sin x$
5. $x^2 + y^2$
6. $a^4 - a^2 - 12$
7. $(t+3)^2 - 16$
8. $x^2h + 12 - 4hx - 3x$
9. $r(r-s) - 2(s-r)$
10. $\cos^2 x - 2\cos x + 1$
11. $y^2 - 3xy + 2x^2$
12. $x^2e^{-3x} + 2xe^{-3x}$
13. $e^{2x} + 2e^x + 1$
14. $P(1+r)^2 + P(1+r)^2 r$

## 4. Fractions

(We assume no denominators are zero.)

### *Rules for fractions*

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$$ adding fractions with the same denominator

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$ adding fractions with different denominators

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$ multiplying fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$ dividing fractions

***If either the numerator or the denominator is not a fraction, the rule for dividing gives,***

$$\frac{\frac{a}{b}}{c}=\frac{a}{b}\cdot\frac{1}{c}=\frac{a}{bc} \quad \text{and} \quad \frac{a}{\frac{b}{c}}=\frac{a}{1}\cdot\frac{c}{b}=\frac{ac}{b}$$

***Fractions involving 0 and 1***

$$a=\frac{a}{1} \qquad \frac{a}{a}=1 \qquad \frac{a}{0} \text{ is not defined} \qquad \frac{0}{0} \text{ is not defined}$$

***Changing the sign of a fraction***

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$$

**Example 1:** $$\frac{4}{x^2+1}-\frac{1-x}{x^2+1}=\frac{4-(1-x)}{x^2+1}=\frac{3+x}{x^2+1}$$

**Example 2:** $$\frac{M}{M^2-M+3}+\frac{1}{M}=\frac{M^2+M^2-M+3}{M(M^2-M+3)}=\frac{2M^2-M+3}{M(M^2-M+3)}$$

**Example 3:** $$2x^{-\frac{1}{2}}+\frac{\sqrt{x}}{3}=\frac{2}{\sqrt{x}}+\frac{\sqrt{x}}{3}=\frac{2\cdot 3+\sqrt{x}\sqrt{x}}{3\sqrt{x}}=\frac{6+x}{3\sqrt{x}}$$

**Example 4:**

$$2\sqrt{t+3}+\frac{1-2t}{\sqrt{t+3}}=\frac{2\sqrt{t+3}}{1}+\frac{1-2t}{\sqrt{t+3}}=\frac{2\sqrt{t+3}\sqrt{t+3}+1-2t}{\sqrt{t+3}}=\frac{2(t+3)+1-2t}{\sqrt{t+3}}=\frac{7}{\sqrt{t+3}}$$

**Example 5:** $\dfrac{-H^2P}{17}\cdot\dfrac{\left(PH^{\frac{1}{3}}\right)^2}{K^{-1}} = \dfrac{-H^2P\left(P^2H^{\frac{2}{3}}\right)}{17K^{-1}} = -\dfrac{H^{\frac{8}{3}}P^3K}{17}$

**Example 6:** $\dfrac{\frac{2z}{w}}{w(w-3z)} = \dfrac{2z}{w}\cdot\dfrac{1}{w(w-3z)} = \dfrac{2z}{w^2(w-3z)}$

### *Removing a factor of one (sometimes called canceling)*

We can reduce a fraction by removing a factor which is equivalent to $+1$, but we can only reduce when we have the same (non-zero) factor in both the numerator and the denominator.

For example, $\dfrac{ac}{bc} = \dfrac{a}{b}\cdot\dfrac{c}{c} = \dfrac{a}{b}\cdot 1 = \dfrac{a}{b}$.

**Example 7:** $\dfrac{2x}{2y} = \dfrac{2}{2}\cdot\dfrac{x}{y} = \dfrac{x}{y}$

**Example 8:** $\dfrac{2+x}{2+y}$ cannot be reduced

**Example 9:** $\dfrac{5n-5}{1-n} = \dfrac{5(n-1)}{(-1)(n-1)} = \dfrac{5}{(-1)}\left(\dfrac{n-1}{n-1}\right) = -5$

**Example 10:** $\dfrac{x^3(-2x)-(4-x^2)(3x^2)}{x^6} = \dfrac{\left[-2x^2-(4-x^2)(3)\right]}{x^4}\left(\dfrac{x^2}{x^2}\right) = \dfrac{-2x^2-12+3x^2}{x^4} = \dfrac{x^2-12}{x^4}$

### *Inserting a factor of one*

Inserting a factor which is equivalent to $+1$ is the opposite of canceling, and it can be useful in adding fractions.

**Example 11:**

$$3-\frac{1}{x-1} = 3\frac{(x-1)}{(x-1)}-\frac{1}{x-1} = \frac{3(x-1)-1}{x-1} = \frac{3x-3-1}{x-1} = \frac{3x-4}{x-1}$$

**Example 12:**

$$\frac{2}{x^2+x}+\frac{x}{x+1} = \frac{2}{x(x+1)}+\frac{x}{x+1} = \frac{2}{x(x+1)}+\frac{x(x)}{(x+1)(x)} = \frac{2+x^2}{x(x+1)}$$

We can multiply both the numerator and denominator of a fraction by the same (non-zero) number without changing the fraction's value (or divide both by the same (non-zero) number). It is not "*legal*" to add the same number to the numerator and denominator, or square both, or take the logarithm of both, etc.

### *Complex Fractions*

To simplify a more complicated fraction we change the numerator and denominator to single fractions and then divide.

**Example 13:**

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\frac{\frac{x-(x+h)}{x(x+h)}}{h}=\frac{\frac{-h}{x(x+h)}}{\frac{h}{1}}=\frac{-h}{x(x+h)}\cdot\frac{1}{h}=\frac{-1}{x(x+h)}\frac{(h)}{(h)}=\frac{-1}{x(x+h)}$$

**Example 14:**

$$\frac{a+b}{a^{-2}-b^{-2}}=\frac{a+b}{\frac{1}{a^2}-\frac{1}{b^2}}=\frac{a+b}{\frac{b^2-a^2}{a^2b^2}}=\frac{a+b}{1}\cdot\frac{a^2b^2}{b^2-a^2}=\frac{(a+b)(a^2b^2)}{(b+a)(b-a)}=\frac{a^2b^2}{b-a}$$

### *Splitting Expressions*

We can reverse the rule for adding fractions to split up an expression into two fractions, $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$.

**Example 15:**

$$\frac{3x^2+2}{x^3}=\frac{3x^2}{x^3}+\frac{2}{x^3}=\frac{3}{x}+\frac{2}{x^3}$$

Sometimes we can alter the form of a fraction even further if we can create a duplicate of the denominator in the numerator.

**Example 16:**

$$\frac{x+3}{x-1}=\frac{(x-1+1)+3}{x-1}=\frac{(x-1)+4}{x-1}=\frac{x-1}{x-1}+\frac{4}{x-1}=1+\frac{4}{x-1}$$

## Problems for section 4

***Perform the following operations and simplify when possible.***

1. $\frac{3}{x-4}-\frac{2}{x+4}$

2. $\frac{x^2}{x-x}-\frac{1}{1-x}$

3. $\frac{1}{2r+3}-\frac{3}{4r^2+6r}$

4. $u+a+\frac{u}{u+a}$

5. $\frac{1}{\sqrt{x}}-\frac{1}{\left(\sqrt{x}\right)^3}$

6. $\frac{1}{e^{2x}}+\frac{1}{e^x}$

7. $\frac{a+b}{2}\cdot\frac{8x+2}{b^2-a^2}$

8. $\frac{0.07}{M}+\frac{3}{4}M^2$

9. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$

10. $\frac{x^3}{x-4}\div\frac{x^2}{x^2-2x-8}$

***Simplify.***

11. $$\frac{\frac{1}{x+y}}{x+y}$$

12. $$\frac{\frac{w+1}{2}}{w+1}$$

13. $$\frac{a^{-2}+b^{-2}}{a^2+b^2}$$

14. $$\frac{a^2-b^2}{a^2+b^2}$$

15. $$p-\frac{q}{\frac{p}{q}+\frac{q}{p}}$$

16. $$\frac{\left[4-(x+h)^2\right]-\left[\left(4-x^2\right)\right]}{h}$$

17. $$\frac{2x\left(x^3+1\right)^2-x^2(2)\left(x^3+1\right)\left(3x^2\right)}{\left[\left(x^3+1\right)^2\right]^2}$$

18. $$\frac{\frac{1}{2}(2x-1)^{-\frac{1}{2}}(2)-(2x-1)^{\frac{1}{2}}(2x)}{\left(x^2\right)^2}$$

19. $$\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

20. $$\frac{\frac{1}{x}\left(3x^2\right)-(\ln x)(6x)}{\left(3x^2\right)^2}$$

***Split into a sum of fractions.***

21. $$\frac{26x+1}{2x^3}$$

22. $$\frac{\sqrt{x}+3}{3\sqrt{x}}$$

23. $$\frac{6l^2+3l-4}{3l^4}$$

24. $$\frac{7+p}{p^2+11}$$

25. $$\frac{\frac{1}{3}x+\frac{1}{2}}{2x}$$

26. $$\frac{t^{\frac{-1}{2}}+t^{\frac{1}{2}}}{t^2}$$

***Rewrite in the form*** $1+\frac{(\ )}{(\ )}$.

27. $$\frac{x-2}{x+5}$$

28. $$\frac{q-1}{q-4}$$

29. $$\frac{R+1}{R}$$

30. $$\frac{3+2u}{2u+1}$$

31. $$\frac{\cos x+\sin x}{\cos x}$$

32. $$\frac{1+e^x}{e^x}$$

## 5. Changing the Form of Expressions

### *Rearranging coefficients and exponents*

Often a simple change in the form of an expression can make the expression look quite different. As we work through a problem, changing forms can be useful.

**Example 1:** $\frac{x}{2} = \left(\frac{1}{2}\right)x$

**Example 2:** $\frac{3}{4(2r+1)^{10}} = \frac{3}{4}(2r+1)^{-10}$

**Example 3:** $2^{-n} = \left(\frac{1}{2}\right)^n$

**Example 4:** $2^{x+3} = 2^x \cdot 2^3 = 8(2^x)$

**Example 5:** $\frac{3x+\sqrt{2x}}{\sqrt{x}} = \frac{3x}{\sqrt{x}} + \frac{\sqrt{2x}}{\sqrt{x}} = \frac{3x}{\sqrt{x}} + \frac{\sqrt{2}\sqrt{x}}{\sqrt{x}} = 3x^{\left(1-\frac{1}{2}\right)} + \sqrt{2} = 3x^{\frac{1}{2}} + \sqrt{2}$

### *Completing the Square*

We can convert any expression of the form $ax^2 + bx + c$ into the form $a(x-h)^2 + k$, This form is useful in graphing parabolas.

**Example 6:** $x^2 - 10x + 4 = x^2 - 10x + (25-25) + 4 = (x^2 - 10x + 25) - 25 + 4 = (x-5)^2 - 21$
where $a = +1, h = +5, k = -21$.

**Example 7:** $-2x^2 - 8x + 1 = -2(x^2+4x) + 1 = -2(x^2 + 4x + 4 - 4) + 1 = -2(x^2 + 4x + 4) + 8 + 1$
$= -2(x+2)^2 + 9$

where $a = -2,\ h = +2,\ k = +9$.

## Problems for section 5

***Rewrite each expression as a sum of powers of the variable with constant coefficients.***

1. $3x^2(x^{-1}) + \frac{1}{2x} + x^2 + \frac{1}{5}$
2. $10(3q^2 - 1)(6q)$
3. $(y - 3y^{-2})^2$
4. $x(x + x^{-1})^2$
5. $2P^2(P) + (9P)^{\frac{1}{2}}$
6. $\frac{(1+3\sqrt{t})^2}{2}$
7. $\frac{18 + x^2 - 3x}{-6}$
8. $\left(\frac{1}{N} - N\right)^2$
9. $\frac{-3(4x - x^2)}{7x}$
10. $\frac{x^4 + 2x + 1}{2\sqrt{x}}$

***Rewrite each expression in the form $a(bx+c)^n$.***

11. $\dfrac{12}{\sqrt{3x+1}}$

12. $\dfrac{250\sqrt[3]{10-s}}{0.25}$

13. $0.7(x-1)^3(1-x)$

14. $\dfrac{1}{2(x^2+1)^3}$

15. $4(6R+2)^3(6)$

16. $\sqrt{\dfrac{28x^2-4\pi x}{x}}$

***Rewrite each expression in the form $ab^x$.***

17. $\dfrac{1^x}{2^x}$

18. $\dfrac{1}{2^x}$

19. $10{,}000(1-.24)^t$

20. $e^{2x+1}$

21. $2\cdot 3^{-x}$

22. $2^x\cdot 3^{x-1}$

23. $16^{\left(\frac{t}{2}\right)}$

24. $\dfrac{e^3}{e^{-x+4}}$

25. $\dfrac{5^x}{-3^x}$

26. $\dfrac{e\cdot e^x}{0.2}$

***Rewrite each expression in the form $a(x-h)^2+k$.***

27. $x^2-2x-3$

28. $10-6x+x^2$

29. $-x^2+6x-2$

30. $3x^2-12x+13$

***Rewrite using positive exponents only and simplify if possible.***

31. $-3(x^2+7)^{-4}(2x)$

32. $-2(1+3^x)^{-3}(\ln 2)(2^x)$

33. $-(\sin\pi t)^{-1}(-\cos\pi t)\pi$

34. $-(\tan z)^{-2}\left(\dfrac{1}{\cos^2 z}\right)$

35. $\dfrac{-e^x(x^2)-e^{-x}(2x)}{(x^2)^2}$

36. $-x^{-2}(\ln x)+x^{-1}\left(\dfrac{1}{x}\right)$

***Rewrite in radical form and simplify if possible.***

37. $\dfrac{1}{2}(x^2+16)^{-\frac{1}{2}}(2x)$

38. $\dfrac{1}{2}(x^2+10x+1)^{-\frac{1}{2}}(2x+10)$

39. $\dfrac{1}{2}(\sin 2x)^{-\frac{1}{2}}(2)(\cos 2x)$

40. $\dfrac{2}{3}(x^3-e^{3x})^{-\frac{5}{3}}(3x^2-e^{3x}(3))$

## 6. Logarithms

We use an abbreviated notation to write logarithms in base 10 and base $e$. The notation $\log M$ means $\log_{10} M$ and $\ln M$ means $\log_e M$.

***Properties of logarithms (M, N > 0)***

| | | |
|---|---|---|
| $\log MN = \log M + \log N$ | $\ln MN = \ln M + \ln N$ | logarithm of a product |
| $\log \frac{M}{N} = \log M - \log N$ | $\ln \frac{M}{N} = \ln M - \ln N$ | logarithm of a quotient |
| $\log M^p = p \log M$ | $\ln M^p = p \ln M$ | logarithm of a power |
| $\log 1 = 0$ | $\ln 1 = 0$ | logarithm of +1 |
| $\log 10 = 1$ | $\ln e = 1$ | logarithm of the base |

Be careful not to make the mistake of trying to use these properties when they do not apply. None of the following forms can be changed.

| | | |
|---|---|---|
| $\log(M+N)$ | $\ln(M+N)$ | no rules for the logarithm of a sum |
| $\dfrac{\log M}{\log N}$ | $\dfrac{\ln M}{\ln N}$ | no rules for the quotient of logarithms |

***Relationship between logarithms and exponentials***

| | |
|---|---|
| $\log N = x \iff 10^x = N$ | $\ln N = x \iff e^x = N$ |
| $\log 10^x = x$ | $\ln e^x = x$ |
| $10^{\log x} = x$ | $e^{\ln x} = x$ |

**Example 1:** ***Evaluate the logarithms.***

a) $\log 10{,}000$

Base 10 logarithms are powers of 10.

The power of 10 needed to get 10,000 is 4, so $\log 10{,}000 = 4$.

b) $\ln 1$

Natural logarithms are powers of $e$.

The power of $e$ needed to get 1 is 0, so $\ln 1 = 0$.

**Example 2:** ***Write the equations in exponential form.***

$\log x = -3$ means $10^{-3} = x$

$\ln x = \sqrt{2}$ means $e^{\sqrt{2}} = x$

**Example 3:** ***Write the equations in logarithmic form.***

$10^3 = 1000$ means $\log 1000 = 3$

$10^{-2} = 0.01$ means $\log 0.01 = -2$

**Example 4:** ***Write the expressions using sums and differences.***

a) $\log 10x = \log 10 + \log x$

$= 1 + \log x$

b) $\ln \frac{e}{\sqrt{x}} = \ln e^2 - \ln \sqrt{x}$

$= \ln e^2 - \ln x^{\frac{1}{2}}$

$= 2\ln e - \frac{1}{2}\ln x$

$= 2 - \frac{1}{2}\ln x$

**Example 5:** ***Write the expression as a single logarithm.***

$$3\left(2\log M + \frac{4}{3}\log N\right) = 6\log M + 4\log N$$

$$= \log M^6 + \log N^4$$

$$= \log M^6 N^4$$

**Example 6:** ***Simplify the expression.***

$$e^{3\ln x} + 2\ln e^{-x} = e^{\ln x^3} + 2\ln e^{-x}$$

$$= x^3 + 2(-x)$$

$$= x^3 - 2x$$

**Example 7:** ***Simplify the expression.***

$$\log 10^{2x} + \log \sqrt{10} = \log 10^{2x} + \log 10^{\frac{1}{2}}$$

$$= 2x\log 10 + \frac{1}{2}\log 10$$

$$= 2x + \frac{1}{2}$$

**Problems for section 6** *(No calculators are necessary)*

***Rewrite the expressions using sums and differences and simplify where possible.***

1. $\log 2x$

2. $\ln\left[u^4(7-u)^3\right]$

3. $\dfrac{\ln x}{2}$

4. $\ln\dfrac{1}{x^2}$

5. $\ln\sqrt{\dfrac{x-1}{x+1}}$

6. $\log\dfrac{(x^2-1)}{x^3}$

***Rewrite the expression as single logarithms and simplify where possible.***

7. $4\log x+\dfrac{1}{2}\log y$

8. $\ln x^2-\ln(x+10)$

9. $\log 3+2\log\sqrt{a}$

10. $3\left[\log(x+1)+\log(x+4)\right]$

11. $\ln A+\ln\dfrac{B}{2}-\ln C^{-1}$

12. $\dfrac{1}{3}\log 8-\dfrac{1}{2}\log 25$

***Simplify the expressions.***

13. $10^{-\log 5}$

14. $e^{-3\ln t}$

15. $2\ln e^{\sqrt{x}}$

16. $\ln(a^2+b^2)$

17. $t\ln e^t$

18. $10^{(2+\log x)}$

19. $\log 10x-\log x$

20. $\ln x^4+2\ln x^{-2}$

21. $\ln\sqrt{x^2+16}$

22. $\ln\sqrt[3]{(3-z)^2}$

23. $\ln x^2-2\ln x$

24. $\log 100^z$

25. $\ln\dfrac{1}{e^x+1}$

26. $\dfrac{\ln e}{\ln e^2}$

# 7 Solving Equations

### *Solving in your head.*

When you first look at an equation, see if you can just guess the answer by mentally trying numbers.

**Example 1:** $\sqrt{x}-4=0$ *"I'm looking for something whose square root is 4."* $\sqrt{16}-4=0$

$2x-3=0$ *"What can I use that will give +3 for the left term?"* $2\left(\frac{3}{2}\right)-3=0$

$\frac{3}{x}+1=0$ *"How can I get a -1 on the left?"* $\frac{3}{(-3)}+1=0$

$e^x=1$ *"What exponent can I use on e to get 1?"* $e^{(0)}=1$

$x^2(x+2)=0$ *"What numbers make a factor zero?"* $0^2=0$ and $(-2)+2=0$

$\frac{(x+1)(3-x)}{(1-x)^2}=0$ *"When will the numerator be 0?"* $(-1)+1=0$ and $3-(3)=0$

$1-\sin x=0$ *"Where is the sine graph up at +1?"* $1-\sin\left(\frac{\pi}{2}\right)=0$

### *Operations on equations*

The rules of equality allow us to perform the same operation on both sides of an equation. If we want to change the form of an equation we can add or subtract the same number from both sides or we can multiply or divide both sides by the same number (as long as we avoid dividing by zero).

### *Linear equations*

To solve a linear equation we clear any parentheses and then isolate the variable.

**Example 2:** *Solve for x*

$$3-[5.4+2(4.3-x)]=2-(0.3x-0.8)$$
$$3-[5.4+8.6-2x]=2-0.3x+0.8$$
$$3-14+2x=1-0.3x+0.8$$
$$2.3x=13.8$$
$$x=6$$

**Example 3:** *Solve for q*

$$p^2q+r(-q-1)=4(p+r)$$
$$p^2q-rq-r=4p+4r$$
$$p^2q-rq=4p+5r$$
$$q(p^2-r)=4p+5r$$
$$q=\frac{4p+5r}{p^2-r}$$

## *Solving by factoring*

Some equations can be put into factored form such that the product of the factors is zero, then we use the fact that if $a \cdot b = 0$ either $a$ or $b$ (or both) is zero.

**Example 4:** ***Solve for x***

$(x+1)(x+3) = 15$ do not make the mistake here of setting $x+1=15$ and $x+3=15$

$$x^2 + 4x + 3 = 15$$

$$x^2 + 4x - 12 = 0$$

$$(x-2)(x+6) = 0$$

$x = 2$ and

$x = -6$ are solutions

**Example 5:** ***Solve for x***

$2(x+3)^2 = 5(x+3)$ dividing by $(x+3)$ here might lead you to miss the solution $x=-3$

$$2(x+3)^2 - 5(x+3) = 0$$

$$(x+3)(2(x+3)-5) = 0$$

$$(x+3)(2x+6-5) = 0$$

$$(x+3)(2x+1) = 0$$

$x = \frac{-1}{2}$ and

$x = -3$ are solutions

**Example 6:** ***Solve for x***

$$e^x + xe^x = 0$$

$e^x(1+x) = 0$ since $e^x$ is never zero, $x=-1$ is the only solution

## *Quadratic formula*

If an equation is in the form $ax^2 + bx + c = 0$ we can use the quadratic formula to find the solutions

$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$

**Example 7:** ***Solve for x***

$$11 + 2x = x^2$$

$$-x^2 + 2x + 11 = 0$$

$$x = \frac{-2+\sqrt{4-4(-1)(11)}}{2(-1)} = \frac{-2+\sqrt{48}}{-2} = \frac{-2+\sqrt{16\cdot 3}}{-2} = \frac{-2+4\sqrt{3}}{-2} = 1-2\sqrt{3}$$

$$x = \frac{-2-\sqrt{4-4(-1)(11)}}{2(-1)} = \frac{-2-\sqrt{48}}{-2} = \frac{-2-\sqrt{16\cdot 3}}{-2} = \frac{-2-4\sqrt{3}}{-2} = 1+2\sqrt{3}$$

## *Fractional Equations*

If the equation involves fractions, we can eliminate them by multiplying both sides of the equation by the least common denominator and then solving as before as long as we check for extraneous solutions.

**Example 8:** ***Solve for x***

$$\frac{2x}{x+1} - 3 = \frac{2}{x^2+x}$$

$$\frac{2x}{x+1} - 3 = \frac{2}{x(x+1)} \qquad \text{multiply both sides by } x(x+1)$$

$$2x^2 - 3x(x+1) = 2$$

$$2x^2 - 3x^2 - 3x = 2$$

$$-x^2 - 3x - 2 = 0$$

$$x^2 + 3x + 2 = 0$$

$$(x+2)(x+1) = 0$$

$x = -2$ and

$x = -1$ are potential solutions, but

since $x = -1$ does not check, the only solution is $x = -2$.

**Example 9:** ***Solve for $P_2$***

$$\frac{1}{P_1} + \frac{1}{P_2} = \frac{1}{P_3}$$

$$P_2P_3 + P_1P_3 = P_1P_2$$

$$P_2P_3 - P_1P_2 = -P_1P_3$$

$$P_2(P_3 - P_1) = -P_1P_3$$

$$P_2 = \frac{-P_1P_3}{P_3 - P_1} = \frac{P_1P_3}{P_1 - P_3}$$

## *Radical Equations*

We solve radical equations by raising both sides of the equation to the same power using the principle that if $a = b$ then $a^r = b^r$.

**Example 10:** ***Solve for x***

$$2\sqrt{x} = x - 3$$

$$\left(2\sqrt{x}\right)^2 = (x-3)^2$$

$$4x = x^2 - 6x + 9$$

$$x^2 - 10x + 9 = 0$$

$$(x-1)(x-9) = 0$$

$x = 1$ and

$x = 9$ are potential solutions.

Since $x = 1$ does not check, the only solution is $x = 9$.

**Example 11:** ***Solve for x***

$$4 = x^{-\frac{1}{2}}$$

$$(4)^{-2} = \left(x^{-\frac{1}{2}}\right)^{-2}$$

$$(4)^{-2} = x$$

$$x = \frac{1}{16}$$

## *Exponential Equations*

When the variable we want to solve for is in the exponent, we again "do the same thing" to both sides of the equation. This time we take logarithms using the property that if $a = b$ then $\log a = \log b$ and $\ln a = \ln b$. We then use the rules of logarithms to simplify the equation.

**Example 12:** ***Solve for x***

$$10^{2x+1} = 3$$

$$\log 10^{2x+1} = \log 3$$

$$2x + 1 = \log 3 \qquad \text{since } \log 10^P = P$$

$$x = \frac{(\log 3) - 1}{2}$$

**Example 13:** ***Solve for x***

$$2e^x = 12$$

$$e^x = 6$$

$$\ln e^x = \ln 6$$

$$x = \ln 6 \qquad \text{since } \ln e^P = P$$

**Example 14:** ***Solve for x***

$$1.07 = 4^{-x}$$

$$\ln 1.07 = \ln\left(4^{-x}\right)$$

$$\ln 1.07 = (-x)(\ln 4)$$

$$x = -\frac{\ln 1.07}{\ln 4}$$

**Example 15:** ***Solve for t.***

$$P = P_0 e^{kt}$$

$$\frac{P}{P_0} = e^{kt}$$

$$\ln\left(\frac{P}{P_0}\right) = \ln e^{kt}$$

$$\ln\left(\frac{P}{P_0}\right) = kt$$

$$t = \frac{1}{k}\ln\left(\frac{P}{P_0}\right)$$

### *Simultaneous equations*

When we have two equations in two variables and are trying to find a solution which fits both, we can substitute.

**Example 16:** ***Solve for x and y.***

$$y + \frac{x}{2} = 3$$

$$2(x+y) = 1-y$$

*Solving the first equation for y and substituting in the second gives*

$$2\left(x + \left(3 - \frac{x}{2}\right)\right) = 1 - \left(3 - \frac{x}{2}\right)$$

$$2x + 6 - x = -2 + \frac{x}{2}$$

$$x + 6 = -2 + \frac{x}{2}$$

$$2x + 12 = -4 + x$$

$$x = -16$$

*Using this value of x in the first equation to find the corresponding y, we have*

$$y = 3 - \left(\frac{-16}{2}\right) = 11$$

*so the solution is* $x = -16, \quad y = 11$

**Example 17:** ***Solve for x and y.***

$$y = x - 1$$

$$x^2 + y^2 = 5$$

*We use the expression (x-1) for y in the second equation.*

$$x^2 + (x-1)^2 = 5$$

$$x^2 + x^2 - 2x + 1 = 5$$

$$2x^2 - 2x - 4 = 0$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0.$$

Thus, $x = 2$ or $x = -1$ are the solutions

*We can then find the corresponding* $y$ *values*

for $x = 2$ , $y = (2) - 1 = 1$

for $x = -1$ , $y = (-1) - 1 = -2$

The solutions to the systemare $x = 2$ and $y = 1$ or $x = -1$ and $y = -2$.

**Example 18:** ***Solve for*** $Q_0$ ***and*** $a$.

$$90.7 = Q_0 a^{10}$$

$$91 = Q_0 a^{13}$$

*Solving the first equation for* $Q_0$

$$Q_0 = \frac{90.7}{a^{10}}$$

$$91 = \left(\frac{90.7}{a^{10}}\right) a^{13}$$

$$91 = 90.7a^3$$

$$a^3 = \frac{91}{90.7}$$

$$a = \sqrt[3]{\frac{91}{90.7}} = 1.0011$$

*To find* $Q_0$ ,

$$90.7 = Q_0 (1.0011)^{10}$$

$$Q_0 = 89.7083$$

*So the simultaneous solution is* $a = 1.0011$ *and* $Q_0 = 89.7083$.

## Problems for section 7

***Solve these linear equations.***

1. $0.06s - 0.001(2471 - s) = 21.5s$
2. $\frac{5}{3}(y + 2) = \frac{1}{2} - y$
3. $3t - \frac{2(t - 1)}{3} = 4$
4. $B - 4[B - 3(1 - B)] = 42$

***Solve these equations by factoring.***

5. $8 + 2x - 3x^2 = 0$
6. $2p^3 + p^2 - 18p - 9 = 0$
7. $N^2 - 2N - 3 = 2N(N - 3)$
8. $\frac{1}{64}t^3 = t$

***Use the quadratic formula.***

9. $x^2 - 1 = 2x$
10. $4x^2 - 13x - 12 = 0$
11. $100 = -16t^2 + 96t + 12$
12. $y^2 + 4y - 2 = 0$

***Solve these fractional equations.***

13. $\dfrac{2}{z-3}+\dfrac{7}{z^2-3z}=0$

14. $\dfrac{x^2+1-2x^2}{\left(x^2+1\right)^2}=0$

15. $4-\dfrac{1}{L^2}=0$

16. $2+\dfrac{1}{q+1}-\dfrac{1}{q-1}=0$

***Solve these radical equations.***

17. $\sqrt{r^2+24}=7$

18. $\dfrac{1}{\sqrt[3]{x}}=-2$

19. $3\sqrt{x}=\dfrac{1}{2}x$

20. $10=\sqrt{\dfrac{v}{7\pi}}$

***Solve these exponential equations.***

21. $5000=2500(0.97)^t$

22. $280=40+30e^{2t}$

23. $\dfrac{1}{2\left(2^x\right)}=16$

24. $1+10^{-x}=4.3$

***Solve these simultaneous equations.***

25. $2x+3y=7$

$y=-\dfrac{3}{5}x+6$

26. $y=2x-x^2$

$y=-3$

27. $y=4-x^2$

$y=2x+1$

28. $y=\dfrac{1}{x}$

$y=4x$

***Solve these equations for the indicated variable.***

29. $x^2+\dfrac{125}{8x}=0$

30. $T=2\pi\sqrt{\dfrac{l}{g}}$ solve for $l$

31. $\left(3x^3-20x\right)(2x)+x^2\left(9x^2-20\right)=0$ solve for $x$

32. $\left(\dfrac{1}{2}\right)^{\frac{t}{1000}}=e^{kt}$ solve for $k$

33. $\dfrac{1}{2}P_0=P_0(0.8)^x$ solve for $x$

34. $y'y^2+2xyy'=4y$ solve for $y'$

35. $l = l_0 + \frac{k}{2}w$ solve for $w$

36. $4\pi r - \frac{256}{r^2} = 0$ solve for $r$

37. $2x - (xy' + yy') + 2yy'$ solve for $y$

38. $-1.5 + 216x^{-2} = 0$ solve for $x$

39. $by - d = ay + c$ solve for $y$

40. $u(v+z) + w(v-3) = z(v-1)$ solve for $v$

# 8. Solving Inequalities

## *Solving Inequalities*

Just as we can ask "What number will make the expression $(x - 1)$ be zero? ", we can also ask "What numbers will make $(x - 1)$ positive? ".

**Example 1:**

| | | |
|---|---|---|
| $x - 1 > 0$ | *"If x is larger than +1, the left hand side is positive."* | $x > 1$ |
| $x + 4 < 10$ | *"As long as x stays smaller than +6, the left hand side will be less than the right hand side."* | $x < 6$ |
| $3 - x < 0$ | *"When x gets larger than +3, the left hand side will be negative."* | $x > 3$ |
| $x - 2 \geq 0$ | *"I need x to be +2 or larger."* | $x \geq 2$ |
| $x^2(x+5) > 0$ | *"The $x^2$ is always positive, so the (x+5) needs to be positive."* | $x > 5$ |

## *Notation for double inequalities*

If we want to describe all the $x$ values between $a$ and $b$, we write $a < x < b$.

## *Deciding when an expression is positive, negative, or zero*

When an expression is a product or a quotient we can determine the sign of the expression on an interval by looking at the sign of each factor.

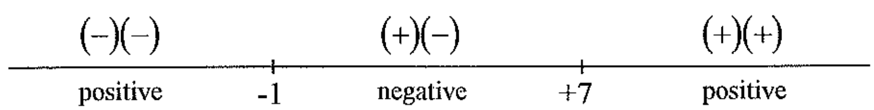
**Example 2:** ***Determine the sign of the expression*** $(4-x)^2 e^{-x}$.

The values of $e^{-x}$ are always positive, and $(4-x)^2$ is positive for any value of $x$ (except $x = 4$) because it is squared. So $(4-x)^2 e^{-x} = 0$ for $x = 4$, and

$$(4-x)^2 e^{-x} > 0 \quad \text{for} \quad x \neq 4.$$

**Example 3:** ***Determine where the expression*** $(x+1)(x-7)$ ***is positive, negative, or zero.***

The expression will be zero when $x = -1$ or $x = +7$. It will be positive when both factors have the same sign. We can check sample $x$ values easily by visualizing a number line and checking the expression's sign in each interval.




Using inequalities,

$$(x+1)(x-7) < 0 \quad \text{negative for} \quad -1 < x < 7$$
$$(x+1)(x-7) = 0 \quad \text{zero for} \quad x = -1 \text{ or } x = 7$$
$$(x+1)(x-7) > 0 \quad \text{positive for} \quad x < -1 \text{ or } x > 7.$$

**Example 4:** ***Determine the signs of the expression*** $\dfrac{2r+5}{(r-1)(r-3)}$.

A fraction is 0 when the numerator is 0, so this expression is 0 when $r = \dfrac{-5}{2}$. Furthermore, the values $r = 1$ and $r = 3$ make the expression undefined. Using a number line,

| $\dfrac{(-)}{(-)(-)}$ | | $\dfrac{(+)}{(-)(-)}$ | | $\dfrac{(+)}{(+)(-)}$ | | $\dfrac{(+)}{(+)(+)}$ |
|---|---|---|---|---|---|---|
| negative | $\dfrac{-5}{2}$ | positive | $+1$ | negative | $+3$ | positive |

So

$$\frac{2r+5}{(r-1)(r-3)} < 0 \quad \text{negative for} \quad r < \frac{-5}{2} \quad \text{or} \quad 1 < r < 3$$

$$\frac{2r+5}{(r-1)(r-3)} = 0 \quad \text{zero for} \quad r = \frac{-5}{2}$$

$$\frac{2r+5}{(r-1)(r-3)} > 0 \quad \text{positive for} \quad \frac{-5}{2} < r < 1 \quad \text{or} \quad r > 3$$

$$\frac{2r+5}{(r-1)(r-3)} \quad \text{not defined for} \quad r = 1 \quad \text{or} \quad r = 3\ .$$

**Example 5:** ***Determine the signs of the expression*** $\dfrac{\sqrt{y+2}}{y^3}$.

The expression is zero when $y = -2$. It is not defined when $y$ is less than -2 because we cannot take the square root of a negative number. Furthermore, it is not defined when $y = 0$ because of the zero in the denominator. The radical is always positive so the numerator will be positive. Because the cube of a negative number is negative, we can visualize signs in each interval,

| | | $\dfrac{(+)}{(-)}$ | | $\dfrac{(+)}{(+)}$ |
|---|---|---|---|---|
| undefined | -2 | negative | 0 | positive |

So

$$\frac{\sqrt{y+2}}{y^3} \quad \text{undefined for} \quad y < -2 \quad \text{or} \quad y = 0$$

$$\frac{\sqrt{y+2}}{y^3} < 0 \quad \text{negative for} \quad -2 < y < 0$$

$$\frac{\sqrt{y+2}}{y^3} = 0 \quad \text{zero for} \quad y = -2$$

$$\frac{\sqrt{y+2}}{y^3} > 0 \quad \text{positive for} \quad y > 0$$

### *Operations on inequalities*

If we compare operations on inequalities with operations on equations, we find many similarities and a few important differences.

| ***If*** $a = b$ | ***If*** $a < b$ | | |
|---|---|---|---|
| $a+2=b+2$ | $a+2<b+2$ | | |
| $a-2=b-2$ | $a-2<b-2$ | | |
| $(+2)a=(+2)b$ | $(+2)a<(+2)b$ | | |
| $(-2)a=(-2)b$ | $(-2)a>(-2)b$ | | (inequality reverses) |
| $a^2=b^2$ | $a^2<b^2$ | if $a,\ b$ positive | |
| $\frac{1}{a}=\frac{1}{b}$ | $\frac{1}{a}>\frac{1}{b}$ | if $a,\ b$ positive | (inequality reverses) |

### *Solving linear inequalities*

We solve linear inequalities with the same techniques we use to solve linear equations; however, the inequality is reversed whenever we multiply or divide both sides of the inequality by a negative number.

**Example 6:** ***Solve for x.***

$$1-\frac{3}{2}x\le 16$$

$$1-\frac{3}{2}x-1\le 16-1$$

$$-\frac{3}{2}x\le 15$$

$$\left(-\frac{2}{3}\right)\left(-\frac{3}{2}x\right)\ge\left(-\frac{2}{3}\right)15$$

$$x\ge -10$$

**Example 7:** ***Solve for x.***

$$-6<4x-7<5$$

$$-6+7<4x-7+7<5+7$$

$$1<4x<12$$

$$\frac{1}{4}<x<3$$

### *Solving non-linear inequalities by factoring and determining signs*

We can often solve polynomial and rational inequalities by starting out the same way we would with an equation and then using a number line to find the intervals in which the inequality holds.

**Example 8:** *Solve for* $x$.

$$2x^2 + x^3 \le 3x$$
$$x^3 + 2x^2 - 3x \le 0$$
$$x\left(x^2 + 2x - 3\right) \le 0$$
$$x(x+3)(x-1) \le 0$$

The left hand side equals $0$ for $x = 0$, $x = -3$, or $x = 1$. To solve the inequality we want to select the $x$ values for which the left hand side is negative.

| $(-)(-)(-) = (-)$ | | $(-)(+)(-) = (+)$ | | $(+)(+)(-1) = (-)$ | | $(+)(+)(+) = (+)$ |
|---|---|---|---|---|---|---|
| negative | -3 | positive | 0 | negative | $+1$ | positive |

So $\quad 2x^2 + x^3 \le 3x \quad$ for $\quad 0 \le x \le 1 \quad$ or $\quad x \le -3$ .

**Example 9:** *Solve for* $q$.

$$\frac{1}{2-q} > q$$
$$\frac{1}{2-q} - q > 0$$
$$\frac{1}{2-q} - q\frac{(2-q)}{(2-q)} > 0$$
$$\frac{1 - q(2-q)}{2-q} > 0$$
$$\frac{1 - 2q + q^2}{2-q} > 0$$
$$\frac{(1-q)^2}{2-q} > 0$$

The numerator is never negative, and the denominator will be positive whenever $q$ is less than $+2$. However, the numerator will be zero for $q = 1$, so $q = 1$ must be excluded to preserve the inequality.

Therefore, $\dfrac{1}{2-q} > q \quad$ for $\quad q < 2$ and $q \ne 1$.

## *Radical inequalities*

We can often eliminate radicals by raising both sides to the same power.

**Example 10:** ***Solve for*** $x$.

$$\sqrt{x-6} < 2$$

The expression $\sqrt{x-6}$ is not defined unless $x \geq 6$, so we can limit the $x$ values we need to consider when we are solving an inequalitiy containing this radical.

$$\sqrt{x-6} < 2$$

$$\left(\sqrt{x-6}\right)^2 < 2^2$$

$$x-6 < 4$$

$$x < 10$$

However, we are only considering $x$ values which are $+6$ or larger, so

$$\sqrt{x-6} < 4 \quad \text{for} \quad 6 \leq x < 10$$

**Example 11:** ***Solve for*** $x$.

$$\sqrt{2-x} > 3$$

$$\left(\sqrt{2-x}\right)^2 > 3^2$$

$$2-x > 9$$

$$-x > 7$$

$$x < -7$$

In this case, the fact that the radical is only defined when $x \leq 2$ does not affect our solution because any number which is less than $-7$ is clearly less than $+2$.

## Problems for section 8

***Solve mentally.***

1. $2(x-7) \geq 0$
2. $\sqrt{x} > 4$
3. $x^2 < 25$
4. $x - 3 > 2$
5. $1+\sqrt{x+4} \neq 0$
6. $x^2 \geq 16$
7. $1+x^2 > 0$
8. $5 - x < 0$
9. $2^{-x} > 0$
10. $2x^2 + 1 < 0$

***Write these descriptions as inequalities.***

11. the $x$ values which are less than $.001$
12. the $y$ values between $-1$ and $+1$
13. all the values of $p$ which are not $5$
14. all the positive values of $k$
15. all the $r$ 's which are not negative
16. the $t$ values which are during or after the year $1995$

***Determine which values of the variable make these expressions (a) undefined (b) zero (c) positive (d) negative.***

17. $3x^2 + 6x$
18. $(2x)e^x + x^2 e^x$
19. $2^{-x}$
20. $6t^2 - 30t + 36$
21. $\frac{1}{3}x^{-\frac{2}{3}}$
22. $\frac{-24}{p^3}$
23. $\frac{1}{2\sqrt{x^2+1}}(2x)$
24. $\frac{\ln x}{x}$
25. $\frac{1-3u^2}{(u^2+1)^3}$
26. $-\frac{2x-1}{(x(x-1))^2}$

***Solve the inequalities.***

27. $4 - x^2 > 0$
28. $1 \leq 4x - 3 \leq 1$
29. $0 \leq \frac{1}{2} - n < 11$
30. $\sqrt{3l} - \frac{1}{4} > 0$
31. $t^2 - 3t - 4 \geq 0$
32. $2(x-1)(x+4) + (x-1)^2 > 0$
33. $2 + \frac{r}{r-3} > 0$
34. $\frac{1}{x} > \frac{1}{x+1}$
35. $\frac{2x^2 - (2x+1)(2x)}{x^4} > 0$
36. $\frac{3(x+2)^2 - 6x(x+2)}{(x+2)^4}$

# 9. Setting Up Functions from Descriptions

### *Deciding What the Letters Stand For*

When an expression contains a combination of letters and numbers some of the letters may represent constants and some may represent variables. For example, in the phrase $f(x) = ax^2 + bx + c$ the $a$, $b$, and $c$ are thought of as constants, the variable is $x$ , and the letter $f$ stands for the quadratic operation itself. Often the letters at the beginning of the alphabet, $a$, $b$, $c$, ....... stands for constants, those in the middle $f, g, h$, .... $F, G$, .... for functions and those at the end $w, x, y, z$ for variables. However, in each specific case you will either be told which is which, or be expected to decide from the context.

**Example 1:** The depth, $y$, of the water (in meters), as a function of time $t$ , in hours since midnight, is given by $y = y_0 + A\cos Bt$ Where $y_0$ is the average depth.

Here the variables are $t$ (independent) and $y$ (dependent) and all the other letters $y_0$ , $A, B$ represent constants. It is common for subscripts to denote a fixed value of the variable.

**Example 2:** The equation for a line through the fixed point $(h, k)$ with specified slope, $m$ , is given by

$$y - k = m(x - h)$$

In this case, the variables are $x$ (independent) and $y$ (dependent). The slope and the coordinates of the point are treated as constants.

### *Coefficients*

We use the word "*coefficient*" to describe the role of certain letters. A constant which multiplies a variable is called a "*coefficient*".

**Example 3:** In the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, the coefficient of $r^3$ is the constant $\frac{4}{3}\pi$ .

### *The Phrases "as a function of" and "in terms of"*

Many problems begin by asking us to set up an equation which describes a relationship between two quantities. Some of these relationships are well known formulas, like the one relating the area of a circle to its radius. The formula $A = \pi r^2$ is a convenient form for finding $A$ from $r$ ; $r$ is the independent variable and $A$ is the dependent variable. We say that the formula gives $A$ *as a function of* $r$ or, as it's sometimes stated, $A$ *in terms of* $r$ . The functional notation $A(r)$ is read $A$ *as a function of* $r$ . We could equally well have chosen to write an equation for the radius as a function of the area, $r(A)$ , which would mean having $A$ as the independent variable and writing $r = \sqrt{\frac{A}{\pi}}$ .

**Example 4:** After a liquid fertilizer is applied to a house plant, the concentration of fertilizer in the plant is found to be twice the reciprocal of the amount of time elapsed since the application. Express the concentration, $C$ , as a function of the time, $t$ .

We are trying to write an equation which is solved for $C$ and has $t$ as the only other variable.

$$C = (\text{twice})(\text{reciprocal of time elapsed})$$

so $C = (2)\left(\frac{1}{t}\right)$,

or $C = \frac{2}{t}$.

**Example 5:** A bus rental company charges $150 initial fee for a bus. In addition to this initial fee the company charges $23.00 an hour for operating costs and $10.00 an hour for the driver's wages. Write a function $D(h)$ for the total cost in dollars, $D$, in terms of the hours of rental time, $h$.

$$D = \text{initial charge} + \text{operating cost} + \text{wages}$$
$$D = \$150 + \$23.00h + \$10.00h$$
$$D = 150 + 33h$$

**Example 6:** The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$. Give a formula for the volume as a function of the radius for a cone whose height is double its radius.

We are to write a formula for $V$ which contains only the variable $r$ and constants. We can eliminate the $h$ variable by using the fact that,

$$\text{height} = (\text{twice})(\text{radius})$$
$$h = 2r.$$

Substituting in the volume formula gives

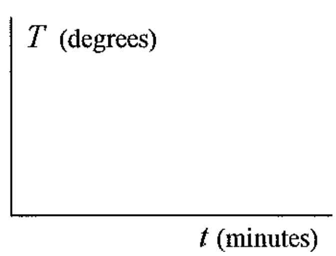
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(2r)$$
$$V = \frac{2}{3}\pi r^3.$$

### *Setting up the axes for graphs*

In addition to setting up a function, we often want to sketch it's graph. We customarily use the horizontal axis for the independent variable and the vertical axis for the dependent variable.

**Example 7:** How would you label the axes to sketch a graph of the temperature, $T$ (in degrees), of a glass of water, $t$ minutes after it is placed in a refrigerator?

Here the independent variable is $t$ (in minutes) and the dependent variable is $T$ (in degrees), so the axes would be labeled,



## Problems for section 9

***Specify which letters represent functions and which represent variables and wich represent constants in the situations described.***

1. The function is a decreasing exponential, $f(x) = a^{-x}$.

2. The height function, $H(t)$, is increasing rapidly, $H(t) = AB^t$.

3. A specified interval on the *x*-axis is given by $a < x < b$.

4. The difference of two functions is constant, $f(x) - g(x) = c$.

5. The general form of a polynomial function is $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$.

6. The voltage, $V(t)$, fluctuates according to $V = V_0 \cos(120\pi t)$, where $V_0$ represents the initial voltage.

7. The bank balance resulting from an initial deposit of $P_0$ dollars after $t$ years $B(t) = P_0\left(1 + \frac{r}{12}\right)^{12t}$ where $r$ is a fixed annual interest rate compounded monthly.

8. The amount of a medicine remaining in the bloodstream $t$ hours after it as been administered is $M(t) = M_0 e^{\left(\frac{-8.3}{h}\right)t}$, where $h$ is the medicine's half-life and $M_0$ is the original dose.

9. The bank balance after $t$ years when $P$ dollars are deposited in an account with a fixed annual interest rate, $r$, compounded continuously $B = Pe^{rt}$.

10. A formula for vertical motion of an object under the effect of gravity, $g$, states that the distance $s$ (in feet) of an object above the ground at time $t$ (in seconds) is given by $s = \frac{1}{2} g t^2 + v_0 t + s_0$.

***Identify the specified coefficients.***

11. $A = \pi r^2$ — What is the coefficient of $r^2$.

12. $\sin\left(2\pi\Theta + \frac{\pi}{2}\right)$ — What is the coefficient of $\Theta$.

13. $P = P_0 e^{kt}$ — What is the coefficient of $t$?

14. $s = -16t^2 - 2t + 1$ — What is the coefficient of $t$?

15. $30\cos\left(\frac{x}{2}\right)$ — What is the coefficient of $x$?

***Write the formulas (common geometric formulas are listed in section 16).***

16. Write a formula for $A$ in terms of $l$ if $A=lw$ and $2l+2w=100$.

17. Write a formula for the function $V(h)$ if $V=\frac{1}{3}\pi r^2 h$ and $\frac{r}{6}=\frac{h}{10}$ .

18. If $A=\pi r^2$ and $r=3t^2$ , write a formula for $A$ as a function of $t$ .

19. For a right triangle $x^2+y^2=z^2$ , write a formula for $z$ in terms of $t$ if $x=3t$ and $y=4\sqrt{t}$ .

20. A refrigerator has a 4 ft. square base. Express it's height, $h$ , as a function of it's volume, $V$.

21. A soda can has volume $V=\pi r^2 h$. Write a formula for it's volume, $V(r)$, if the can's height is 5 inches longer than it's radius.

22. Express the area of a circle, $A$ , in terms of its circumference, $C$.

23. Express the volume, $V$, of a cube as a function of the area, $A$ , of its base.

***Label the axes.***

24. Label the axes for a sketch of a problem which begins "Over the past century we have seen changes in the population, $P$ (in millions), of the city ...."

25. Label the axes for a problem which says "Sketch a graph of the cost of manufacturing $q$ items..."

26. Label the axes for a problem which says "Graph the pressure, $p$ , of a gas as a function of its volume, $v$ , where $p$ is in pounds per square inch and $v$ is in cubic inches."

27. Label the axes for a graph of the function $f(z)=z_0(a)^z$.

28. Label the axes for a problem which asks you to "Graph $D$ in terms of $y$ ..."

## 10. Understanding Function Notation

It is easier to say "$f(x)$ is linear" or "$f(x) = mx + b$" than to say *"you get $y$ by multiplying $x$ by some constant and then adding some other constant to your answer."* The language of function notation gives you a name, $f$, for the operation and at the same time identifies the independent variable, $x$.

**Example 1:** Let $f(x) = x^2 + 6$, then

$$f(0) = (0)^2 + 6$$
$$f(-3) = (-3)^2 + 6$$
$$f(t) = (t)^2 + 6$$
$$f(a+h) = (a+h)^2 + 6$$
$$f(\underbrace{\quad) = (\quad}_{\text{same number}})^2 + 6$$

**Example 2:** Let $f(x) = \sqrt{x}$, then

$f(3.7) = \sqrt{3.7}$

$f(2+h) = \sqrt{2+h}$ (note $f(2+h) \neq f(2) + f(h)$)

$f(2) + h = \sqrt{2} + h$

$f(2a) = \sqrt{2a}$ (note $f(2a) \neq 2f(a)$)

$2f(a) = 2\sqrt{a}$

When we use function notation, it is crucial to "translate" the meaning of the notation correctly. Similar-appearing forms can have quite different meanings.

**Example 3:** ***"Translate" the following function sentences using the function*** $f(x) = (x-1)^3$.

a) Let $f(x) = (x-1)^3$. This tells you the rule for forming the function "$f$" using the variable $x$.

b) Solve $f(x) = 27$. This tells you to find the $x$ value which makes $y = 27$. In other words, solve the equation $(x-1)^3 = 27$ getting $x = 6$

c) Find $f(3)$. This tells you to find the $y$ value corresponding to $x = 3$, $(3-1)^3 = 8$.

**Example 3:** Let $f(x)=x^2+3x-10$. Find the $y$-intercept of the graph of $f$ and the places where the graph crosses the $x$-axis.

To find the $y$-intercept, we let $x=0$, so we are looking for the value:

$$f(0)= (0)^2+3(0)-10= -10.$$

This gives us a $y$-intercept of $(0,-10)$. The graph will cross the $x$-axis wherever $y=0$, in other words where $f(x)=0$. Thus,

$$x^2+3x-10=0,$$
$$(x+5)(x-2)=0,$$
$$\text{so}\quad x=-5 \text{ or}$$
$$x=2.$$

The graph crosses the $x$-axis at $(-5,0)$ and $(2,0)$.

**Example 4:** Let $f(x)=4e^x$. Find $x$ such that $f(x)=12$.

$$4e^x=12$$
$$e^x=3$$
$$\ln e^x=\ln 3$$
$$x=\ln 3$$

## Problems for section 10

1. Let $f(x)=\dfrac{3x}{1+x}$
   a. If $x=1$, what is $f(x)$?
   b. If $x=0$, what is $f(x)$?
   c. Find an $x$ such that $f(x)=4$.
   d. Solve the equation $f(x)=\dfrac{1}{2}$ for $x$.

2. Let $g(z)=\ln z$
   a. Find $g(e^2)$.
   b. Find a value which makes $g(z)$ zero.
   c. What is $g(1)$?
   d. Where does the graph of $g(z)$ cross the $z$ axis?

3. Let $M(r)=20(1.02)^r$
   a. If $r=1$, what is $M(r)$?
   b. If $r=0$, what is $M(r)$?
   c. Find an $r$ such that $M(r)=40$.
   d. Solve the equation $M(r)=10$ for $r$.
   e. What is the $M$ intercept of the graph of $M(r)$?

4. Let $f(x)=\sqrt{x^2+16}-5$.
   a. Find $f(0)$.
   b. For what $x$ is $f(x)$ zero?
   c. Find $f(3)$.
   d. What is the $y$ intercept of the graph of $f(x)$?
   e. Where does the graph cross the $x$ axis?

5. Let $v(t)=t^2-2t$ where $v$ is the velocity (ft./sec.) of an object at time $t$.
   a. What is the initial velocity, $v(0)$?
   b. When is the object stopped? (This is the time when $v(t)=0$).
   c. What is the meaning of the notation $v(3)$? What are its units?

6. Let $s(t)=2t^4+320$ be the position (kilometers) of an object at time $t$ (hours).

   a. What is the value of $s(0)$?

   b. What is meaning of $s(0)$ in practical terms?

   c. What is the sign of $s(t)$?

7. Let $v(t)=t^2-7t+12$ where $v$ is the velocity (inches/year) of an object at time $t$.

   a. How fast is the object moving after 42 months?
   b. When is the object stopped?
   c. For what $t$ values is $v(t)>0$?

8. Let $s(t)=11t^2+t+100$ be the position (miles) of a car driving on a straight road at time $t$ (hours), then the car's velocity (miles/hour) at any time $t$ will be given by $v(t)=22t+1$.

   a. Use function notation to express the car's position after 2 hours. Where is the car then?
   b. Use function notation to express the question; *"When is the car going 65 mph?"*
   c. Where is the car when it is going 67 mph?

## 11. Evaluating Functions

There are certain common values of functions that you should memorize before you do the problems in this section. These values are often used in examples, and you need to be able to recognize them on sight. Section 17 lists the values you should memorize.

### *Calculate in your head*

If an expression is not too complicated you can sometimes substitute a number into the function mentally.

**Example 1:**

$\frac{2}{3}x^{-\frac{1}{3}}$ at $x = 1$ "*+1 raised to any power is still +1*" $\frac{2}{3}(1)^{-\frac{1}{3}} = \frac{2}{3}$

$\frac{2}{3}x^{-\frac{1}{3}}$ at $x = 0$ "$x^{-\frac{1}{3}}$ *means* $\frac{1}{\sqrt[3]{x}}$, *so zero is in the denominator*" $\frac{2}{3}\frac{1}{\sqrt[3]{0}}$ is undefined

**Example 2:**

$x^2 \sin x$ at $x = \pi$ "*$\sin \pi$ is zero, so the product will be zero*" $(\pi)^2 \sin \pi = \pi^2(0) = 0$

### *Formal Substitution*

When the calculations are more complicated, you may need to write out some steps.

**Example 3:**

For $f(x) = 3x^2 + 2x$, find $\frac{f(b) - f(a)}{b - a}$ when $a = -2$ and $b = 1$.

Here it is important to use enough parentheses to keep the minus signs straight.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{\left[3(1)^2 + 2(1)\right] - \left[3(-2)^2 + 2(-2)\right]}{1 - (-2)} = \frac{5 - [3(4) - 4]}{3} = \frac{5 - 8}{3} = -1$$

**Problems for section 11**

***Evaluate these expressions mentally.***

1. $\ln x^2$ at $x = -1$
2. $\sqrt[3]{x - 27}$ at $x = 26$
3. $\sin x + \cos x$ at $x = 0$
4. $(\cos 2x)2$ at $x = \pi$
5. $2\cos x$ at $x = \pi$
6. $-\ln x$ at $x = e$
7. $10^{(2x+3)}$ at $x = 0$
8. $2x^2 - (x - 3)$ at $x = 3$
9. $2^y$ at $y = -2$
10. $3^{-x}$ at $x = -1$
11. $(\ln 2)2^x$ at $x = 0$
12. $3(x^2 + 1)^2(2x)$ at $x = 1$
13. $2e^x$ at $x = \ln 3$
14. $\log 2x$ at $x = 5$
15. $\frac{\sin x}{x}$ at $x = \frac{\pi}{2}$
16. $x \tan x$ at $x = \frac{\pi}{4}$

## Problems for section 11 cont.

17. $\frac{1}{2}x^{-\frac{1}{2}}$ at $x=1$

18. $\frac{1}{2\sqrt{x}}$ at $x=36$

19. $\frac{(x^2+1)(x-1)+(2x+1)}{(x+1)^2}$ at $x=1$

20. $\sin^2 x$ at $x=\frac{3\pi}{2}$

***Find the following values.***

21. If $F(x)=-\cos x$ find $F(b)-F(a)$ for $a=0,\ b=\pi$.

22. If $\frac{2x}{3}=\frac{(3x+1)^2}{6}+C$ find $C$ for $x=1$.

23. If $y-1=m(x+4)$ find $m$ for $x=-2,\ y=\frac{1}{2}$.

24. If $V=\pi r^2 h$ find $h$ for $V=12$ cubic inches, $r=\sqrt{2}$ inches.

25. If $F(x)=\frac{1}{4}x^4+x$ find $F(b)-F(a)$ for $a=-1,\ b=1$.

26. If $4x-3(y^2+x^2yy')+4y'=0$ find $y'$ for $x=1,\ y=2$.

27. If $x^2+y^2=z^2$ find $y$ for $x=3,\ z=5,\ y>0$.

28. If $g(w)=(w+1)^{\frac{1}{2}}+\frac{1}{2}w(w+1)^{-\frac{1}{2}}$ find $g(8)$.

29. If $h(z)=\frac{(\cos z)(2z+1)-(\sin z)2}{(2z+1)^2}$ find $h(0)$.

30. If $s(t)=(3t^2)(\ln t)+t^3\left(\frac{1}{t}\right)$ find $s(t)$ when $t=2$.

# 12. Combining Evaluation, Graphing, and Solving

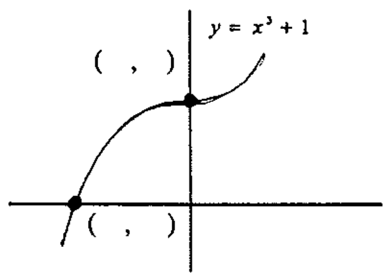
We can use our understanding of functions and our knowledge of the shapes of curves to evaluate some of the key coordinates on graphs.

### *Intercepts*

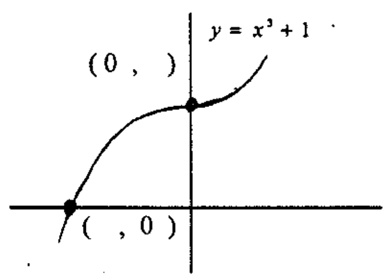
We usually want to label the intercepts of our graphs when we make a sketch. The $y$-intercept will have a zero as its first coordinate (0, ), and any $x$-intercept will have a zero as its second coordinate ( ,0). To decide on the missing coordinates we can think in the following way.

The coordinates of the $y$-intercept are ( 0 , ( the $y$ value corresponding to $x = 0$ ) ).
The coordinates of any $x$-intercept are ( (any $x$ value making $y = 0$), 0 ).

**Example 1:** ***Fill in the missing coordinates.***

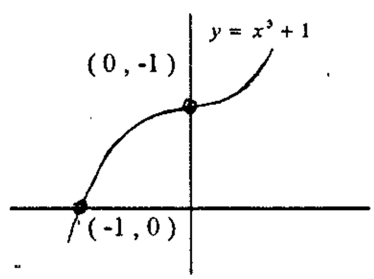



To solve this problem we first fill in what we know without any calculations.




The $y$-intercept will be $\left(0, (0)^3 + 1\right) = (0,1)$ and the $x$-intercept will be in the form

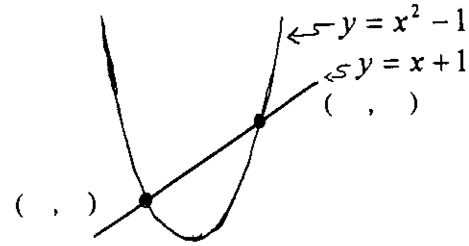
(solution of $x^3 + 1 = 0,\ 0$). The value $x = -1$ will make $x^3 + 1 = 0$, so the $x$-intercept is (-1 , 0 ).



### *Intersections*

When two graphs intersect, the point of intersection lies on both graphs, so the coordinates of the point of intersection will satisfy both of the equations which led to these graphs. We need to solve the equations simultaneously, to find the coordinates of the intersection

**Example 2:** ***Fill in the missing coordinates.***




Solve $y = x^2 - 1$ and $y = x + 1$ simultaneously by setting them equal.

Then $x^2 - 1 = x + 1$,

$x^2 - x - 2 = 0$,

and $(x+1)(x-2) = 0$.

Therefore $x = -1$ and

$x = 2$ are solutions

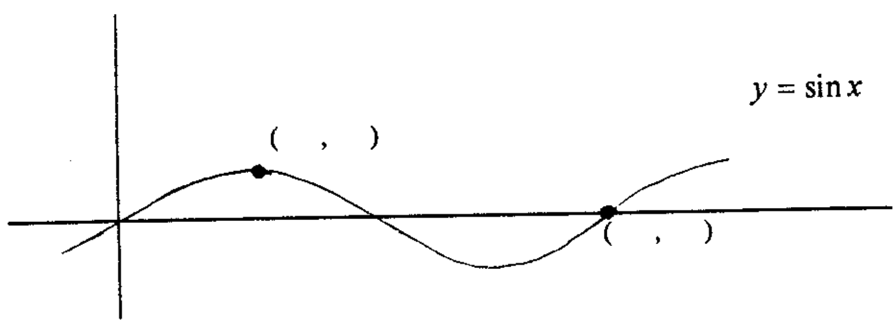
To get the corresponding $y$-values, we use either equation to find

$x = -1$ gives $y = 0$,

and $x = 2$ gives $y = 3$.

The graphs intersect at (-1 , 0 ) and ( 2 , 3 ).

**Example 3:** ***Fill in the missing coordinates.***

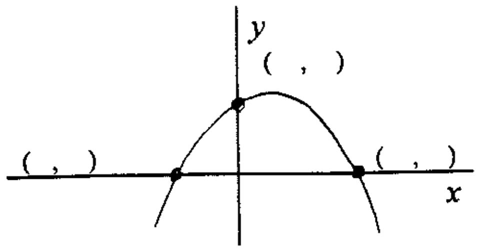



The sine function completes one full cycle between $x = 0$ and $x = 2\pi$ , so the right hand point has coordinates $(2\pi, 0)$. The graph reaches its highest value one quarter of the way through the cycle at $x = \frac{\pi}{2}$ . Since $\sin\frac{\pi}{2} = 1$ , the left hand point has coordinates $\left(\frac{\pi}{2}, 1\right)$.

## Problems for section 12

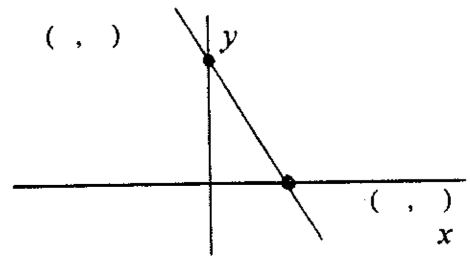
***Fill in all the missing coordinates.***

1.



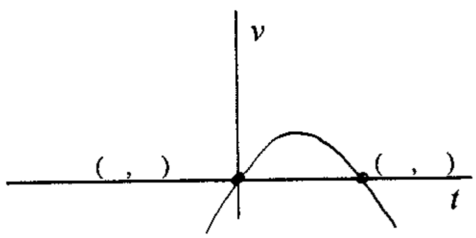

$y = -(x + 1.3)(x - 2.1)$

2.



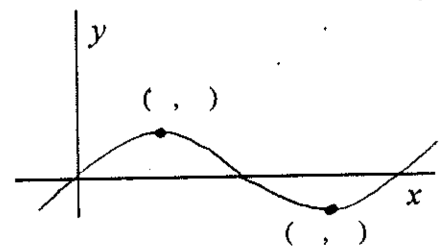

$2x + y - 3.04 =$

3.



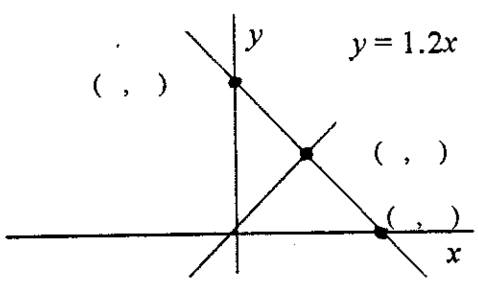

$v = t(t - 2)$

4.



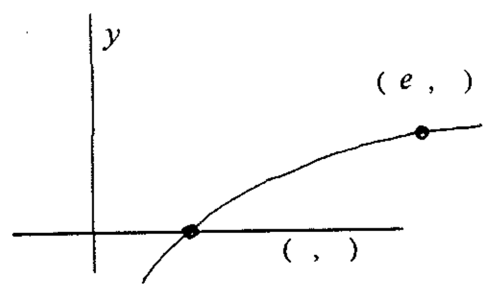

$y = 2\sin x$

5.



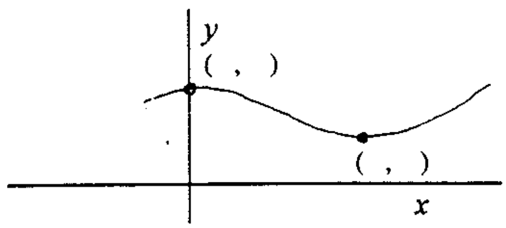

$y = 3 - x$

6.



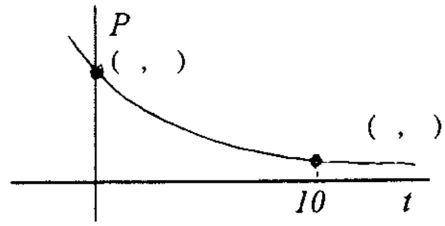

$y = \ln x$

7.




$y = \cos x + 2$

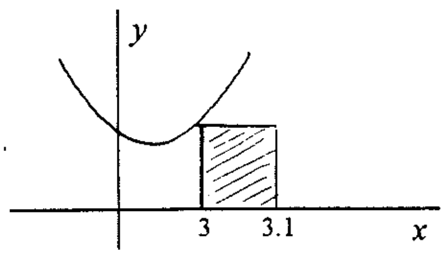
8.




$P = 5e^{-0.2t}$

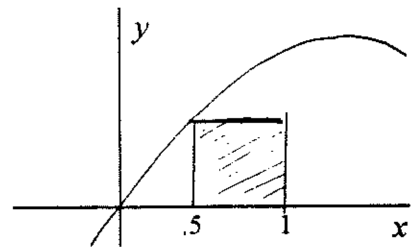
***Calculate the area of the shaded rectangles.***

9.



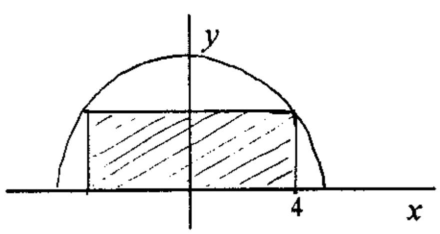

$y = (x-1)^2 + 1$

10.



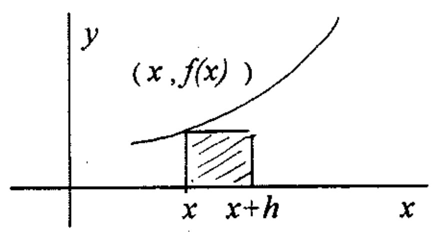

$y = x(5-x)$

11.




$x^2 + y^2 = 25$

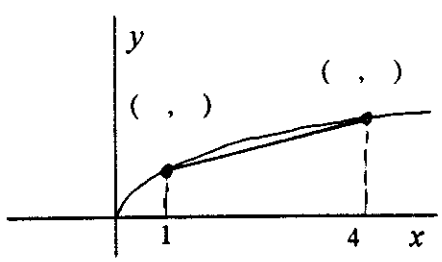
12.




$y = f(x)$

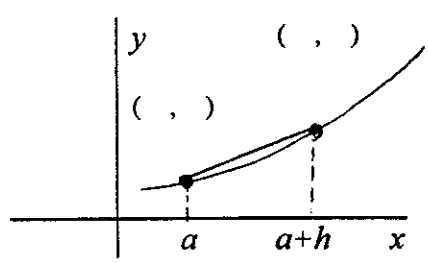
***Fill in the missing coordinates and then write an expression for the slope of the line connecting the two points.***

13.




$y = \sqrt{x}$

14.




$y = f(x)$

## 13. Composition of Functions and Substitution

There are many ways of combining functions to create more complicated functions. For instance, we can subtract one function from another or multiply two or more functions.

### *Combinations*

We can combine functions by adding, subtracting, multiplying, or dividing.

**Example 1:** Let $f(x)=3+x^2$ and $g(x)=\sin x$

$2g(x)+f(x)=2\sin x+(3+x^2)$ "*twice* $\sin x$ *plus the quantity* $(3+x^2)$"

$1+\dfrac{f(x)}{g(x)}=1+\dfrac{\sin x}{3+x^2}$ "*one plus the quotient of* $\sin x$ and $(3+x^2)$"

### *Compositions:*

If we substitute one function into another, we can generate functions in which one function can be thought of as "inside" the other.

**Example 2:** Let $f(x)=3+x^2$ and $g(x)=\sin x$

$f(g(x))=3+(\sin x)^2$ "*three plus the square of* $\sin x$", the inside function is $(\sin x)$

$g(f(x))=\sin(3+x^2)$ "*the sine of the quantity* $(3+x^2)$", the inside function is $(3+x^2)$

### *Simple substitutions*

When we are using the function notation $f(x)$ and something other than the independent variable alone appears in the parentheses, we are being asked to form a new function by substitution.

**Example 3:**

$f(x)=\sqrt{x+1}$ think $f(\ )=\sqrt{(\ )+1}$

$f(x^3)=\sqrt{(x^3)+1}$ the inside function is $(x^3)$

$f(2x)=\sqrt{(2x)+1}$ the inside function is $(2x)$

$f(-x)=\sqrt{(-x)+1}$ the inside function is $(-x)$

$f(x+h)=\sqrt{(x+h)+1}$ the inside function is $(x+h)$

### *Common mistakes*

Do not confuse substitution with performing algebraic operations on the fiction itself. The letters and symbols may be the same but the meaning is very different.

**Example 4:** ***Even though $f(3x)$ and $3f(x)$ each contain both the letter $f$, and the letter $x$, and the number $3$, they describe different operations.***

Let $f(x)=x^2+4$ think $f(\ )=(\ )^2+4$

$f(3x)=(3x)^2+4$ substitute $(3x)$ in $f$

$3f(x)=3(x^2+4)$ multiply $f(x)$ by 3

**Example 5:**

Let $g(x) = \cos x$ — think $g(\ ) = \cos(\ )$

$g(x^2) = \cos(x^2)$ — the cosine of $(x^2)$

$(g(x))^2 = (\cos x)^2$ — the square of $(\cos x)$

**Example 6:**

Let $f(x) = \dfrac{1}{x}$ — think $f(\ ) = \dfrac{1}{(\ )}$

$f(x+h) = \dfrac{1}{(x+h)}$ — the reciprocal of $(x+h)$

$f(x) + h = \left(\dfrac{1}{x}\right) + h$ — $f(x)$ plus the constant, $h$

### *Conventions used in function notation*

There are times when it is customary to omit the parentheses when writing certain functions.

| | | | |
|---|---|---|---|
| $\sin 2x$ | means | $\sin(2x)$ | composition |
| $\sin x^2$ | means | $\sin(x^2)$ | composition |
| $\sin^2 x$ | means | $(\sin x)^2$ | multiplication |
| $\ln 5x$ | means | $\ln(5x)$ | composition |
| $\ln x^4$ | means | $\ln(x^4)$ | composition |

### *Difference quotients*

One common operation which requires both the composition and the combination of functions comes up in the calculation of slope where we have.

$$\frac{\text{difference of } y \text{ values}}{\text{difference of } x \text{ values}} = \frac{f(x+h) - f(x)}{(x+h) - (x)} = \frac{f(x+h) - f(x)}{h}$$

The expression $\dfrac{f(x+h) - f(x)}{h}$ is called a difference quotient.

**Example 7:** ***Let $f(x) = 2 - x^2$. Find and simplify the difference quotient.***

$$\frac{f(x+h) - f(x)}{h} = \frac{\left[2 - (x+h)^2\right] - \left[2 - x^2\right]}{h} = \frac{\left[2 - (x^2 + 2xh + h^2)\right] - \left[2 - x^2\right]}{h}$$

$$= \frac{(2 - x^2 - 2xh - h^2) - 2 + x^2}{h} = \frac{-2xh - h^2}{h} = h\left(\frac{-2x - h}{h}\right)$$

$$= -2x - h$$

In problems like this, be careful to form the composition $f(x+h)$ correctly and be sure that the minus sign in the numerator is applied to the entire function, $f(x)$.

## Problems for section 13

1. Let $f(x) = \frac{1}{x^2}$, find and simplify

   a. $f(2x)$ b. $2f(x)$

   c. $f(\sqrt{x})$ d. $\sqrt{f(x)}$

   e. $f(x^2)$ f. $(f(x))^2$

2. Let $h(x) = \tan x$ , find

   a. $h(\alpha)$ b. $h(\Theta)$

   c. $h(x^2)$ d. $[h(x)]^2$

   e. $xh(x)$ f. $\frac{h(2x)}{2x}$

3. Let $k(t) = e^t$

   a. $k(t^2)$ b. $[k(t)]^2$

   c. $k(\ln t)$ d. $k(t) - k(-t)$

   e. $k(\cos t)$ f. $\frac{1}{k(t)}$

4. Let $h(\Theta) = \sin 2\Theta$.

   The double angle formula gives

   $\sin 2\Theta = 2\sin\Theta\cos\Theta$.

   Find a formula for $\sin 4\Theta$ by forming $h(2\Theta)$.

5. The functions $f(x)$ and $g(x)$ are inverses of each other if $f(g(x)) = x$ and $g(f(x)) = x$. Check that the following pairs of functions are inverses by forming the compositions.

   a. $f(x) = 1 + \ln x$, $g(x) = e^{(x-1)}$

   b. $f(x) = x^2 + 2$, $g(x) = \sqrt{x-2}$, $x \geq 2$

6. Let $f(x) = x^2 + 3x - 5$.

   Find and simplify $\frac{f(1+h) - f(1)}{h}$.

7. Let $s(t) = \frac{1}{t}$.

   Find and simplify $\frac{s(t+h) - s(t)}{h}$.

# 14. Recognizing the Common Types of Functions

Whenever we can classify a function as one of the familiar types such as "*quadratic*" or "*trigonometric*" we know it will have certain well-known properties; for example, a trigonometric function will be periodic. As you learn to use the vocabulary of functions more and more precisely, it will be easier for you to read mathematics with genuine understanding. When you are first presented with any formula, you should try to establish what type of a function it describes. This may require you to rewrite the function in order to convert it into standard form.

### *Linear functions*

$$x,\quad \frac{1}{2}t,\quad -.04P,\quad R+3,\quad 7s+4.6,\quad 4(z-1)+2,\quad \pi-2\Theta,\quad \text{etc.}$$

The form of these linear functions is easily recognizable,

$f(x)=ax+b$ where $a,\ b$ are constants.

### *Quadratic functions*

$$x^2,\quad .0001L^2,\quad -\frac{1}{2}r^2,\quad x^2-4,\quad x(x+2),\quad 3x^2+x-1,\quad \text{etc.}$$

A function is quadratic if it can be put in the form

$f(x)=ax^2+bx+c$ where $a,\ b,\ c$ are constants and $a\neq 0$

### *Cubic functions*

$$x^3,\quad 1+5t^3,\quad x^3+2x^2-x+\frac{1}{7},\quad 3q(q^2+2),\quad (P+2)^3,\quad \text{etc.}$$

A function is cubic if it can be put in the form

$f(x)=ax^3+bx^2+cx+d$ where $a,\ b,\ c,\ d$ are constants and $a\neq 0$

### *Polynomial functions*

$$4x^5+3x^4+2x^2+1,\quad 1+M+M^4,\quad \frac{1}{2}l(l-1)^3,\quad -x^7+16.2x,\quad \text{etc.}$$

A polynomial function can be written in the form

$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots\ldots+a_1x+a_0$ ( $n$ is a non-negative integer )

Where $n$ is the degree of the polynomial and the coefficients $a_n$ , $a_{n-1}$ , …… , $a_0$ are constants.

Of course the functions which we have previously labeled cubic, quadratic, and linear are also polynomials.

### *Rational functions*

$$\frac{1}{x+1},\quad \frac{x^2+x}{3x^3+2x+14},\quad \frac{1-t}{t^2-3t+4},\quad \frac{p^3}{2p+7},$$

All of the four preceding types (linear, quadratic, cubic, polynomial) are special cases of the broader category called *rational functions* which is made up of all quotients of polynomials. The rational functions can be written in the form.

$f(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

### *Power functions*

$$x^2,\quad x^3,\quad t^{-2},\quad z^{\frac{1}{2}},\quad 2R^9,\quad \sqrt{s},\quad .03x^{\frac{2}{3}},\quad T^4,\quad x^{\pi},\quad \text{etc.}$$

These are functions formed by raising a variable to a constant power. A power function can be written in the form

$f(x) = a(x)^r$ where $a$, $r$ are real constants

### *Exponential functions*

$$10^x,\quad e^x,\quad 2^{-x},\quad 100e^{2t},\quad .001(1.07)^t,\quad \text{etc.}$$

These functions are formed in a different way from power functions. An exponential function is a constant raised to a variable power. An exponential function can be written in the form

$f(x) = a(b)^x$ where $a$, $b$ are constants and $b > 0$.

### *Logarithmic functions*

$$\log x,\quad \ln x,\quad 56\log 3x,\quad \frac{1}{2}\ln x,\quad \text{etc.}$$

The logarithmic functions are easy to recognize because the name of the function is always included in the expression. The definitions are

$f(x) = \log x$ is the inverse of the exponential $10^x$.

$g(x) = \ln x$ is the inverse of the exponential $e^x$.

Remember that an expression like $\ln 3$ is a constant.

### *Trigonometric functions*

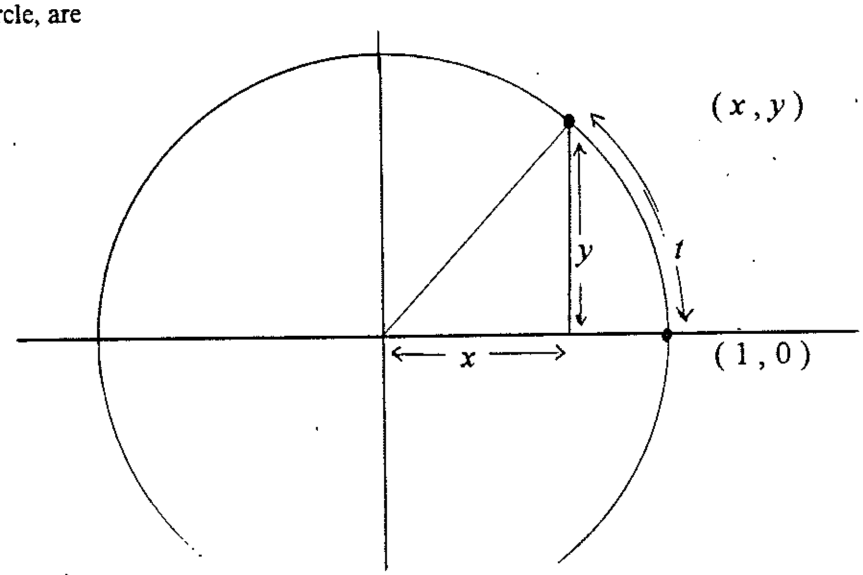
$$\sin x,\quad \cos x,\quad \tan x,\quad \sin \pi t,\quad 4\cos\left(2\Theta - \frac{\pi}{2}\right),\quad \text{etc.}$$

Trigonometric functions are designated by name in mathematical writing. The definitions which refer to the unit circle, are

$\cos t = x$

$\sin t = y$

$\tan t = \frac{x}{y}$




**Example 1:** ***Which of these functions are polynomials?***

$$f(x) = 3x^3 + 2x^2 + x^{\frac{1}{2}} + 2$$

$$g(x) = 5x^2 - x^{-1}$$

Polynomials do not have terms with negative exponents or fractional exponents so neither $f(x)$ nor $g(x)$ is a polynomial.

**Example 2:** ***Which of these functions are rational functions?***

$$f(t) = \frac{t^2 + 2}{\sqrt{1+t}}$$

$$g(y) = 3y(y+1)^{-2}$$

The function $f(t)$ is not rational because the denominator, $\sqrt{1+t}$ , is not a polynomial. On the other hand, $g(y)$ can be rewritten as

$$g(y) = \frac{3y}{(y+1)^2} = \frac{3y}{y^2 + 2y + 1}$$

which is rational.

**Example 3:** ***What type of function is $h(x)$ ?***

$$h(x) = (e^x)^5$$

We must decide whether $h(x)$ has the exponential form $(\text{constant})^{(\text{variable})}$ or the power form $(\text{variable})^{(\text{constant})}$. Using the rules of exponents we can write

$$h(x) = (e^x)^5 = e^{(5x)} = (e^5)^x.$$

Thus, $h(x)$ is the constant, $(e^5)$ , raised to a variable power. So $h(x)$ can be classified as an exponential function.

**Example 4:** ***What type of function is $l(q)$ ?***

$$l(q) = 2q(\ln e^q + 1)$$

The logarithm term simplifies to just $q$ , so

$$\begin{aligned} l(q) &= 2q(\ln e^q + 1) \\ &= 2q(q+1) \end{aligned}$$

which is a quadratic function.

## Problems for section 14

***Determine whether the following functions are linear, quadratic, cubic, or none of these.***

1. $f(x) = x^3(x^{-3} + x^{-2})$
2. $r(x) = (x+1)(x+2)(x+3)$
3. $G(x) = mx^2 + b$
4. $s(t) = (t-1)(t^2 + 4x + 4)$
5. $S(r) = 4r + 2^3$
6. $K(x) = \ln 5 - x$
7. $V(t) = (t-a)(t-b)$
8. $M(p) = \frac{1}{2}p(2-p)$
9. $h(x) = \frac{x^2}{4} + \frac{x}{5}$
10. $Q(x) = \left(x - x^{\frac{1}{2}}\right)^3$

***Determine whether the following functions are power functions, exponential functions or neither.***

11. $S(x) = (\sin x)^2$
12. $Y(x) = (2x)^5$
13. $g(z) = \left(\sqrt{3}\right)^z$
14. $F(x) = x^x$
15. $h(t) = e^{\sin t}$
16. $L(x) = 10^{\log x^2}$
17. $P(t) = B\left(1 - \frac{r}{n}\right)^{nt}$
18. $r(y) = \frac{1}{2^y}$
19. $f(x) = e^{x^2}$
20. $T(x) = \frac{1}{x^2}$

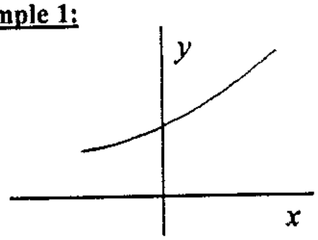
# 15. Common Graphs to Memorize

### *Graph basics*

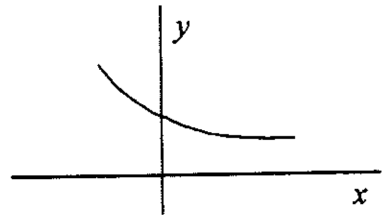
There are certain descriptions which are often used when we talk about graphs.

| | |
|---|---|
| *"The graph is positive",* | means that the <u>$y$-values</u> of the graph are positive. |
| *"The graph is negative",* | means that the <u>$y$-values</u> of the graph are negative. |
| *"The graph is increasing",* | means that as we move from <u>left to right</u> the $y$-values climb. |
| *"The graph is decreasing",* | means that as we move from <u>left to right</u> the $y$-values drop. |

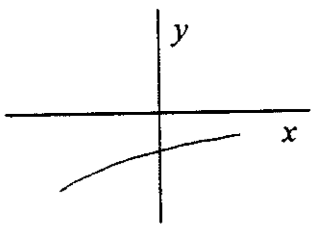
**Example 1:**



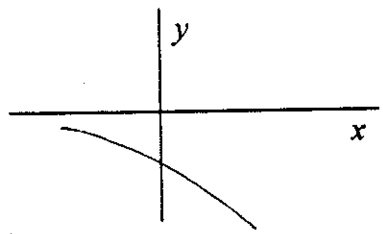

$y$'s positive, graph lies above $x$ axis.
$y$'s increase from left to right




$y$'s positive, graph lies above the $x$ axis.
$y$'s decrease from left to right.



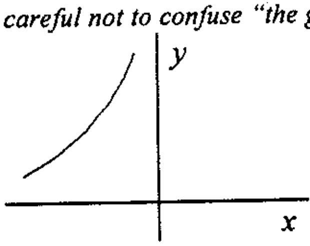

$y$'s negative, graph lies below $x$ axis.
$y$'s increase from left to right




$y$'s negative, graph lies below the $x$ axis.
$y$'s decrease from left to right.

**Example 2:**

*Be careful not to confuse "the graph is negative" with "the $x$-values of the graph are negative".*

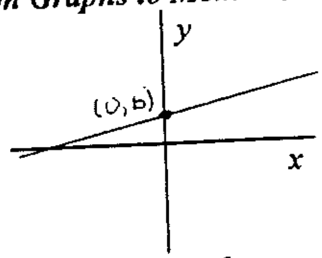



*Here the $x$-values themselves are negative, but the graph is positive.*

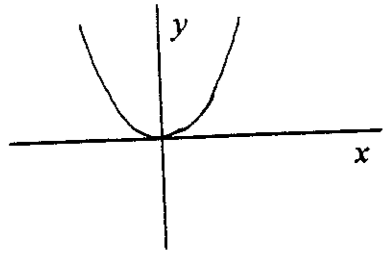
## Essential information to memorize for each graph

a. domain
b. range
c. general shape
d. intercepts and key values
e. asymptotes

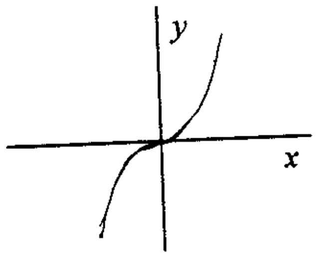
***Common Graphs to Memorize***



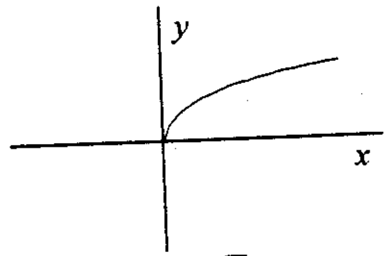

$y = mx + b$



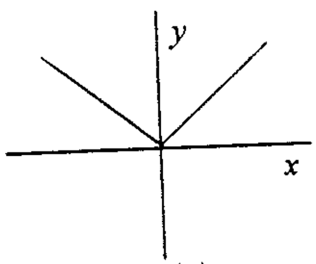

$y = x^2$



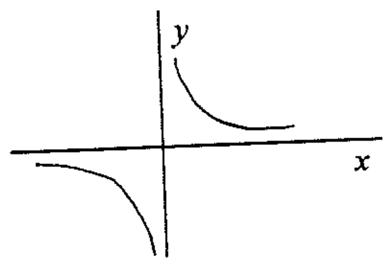

$y = x^3$



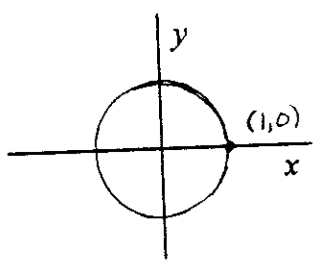

$y = \sqrt{x}$



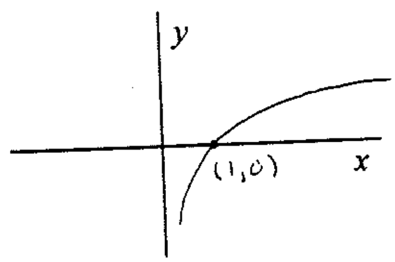

$y = |x|$



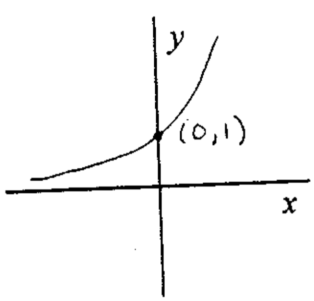

$y = \frac{1}{x}$



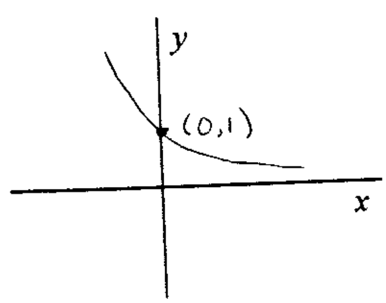

$x^2 + y^2 = 1$



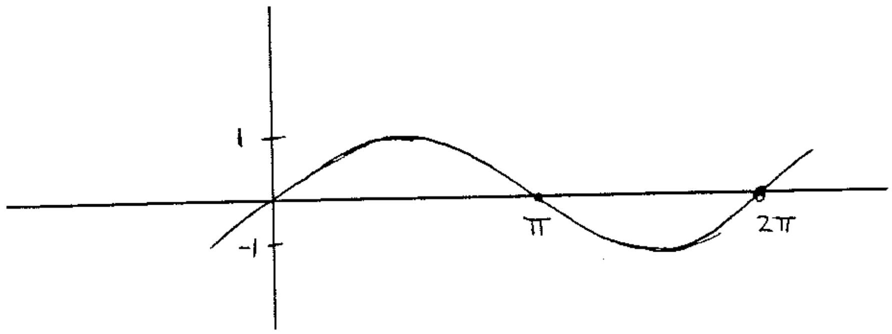

$y = \ln x$



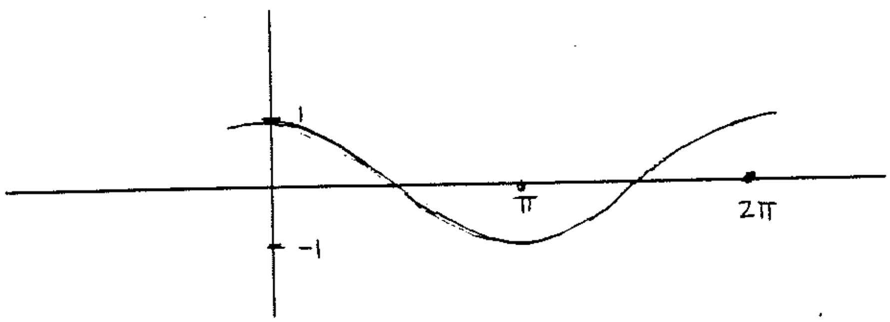

$y = e^x$



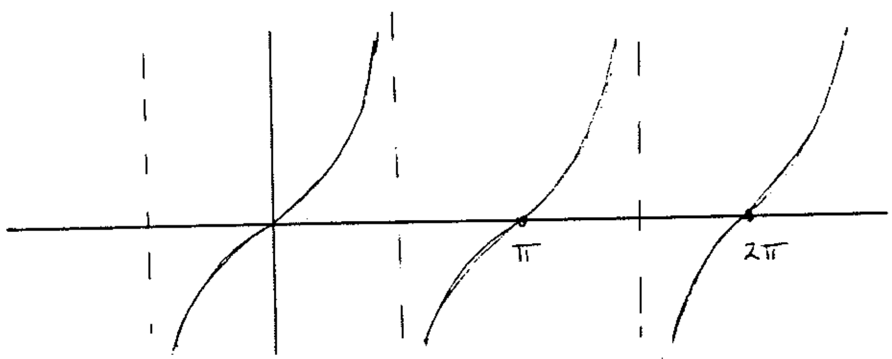

$y = e^{-x}$




$y = \sin x$




$y = \cos x$




$y = \tan x$

## Problems for section 15

1. Which of the common graphs are positive for all $x$ ?
2. Which of the common graphs are positive or zero for all $x$ ?
3. Which of the common graphs is not the graph of a function?
4. Which of the common graphs have domain $x \geq 0$ ?
5. Which of the common graphs are always increasing?
6. Which of the common graphs are always decreasing?
7. Which of the common graphs have horizontal asymptotes? What are the equations of their asymptotes?
8. Which of the common graphs have vertical asymptotes? What are the equations of their asymptotes?

# 16. Formulas and Key Function Values to Memorize

***Integer powers***

| squares | cubes | $4^{th}$ powers | $5^{th}$ powers |
|---|---|---|---|
| $2^2 = 4$ | $2^3 = 8$ | $2^4 = 16$ | $2^5 = 32$ |
| $3^2 = 9$ | $3^3 = 27$ | $3^4 = 81$ | |
| $4^2 = 16$ | $4^3 = 64$ | | |
| $5^2 = 25$ | $5^3 = 125$ | | |
| $6^2 = 36$ | | | |
| $7^2 = 49$ | | | |
| $8^2 = 64$ | | | |
| $9^2 = 81$ | | | |
| $10^2 = 100$ | $10^3 = 1000$ | $10^4 = 10{,}000$ | $10^5 = 100{,}000$ |
| $11^2 = 121$ | | | |
| $12^2 = 144$ | | | |

***Important constants***

$\pi \approx 3.142\ldots$

$e \approx 2.718\ldots$

$\sqrt{2} \approx 1.414\ldots$

***Key logarithm values***

$\log 1 = 0$ $\qquad \log 10 = 1$

$\ln 1 = 0$ $\qquad \ln e = 1$

***Key trigonometric values***

| $x$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|
| $\sin x$ | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ | $-1$ | $0$ |
| $\cos x$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ | $0$ | $1$ |
| $\tan x$ | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | undefined | $0$ | undefined | $0$ |

### *Trigonometric identities*

$$\tan x = \frac{\sin x}{\cos x} \qquad \sin^2 x + \cos^2 x = 1$$

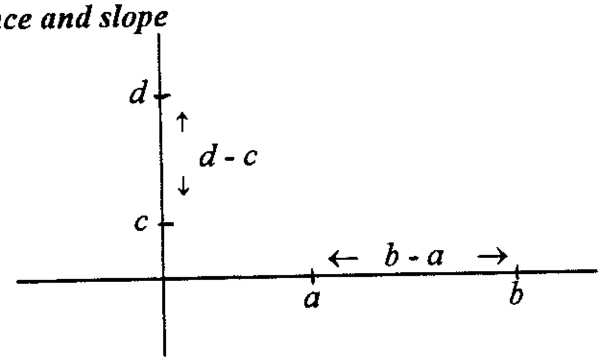
$$\sin(-x) = -\sin x \qquad \sin 2x = 2\sin x \cos x$$

$$\cos(-x) = \cos x$$

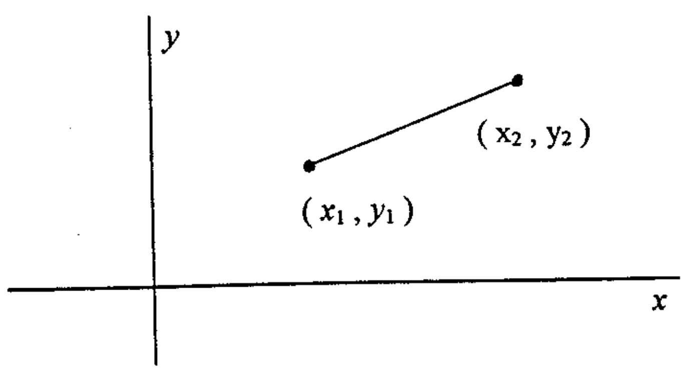
### *Geometric formulas*

| | | | | |
|---|---|---|---|---|
| area of a circle | $A = \pi r^2$ | $r$ = radius | | |
| circumference of a circle | $C = 2\pi r$ | $r$ = radius | | |
| area of a rectangle | $A = lw$ | $l$ = length | $w$ = width | |
| perimeter of a rectangle | $P = 2l + 2w$ | | | |
| area of a triangle | $A = \frac{1}{2}bh$ | $b$ = base | $h$ = height | |
| volume of a rectangular solid | $V = lwh$ | $l$ = length | $w$ = width | $h$ = height |
| volume of a right circular cylinder | $V = \pi r^2 h$ | $r$ = radius | $h$ = height | |
| volume of a right circular cone | $V = \frac{1}{3}\pi r^2 h$ | $r$ = radius | $h$ = height | |
| volume of a sphere | $V = \frac{4}{3}\pi r^3$ | $r$ = radius | | |

### *Distance and slope*




vertical and horizontal distance




$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

## Answers to Problems for section 1

1. 49
2. 16
3. -1
4. -121
5. -18
6. 1
7. 2100
8. -5
9. 4
10. -6
11. .2
12. 4
13. 3
14. .5
15. $.16x^2y^4$
16. $3^{x+1}$
17. $8L$
18. $70\left(\sqrt[6]{W}\right)^5 = 70x^{\frac{5}{6}}$
19. $16s^3t^2$
20. $e^x$
21. $\dfrac{A^3}{3B^3}$
22. $e^{kt+4}$
23. $(M+2)^2$
24. $9x^5$
25. $x^{3e}$
26. $\dfrac{e^{2y}}{y^4}$
27. $4x^{(3\pi-1)}$
28. $25(2b+1)^{20}$
29. $3a$
30. $4u^4v^2w^4$
31. $\left(\sqrt[5]{-32}\right)^3 = (-2)^3 = -8$
32. $-\left(\sqrt[5]{32}\right)^3 = -(2)^3 = -8$
33. $-\left(\sqrt[4]{625}\right)^3 = -(25)^3 = -125$
34. $\left(\sqrt[4]{-625}\right)^3$ not a real number
35. $\left(\sqrt[3]{-1728}\right)^4 = (12)^4 = 20{,}736$
36. $\left(\sqrt{64}\right)^{-3} = 8^{-3} \cong .002$
37. $-\left(\sqrt{64}\right)^3 = -8^3 = -512$
38. $\left(\sqrt{-64}\right)^3$ not a real number

## Answers to Problems for section 2

1. $4x^2 + 11x - 20$
2. $Pp^2 - 6Ppq + 9Pq^2$
3. $A^4 - 2A^2B^2 + B^4$
4. $4x^2 - 24x + 43$
5. $-2x - 2\sqrt{2x} - 1$
6. $-250t$
7. $2^u + u2^{2u}$ or $2^u + u4^u$
8. $KRr^2 - Kr^3$
9. $30 + \dfrac{72}{k} + 2x$ or $30 + 72x^{-1} + 2x$
10. $\dfrac{e^{2x} + 2 + e^{-2x}}{4}$ or $\dfrac{e^{2x}}{4} + \dfrac{1}{2} + \dfrac{1}{4e^{2x}}$

## Answers to Problems for section 3

1. $2(x-3)(x-2)$
2. $\pi r(r+2h)$
3. $(6-B)(4-B)$
4. $\sin x(x-1)$
5. does not factor
6. $(a+2)(a-2)(a^2+3)$
7. $((t+3)-4)((t+3)+4)=(t-1)(t+7)$
8. $(x-4)(hx-3)$
9. $(r-2)(r+2)$
10. $(\cos x-1)^2$
11. $(y-2x)(y-x)$
12. $xe^{-3x}(x+2)$
13. $(e^x+1)^2$
14. $(1+r)^2(P+\Pr)$

## Answers to Problems for section 4

1. $\frac{x+20}{x^2-16}$
2. $\frac{x^2+1}{x-1}$
3. $\frac{1}{2r}$
4. $\frac{(u+a)^2+u}{u+a}$
5. $\frac{x-1}{x\sqrt{x}}$
6. $\frac{1+e^x}{e^{2x}}$
7. $\frac{4x+1}{b-a}$
8. $\frac{.28+3M^3}{4M}$
9. $\frac{r_2r_3+r_1r_3+r_1r_2}{r_1r_2r_3}$
10. $x(x+2)$
11. $\frac{1}{(x+y)^2}$
12. $\frac{1}{2}$
13. $\frac{1}{a^2b^2}$
14. cannot be simplified
15. $\frac{p^3}{p^2+q^2}$
16. $-2x-h$
17. $\frac{2x(1-2x^3)}{(x^3+1)^3}$
18. $\frac{2x(1-2x^3)}{x^4\sqrt{2x-1}}$
19. $\frac{-2x-h}{x^2(x+h)^2}$
20. $\frac{1-2\ln x}{3x^3}$
21. $\frac{13}{x^2}+\frac{1}{2x^3}$
22. $\frac{1}{3}+\frac{1}{\sqrt{x}}$
23. $\frac{2}{l^2}+\frac{1}{l^3}-\frac{4}{3l^4}$
24. $\frac{7}{p^2+11}+\frac{p}{p^2+11}$
25. $\frac{1}{6}-\frac{1}{4x}$
26. $\frac{1}{t^{\frac{5}{2}}}+\frac{1}{t^{\frac{3}{2}}}$

27. $1+\frac{-7}{x+5}$

28. $1+\frac{3}{q-4}$

29. $1+\frac{1}{R}$

30. $1+\frac{2}{2u+1}$

31. $1+\frac{\sin x}{\cos x}$

32. $1+\frac{1}{e^x}$

**Answers to Problems for section 5**

1. $3x+\frac{1}{2}x^{-1}+x^2+\frac{1}{5}$

2. $180q^3-60q$

3. $y^2-18y^{-1}+9y^{-4}$

4. $x^3+2x+x^{-1}$

5. $2P^3+3P^{\frac{1}{2}}$

6. $\frac{1}{2}+3t^{\frac{1}{2}}+\frac{9}{2}t$

7. $-3-\frac{1}{6}x^2+\frac{1}{2}x$

8. $N^{-2}-2+N^2$

9. $\frac{-12}{7}+\frac{3}{7}x$

10. $\frac{1}{2}x^{\frac{7}{2}}+x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}$

11. $12(3x+1)^{-\frac{1}{2}}$

12. $1000(10-s)^{\frac{1}{3}}$

13. $-0.7(x-1)^4$

14. $\frac{1}{2}(x^2+1)^{-3}$

17. $\left(\frac{1}{2}\right)^x$

18. $\left(\frac{1}{2}\right)^x$

19. $10{,}000(0.76)^t$

20. $e(e^2)^x$

21. $2\left(\frac{1}{3}\right)^x$

22. $\frac{1}{3}(6)^x$

23. $4^t$

24. $\left(\frac{1}{e}\right)e^x$

25. $\left(-\frac{5}{3}\right)^x$

26. $(5e)e^x$

27. $(x-1)^2-4$

28. $(x-3)^2+1$

29. $-(x-3)^2+7$

30. $3(x-2)^2+1$

31. $\frac{-6x}{x^2+7}$

32. $\frac{(-2\ln 2)2^x}{(1+3^x)^3}$

33. $\frac{\pi\cos\pi t}{\sin\pi t}$

34. $-\frac{1}{\sin^2 z}$

35. $\frac{-xe^{2x}-2}{x^3}$

36. $\frac{1-\ln x}{x^2}$

37. $\dfrac{x}{\sqrt{x^2+16}}$

38. $\dfrac{x+5}{\sqrt{x^2+10x+1}}$

39. $\dfrac{\cos 2x}{\sqrt{\sin 2x}}$

40. $\dfrac{2}{\left(\sqrt[3]{x^3-e^{3x}}\right)^2}$

**Answers to Problems for section 6**

1. $\log 2x + \log x$
2. $4\ln u + 3\ln(7-u)$
3. cannot be rewritten
4. $-2\ln x$
5. $\frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$
6. $\log(x^2-1) - 3\log x$
7. $\log x^4\sqrt{y}$
8. $\ln\dfrac{x^2}{x+10}$
9. $\log 3a$
10. $\log(x+1)^3(x+4)^3$
11. $\ln\dfrac{ABC}{2}$
12. $\log\dfrac{2}{5}$
13. $\dfrac{1}{5}$
14. $t^{-3}$
15. $2\sqrt{x}$
16. no simplification possible
17. $t^2$
18. $100x$
19. 1
20. 0
21. $\frac{1}{2}\ln(x^2+16)$
22. $\frac{2}{3}\ln(3-z)$
23. 0
24. $2x$
25. $-\ln(e^x+1)$
26. $\dfrac{1}{2}$

**Answers to Problems for section 7**

1. $s \cong -.115$

2. $x = -\frac{17}{16}$

3. $t = \frac{10}{7}$

4. $B = -2$

5. $x = 2, \quad x = -\frac{4}{3}$

6. $p = 3, \quad p = -3, \quad p = -\frac{1}{2}$

7. $N = 3, \quad n = 1$

8. $t = 0, \quad t = 8, \quad t = -8$

9. $x = 1 + \sqrt{2}, \quad x = 1 - \sqrt{2}$

10. $x = 4, \quad x = -\frac{3}{4}$

11. $t \cong 1.13, \quad t \cong 4.87$

12. $y = -3 + \sqrt{6}, \quad y = -3 - \sqrt{6}$

13. $z = -\frac{7}{2}$

14. $x = 1, \quad x = -1$

15. $L = \frac{1}{2}, \quad L = -\frac{1}{2}$

16. $q = \sqrt{2}, \quad q = -\sqrt{2}$

17. $r = 5, \quad r = -5$

18. $x = -\frac{1}{8}$

19. $x = 0, \quad x = 36$

20. $v = 700\pi$

21. $t \cong -22.76$

22. $t \cong 1.04$

23. $x = 5$

24. $x \cong -.52$

25. $x = -55, \quad y = 39$

26. $x = 3, \quad y = -3$
    $x = -1, \quad y = -3$

27. $x = -3, \quad y = -5$
    $x = 1, \quad y = 3$

28. $x = \frac{1}{2}, \quad y = 2$
    $x = -\frac{1}{2}, \quad y = -2$

29. $A = -\frac{5}{2}$

30. $l = g\left(\frac{T}{2\pi}\right)^2$

31. $x = 0, \quad 2, \quad -2$

32. $k = \frac{\ln(5)}{1000} = -.000693$

33. $x = \frac{\ln(.5)}{\ln(.8)} \cong 3.106$

33. $x = \dfrac{\ln(.5)}{\ln(.8)} \cong 3.106$

34. $y' = \dfrac{4}{y + 2x}$

35. $w = \dfrac{2}{k}(l - l_0)$

36. $r = \dfrac{4}{\sqrt[3]{\pi}}$

37. $y' = \dfrac{2x}{x - y}$

38. $x = 12, \quad x = -12$

39. $y = \dfrac{c + d}{b - a}$

40. $v = \dfrac{3w - 2u - z}{u + w - z}$

## Answers to Problems for section 8

1. $x \geq 7$
2. $x > 16$
3. $-5 < x < 5$
4. $x > 5$
5. $x \geq -4$
6. $x < -4, \quad x > 4$
7. all $x$
8. $x > 5$
9. all $x$
10. no solution
11. $x < .001$
12. $-1 < y < +1$
13. $p \neq 55$
14. $k > 0$
15. $r \geq 0$
16. $t \geq 1995$

17. (a) none
(b) $x = 0, \quad x = -2$
(c) $x > 0, \quad x < -2$
(d) $-2 < x < 0$

18. (a) none
(b) $x = 0, \quad x = -2$
(c) $x > 0, \quad x < -2$
(d) $-2 < x < 0$

19. (a) none
(b) no $x$
(c) all $x$
(d) no $x$

20. (a) none
(b) $t = 2, \quad t = 3$
(c) $t < 2, \quad t > 3$
(d) $2 < t < 3$

21. (a) $x = 0$
(b) no $x$
(c) all $x$
(d) no $x$

22. (a) $p = 0$
(b) no $p$
(c) $p < 0$
(d) $p > 0$

23. (a) none
(b) $x = 0$
(c) $x > 0$
(d) $x < 0$

24. (a) $x \leq 0$
(b) $x = 1$
(c) $x > 1$
(d) $0 < x < 1$

25. (a) none
(b) $u = \frac{1}{\sqrt{3}}, \quad u = -\frac{1}{\sqrt{3}}$
(c) $-\frac{1}{\sqrt{3}} < u < \frac{1}{\sqrt{3}}$
(d) $u < -\frac{1}{\sqrt{3}}, \quad u > \frac{1}{\sqrt{3}}$

26. (a) $x = 0, \quad x = 1$
(b) $x = \frac{1}{2}$
(c) $x < \frac{1}{2}$
(d) $x > \frac{1}{2}$

27. $-2 < x < 2$

28. $\frac{1}{2} \leq x \leq 1$

29. $-\frac{21}{2} < n \leq \frac{1}{2}$

30. $l > \frac{1}{48}$

31. $t \leq -1, \quad t \geq 4$

32. $x < -\frac{7}{3}, \quad x > 1$

33. $r < 2, \quad r > 3$

34. $x < -1, \quad x > 0$

35. $x < -1, \quad x > 0$

36. $-2 < x < 2$

## Answers to Problems for section 9

| | Functions | Variables | Constants |
|---|---|---|---|
| 1. | $f$ | $x$ | $a$ |
| 2. | $H$ | $t$ | $A, B$ |
| 3. | . | $x$ | $a, b$ |
| 4. | $f, g$ | $x$ | $c$ |
| 5. | $P$ | $x$ | $n, a_0, a_1, \ldots, a_n$ |
| 6. | $B$ | $t$ | $\pi, V_0$ |
| 7. | $B$ | $t$ | $V, P_0$ |
| 8. | $M$ | $t$ | $M_0, h, e$ |
| 9. | | $B, t$ | $P, e$ |
| 10. | | $s, t$ | $g, v_0, s_0$ |

11. $\pi$

12. $2\pi$

13. $k$

14. -2

15. $\frac{1}{2}$

16. $A = l(50 - l)$

17. $V = \frac{3}{25}\pi h^3$

18. $A = 9\pi t^4$

19. $z = \sqrt{9t^2 + 16t}$.

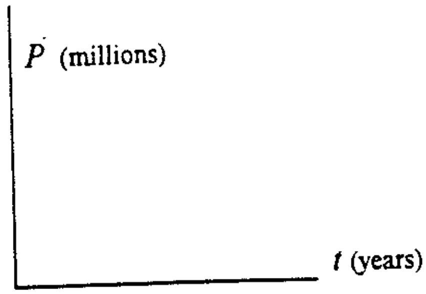
20. $h = \frac{V}{16}$

21. $V = \pi r^2(r + 5)$

22. $A = \frac{C^2}{4\pi}$

23. $V = A\sqrt{A}$

24.

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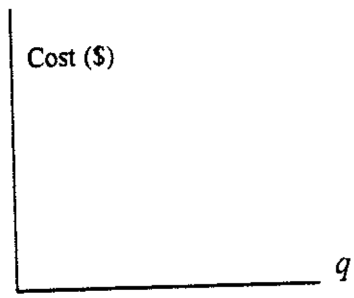


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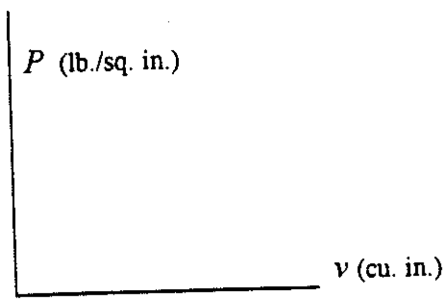


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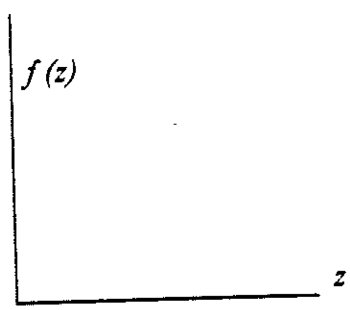


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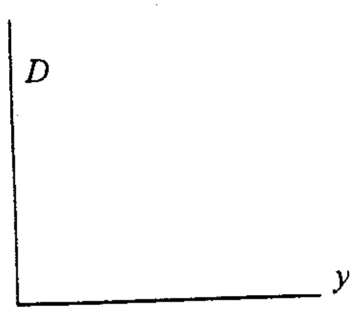


## Answers to Problems for section 10

1.
a. $\frac{3}{2}$
b. 0
c. $x = -4$
d. $x = \frac{1}{5}$

2.
a. 2
b. $z = 1$
c. 0
d. $z = 1$

3.
a. 20.4
b. 20
c. $r = \frac{\ln 20}{\ln 1.02} \cong$
d. $r = \frac{\ln \frac{1}{2}}{\ln 1.02} \cong$
e. 20

4.
a. -1
b. $x = 3, \quad x = -3$
c. 0
d. -1
e. $x = 3, \quad x = -3$

5.
a. 0
b. $t = 0, \quad t = 2$ sec.
c. the velocity at 3 sec. , ft./sec.

6.
a. 230 km.
b. the position of the object at $t = 0$ .
c. always positive

7.
a. $\left(1\frac{1}{2}\right)^2 - 7\left(1\frac{1}{2}\right) + 12 = 3.75$ in./yr.
b. When is the object stopped?
c. For what $t$ values is $v(t) > 0$ ?

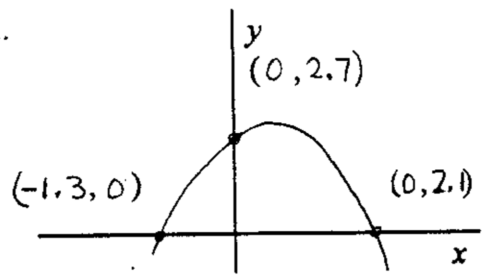
8. Let $s(t) = 11t^2 + t + 100$ be the position (miles) of a car driving on a straight road at time $t$ (hours), then the car's velocity (miles/hour) at any time $t$ will be given by $v(t) = 22t + 1$.
a. Use function notation to express the car's position after 2 hours. Where is the car then?
b. Use function notation to express the question; *"When is the car going 65 mph?"*
c. Where is the car when it is going 67 mph?

## Answers to Problems for section 11

1. $0$
2. $1$
3. $1$
4. $2$
5. $-2$
6. $-1$
7. $1000$
8. $18$
9. $\frac{1}{4}$
10. $3$
11. $\ln 2$
12. $24$
13. $6$
14. $1$
15. $\frac{2}{\pi}$
16. $\frac{\pi}{4}$
17. $\frac{1}{2}$
18. $\frac{1}{12}$
19. $\frac{3}{4}$
20. $1$
21. $2$
22. $-2$
23. $-\frac{1}{4}$
24. $h = \frac{6}{\pi}$
25. $2$
26. $2$
27. $4$
28. $4\frac{1}{3}$
29. $1$
30. $4(3\ln 2 + 1) \cong 12.32$

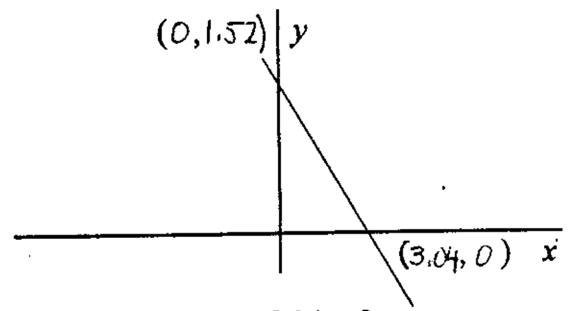
## Problems for section 12

1.



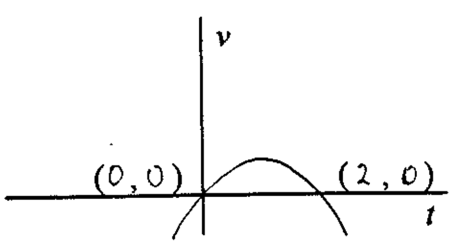

$y = -(x + 1.3)(x - 2.1)$

2.



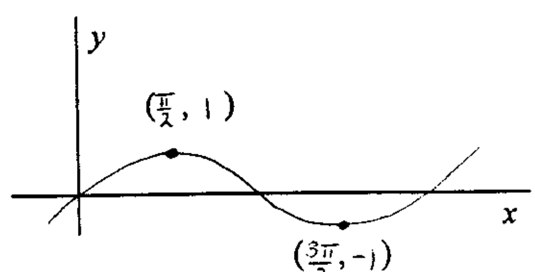

$2x + y - 3.04 = 0$

3.



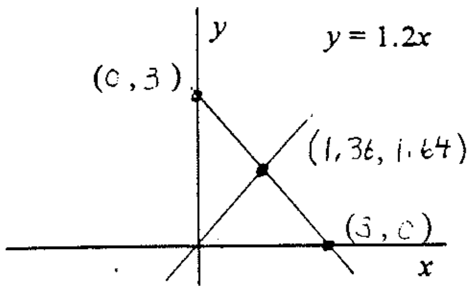

$v = t(t - 2)$

4.



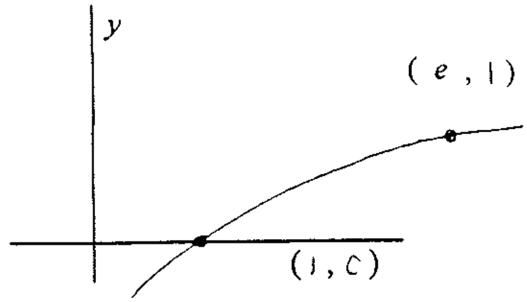

$y = 2\sin x$

5.



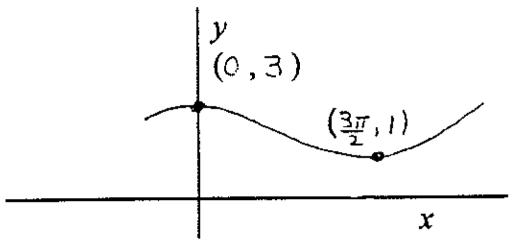

$y = 3 - x$

6.



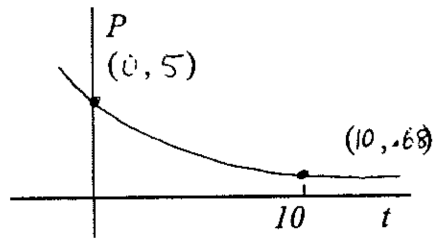

$y = \ln x$

7.



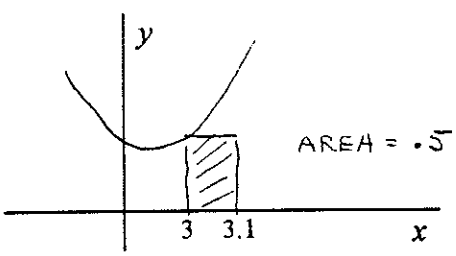

$y = \cos x + 2$

8.



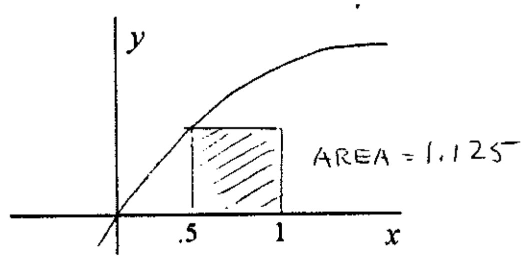

$P = 5e^{-0.2t}$

9.



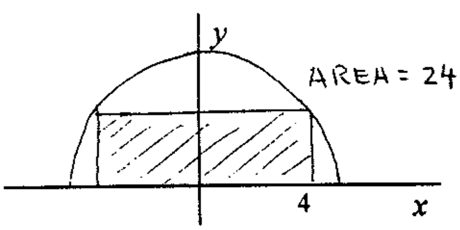

$y = (x-1)^2 + 1$

10.



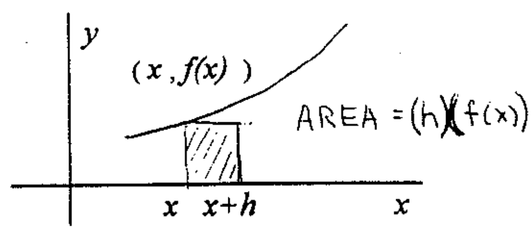

$y = x(5-x)$

11.



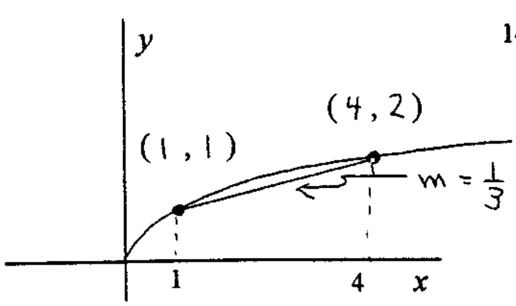

$x^2 + y^2 = 25$

12.



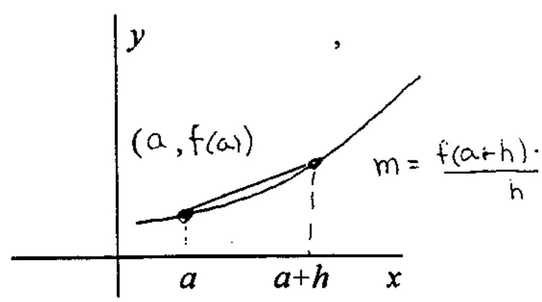

$y = f(x)$

13.




$y = \sqrt{x}$

14.




$y = f(x)$

## Answers to problems in section 13

1\.
a. $\frac{1}{4x^2}$
b. $\frac{2}{x^2}$
c. $\frac{1}{x}$
d. $\frac{1}{|x|}$
e. $\frac{1}{x^4}$
f. $\frac{1}{x^4}$

2\.
a. $\tan a$
b. $\tan \Theta$
c. $\tan(x^2)$
d. $(\tan x)^2$
e. $x \tan x$
f. $\frac{\tan(2x)}{2x}$

3\.
a. $e^{t^2}$
b. $e^{2t}$
c. $t$
d. $e^t - e^{-t}$
e. $e^{\cos t}$
f. $\frac{1}{e^t}$ or $e^{-t}$

4\. $\sin 4\Theta = \sin 2(2\Theta)$

$\sin 4\Theta = 2 \sin 2\Theta \cos 2\Theta$

5\.
a. $f(g(x)) = 1 + \ln\left(e^{(x-1)}\right) = 1 + (x-1) = x$

$g(f(x)) = e^{((1+\ln x)-1)} = e^{\ln x} = x$

b. $f(g(x)) = \left(\sqrt{x-2}\right)^2 + 2 = x - 2 + 2 = x$

$g(f(x)) = \sqrt{(x^2+2)-2} = \sqrt{x^2} = x$ for $x \geq 2$

6\. $5 + h$

7\. $\frac{-1}{t(t+h)}$

## Answers to problems in section 14

1. linear
2. cubic
3. quadratic
4. polynomial
5. linear
6. linear
7. quadratic
8. quadratic
9. quadratic
10. none of these
11. neither
12. power
13. exponential
14. neither
15. neither
16. quadratic
17. exponential
18. exponential
19. neither
20. neither