A. APPENDICES

You should remember the following piece of advice. There is so little that is true in mathematics, that anything you make up is likely to be wrong.

There are seven appendices.

1. **Background Material**. The material in this section is critical for success in any ordinary differential equations course. You will have seen all of this material in previous courses, but, if you are a typical student, you will have forgotten much of it. In fact, you might even think that you haven t seen some of it before.

2. **Partial Fractions**. This material should also be familiar to you. You will use it extensively when integrating and when applying Laplace transforms.

3. **Infinite Series, Power Series, and Taylor Series**. This material should also be familiar to you, but most students don t really master series until they use it in differential equations.

4. **Complex Numbers**. Much of this material will be new to most students.

5. **Elementary Matrix Operations**. Most students will have seen special cases of these results.

6. **Least Squares Approximation**. This will be new to most students. It shows how to find the best straight-line approximation, $y = mx + b$ to a data set consisting of n points.

7. **Proofs of the Oscillation Theorems**. This contains the proofs of the theorems stated in Chapter 8.

A.1 Background Material

Solving Quadratic Equations:
The solutions of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the discriminant that is, $b^2 - 4ac$ is positive, the equation has distinct real roots. If $b^2 - 4ac = 0$, the roots are real, but repeated. If $b^2 - 4ac < 0$, the roots are complex, and are complex conjugates.

Perpendicular Lines:
The two straight lines, $y = m_1x + b_1$ and $y = m_2x + b_2$ are perpendicular if $m_1m_2 = -1$.

Properties of Trigonometric Functions:
All angles are measured in radians where $180^o = \pi$ radians. Thus, $90^o = \pi\ 2$ radians, $60^o = \pi\ 3$ radians, $45^o = \pi\ 4$ radians, and $30^o = \pi\ 6$ radians. An easy way to remember the values of the trig functions at frequently used angles is to use the following table.

$$\begin{array}{rclllll} & & 0^o & 30^o & 45^o & 60^o & 90^o \\ x & = & 0 & \pi\ 6 & \pi\ 4 & \pi\ 3 & \pi\ 2 \\ \sin x & = & \overline{0}\ 2 & \overline{1}\ 2 & \overline{2}\ 2 & \overline{3}\ 2 & \overline{4}\ 2 \\ \cos x & = & \overline{4}\ 2 & \overline{3}\ 2 & \overline{2}\ 2 & \overline{1}\ 2 & \overline{0}\ 2 \end{array}$$

which simplifies to

$$\begin{array}{rclllll} & & 0^o & 30^o & 45^o & 60^o & 90^o \\ x & = & 0 & \pi\ 6 & \pi\ 4 & \pi\ 3 & \pi\ 2 \\ \sin x & = & 0 & 1\ 2 & 1\ \ \overline{2} & \overline{3}\ 2 & 1 \\ \cos x & = & 1 & \overline{3}\ 2 & 1\ \ \overline{2} & 1\ 2 & 0 \end{array}$$

The graphs of $\sin x$ and $\cos x$ are shown in Figure A.1.

empty

Graphs of the functions $\sin x$ and $\cos x$

Figure A.1

$$\begin{array}{rcl} \sin(-x) & = & -\sin x \\ \cos(-x) & = & \cos x \\ \sin^2 x + \cos^2 x & = & 1 \\ \sin(x+y) & = & \sin x \cos y + \cos x \sin y \\ \cos(x+y) & = & \cos x \cos y - \sin x \sin y \end{array}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\begin{aligned}
\tfrac{d}{dx}\sin x &= \cos x \\
\tfrac{d}{dx}\cos x &= -\sin x \\
\tfrac{d}{dx}\tan x &= \sec^2 x
\end{aligned}$$

$$\begin{aligned}
\textstyle\int \sin x\, dx &= -\cos x + C \\
\textstyle\int \cos\ dx &= \sin x + C \\
\textstyle\int \tan x\, dx &= -\ln\ \ \cos x\ + C
\end{aligned}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \text{ for } - \quad < x <$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \text{ for } - \quad < x <$$

The Inverse Trigonometric Functions:
If $x = \sin\theta$ and $-\pi\ \ 2 \leq \theta \leq \pi\ \ 2$ then $\theta = \arcsin x$. The function $\arcsin x$ is sometimes written $\sin^{-1} x$. If $x = \tan\theta$ and $-\pi\ \ 2 < \theta < \pi\ \ 2$ then $\theta = \arctan x$. The function $\arctan x$ is sometimes written $\tan^{-1} x$.

$$\begin{aligned}
\tfrac{d}{dx}\arcsin x &= 1\ \ \ \overline{1 - x^2} \\
\tfrac{d}{dx}\arctan x &= 1\ \ \left(1 + x^2\right)
\end{aligned}$$

Properties of Exponential Functions:

$$\begin{aligned}
e^0 &= 1 \\
e^{x+y} &= e^x e^y \\
e^{x-y} &= e^x e^{-y} \\
e^{-x} &= 1\ \ e^x \\
e^{ax} &= (e^a)^x \\
a^x &= e^{x \ln a}, \text{ for } a > 0
\end{aligned}$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \text{ for } - \quad < x <$$

The graphs of e^x and e^{-x} are shown in Figure A.2.

Properties of Logarithmic Functions:
$\ln x$ is defined only for $x > 0$

$$\begin{aligned}
\ln 1 &= 0 \\
\ln e &= 1 \\
\ln(xy) &= \ln x + \ln y \\
\ln(x\ \ y) &= \ln x - \ln y \\
\ln x^n &= n \ln x \\
\ln x^{-1} &= -\ln x \\
e^{\ln x} &= x, \text{ if } x > 0 \\
\ln e^x &= x
\end{aligned}$$

empty

Graphs of the functions e^x and e^{-x}

Figure A.2

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln\ x\ + C$$

$$\int \ln x\,dx = x\ln x - x + C$$

There are **NO** general formulas that simplify either $\ln(x+y)$ or $\ln(x-y)$.

WRONG: $\ln(x+y) = \ln x + \ln y$

WRONG: $\ln(x-y) = \ln x - \ln y$

The graphs of e^x and $\ln x$ are shown in Figure A.3.

empty

Graphs of the functions e^x and $\ln x$

Figure A.3

The Hyperbolic Functions:

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\begin{aligned} \sinh 0 &= 0 \\ \cosh 0 &= 1 \end{aligned}$$

$$\begin{aligned} \tfrac{d}{dx}\sinh x &= \cosh x \\ \tfrac{d}{dx}\cosh x &= \sinh x \end{aligned}$$

Properties of Derivatives:

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}f(x)$$

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

$$\frac{d}{dx}\left[f(x)g(x)\right] = \frac{d}{dx}\left[f(x)\right]g(x) + f(x)\frac{d}{dx}\left[g(x)\right]$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^{2}(x)}$$

$$\text{If } y = f(u(x)) \quad \text{then } \frac{dy}{dx} = \frac{df}{du}\frac{du}{dx}$$

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x), \text{ if } f(t) \text{ is continuous at } t = x$$

If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$

If $f'(x) > 0$ in the interval $a < x < b$ then $f(x)$ is increasing in that interval.

If $f'(x) < 0$ in the interval $a < x < b$ then $f(x)$ is decreasing in that interval.

If $f''(x) > 0$ in the interval $a < x < b$ then $f(x)$ is concave up in that interval.

If $f''(x) < 0$ in the interval $a < x < b$ then $f(x)$ is concave down in that interval.

The function $f(x)$ has a local, or relative, maximum at x_0, if $f(x_0) \geq f(x)$ for all x near x_0

The function $f(x)$ has a local, or relative, minimum at x_0, if $f(x_0) \leq f(x)$ for all x near x_0

Properties of Integrals:

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int \left[f(x) \pm g(x)\right]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Table A.1 Table of derivatives

$f(x)$	$f\ (x)$
c	0
x^n	nx^{n-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x = 1 \quad \cos^2 x$
$\cot x$	$-\csc^2 x = -1 \quad \sin^2 x$
$\sec x = 1 \quad \cos x$	$\sec x \tan x = \sin x \quad \cos^2 x$
$\csc x = 1 \quad \sin x$	$-\csc x \cot x = -\cos x \quad \sin^2 x$
$\ln\ x$	$1\ \ x$
$\sinh x = (e^x - e^{-x})\ \ 2$	$\cosh x$
$\cosh x = (e^x + e^{-x})\ \ 2$	$\sinh x$
$\arcsin x$	$1 \quad \overline{1 - x^2}$
$\arctan x$	$1\ \ (1 + x^2)$

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b f(x)g\ (x)\,dx = \ f(x)g(x)\ {}_a^b - \int_a^b f\ (x)g(x)\,dx$$

$$\int f(g(x))g\ (x)\,dx = \int f(u)\,du \text{ where } u = g(x)$$

There are **NO** general formulas that simplify either $\int [f(x)g(x)]\ dx$ or $\int [f(x)\ \ g(x)]\ dx$.

$$\textbf{WRONG}\text{:}\ \int f(x)g(x)\,dx = \int f(x)\,dx \int g(x)\,dx$$

$$\textbf{WRONG}\text{:}\ \int \frac{f(x)}{g(x)}\,dx = \frac{\int f(x)\,dx}{\int g(x)\,dx}$$

A.2 Partial Fractions

We sometimes need to express a rational polynomial that is, a function of the type

$$R(x) = \frac{P(x)}{Q(x)} \tag{A.1}$$

where $P(x)$ and $Q(x)$ are polynomials in an alternative form. The standard technique, known as partial fractions, goes as follows. (We should point out that this general explanation is much more involved than doing a particular example.)

Table A.2 Table of integrals

$f(x)$	$\int f(x)\,dx$
x^n	$x^{n+1}\ \ (n+1) + C,\ n = -1$
$1\ \ x$	$\ln\ \ x\ \ + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\tan x$	$-\ln\ \ \cos x\ \ + C = \ln\ \ \sec x\ \ + C$
$\cot x$	$\ln\ \ \sin x\ \ + C$
$\sec x = 1\ \ \ \cos x$	$\ln\ \ \sec x + \tan x\ \ + C$
$\csc x = 1\ \ \ \sin x$	$\ln\ \ \csc x - \cot x\ \ + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\ln x$	$x \ln x - x + C$
$\sinh x = (e^x - e^{-x})\ \ 2$	$\cosh x + C$
$\cosh x = (e^x + e^{-x})\ \ 2$	$\sinh x + C$
$1\ \ [(x-a)(x-b)]$	$(\ln\ \ x - a\ \ - \ln\ \ x - b\ \ )\ \ (a - b) + C,\ a = b$
$1\ \ (1 + x^2)$	$\arctan x + C$
$1\ \ \ \ \overline{1 + x^2}$	$\ln(x +\ \ \ \overline{x^2 + 1}) + C$
$1\ \ \ \ \overline{1 - x^2}$	$\arcsin x + C$
$1\ \ \ \ \overline{x^2 - 1}$	$\ln(x +\ \ \ \overline{x^2 - 1}) + C$
$e^{ax} \sin bx$	$e^{ax}(a \sin bx - b \cos bx)\ \ (a^2 + b^2) + C$
$e^{ax} \cos bx$	$e^{ax}(b \sin bx + a \cos bx)\ \ (a^2 + b^2) + C$

1. If the degree of the polynomial $Q(x)$ is less than or equal to the degree of $P(x)$ then divide $Q(x)$ into $P(x)$, obtaining a polynomial plus a term similar to $R(x)$ in (A.1), but where the degree of the new $Q(x)$ is greater than the degree of the new $P(x)$. From now on we concentrate on this new $R(x)$.

2. Factor $Q(x)$ into linear factors and quadratic factors (that cannot be written as the product of linear factors with real coefficients), so that

   $$Q(x) = (x - r_1)^{n_1} \cdots (x - r_p)^{n_p} \left(a_1 x^2 + b_1 x + c_1\right)^{m_1} \cdots \left(a_q x^2 + b_q x + c_q\right)^{mq}$$

   where n_1 through n_p and m_1 through m_q are positive integers, and $a_1 x^2 + b_1 x + c_1$, and so on, have no real roots. For example, if $Q(x) = x^3 - x$ then $Q(x) = x(x-1)(x+1)$, whereas, if $Q(x) = x^3 + x$, then $Q(x) = x(x^2+1)$.

3. For each linear factor of $Q(x)$ of degree n say, $(x - r)^n$ write down a contribution to $R(x)$ that is an expansion with n terms; namely,

   $$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_n}{(x - r)^n}$$

   where A_1 through A_n are constants to be determined.

4. For each quadratic factor of $Q(x)$ of degree m say, $\left(ax^2 + bx + c\right)^m$ write down a contribution to $R(x)$ that is an expansion with m terms;

namely,

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{\left(ax^2 + bx + c\right)^2} + \cdots + \frac{B_mx + C_m}{\left(ax^2 + bx + c\right)^m}$$

where B_1 through B_m and C_1 through C_m are constants to be determined.

5. Add the contributions to $R(x)$ from all the terms in $Q(x)$ and set them equal to $R(x)$. Now cross-multiply this identity in x by $Q(x)$ to evaluate the constants.

Example 1 :

Write $1 \ \left(x^2 - 1\right)$ as a partial fraction.

Here $P(x) = 1$, and $Q(x) = x^2 - 1$. The degree of $Q(x)$ two exceeds that of $P(x)$ one so we do not divide $P(x)$ by $Q(x)$ but use $R(x) = 1 \ \left(x^2 - 1\right)$ Now $Q(x) = (x-1)(x+1)$, so we have two linear roots, each of degree one. The contribution from $(x-1)$ is $A \ (x-1)$, and the contribution from $(x+1)$ is $B \ (x+1)$. Thus the total contribution to $R(x)$ is

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

where A and B are constants to be determined by making this last equation an identity. Cross-multiplying by $(x-1)(x+1)$ gives $1 = A(x+1) + B(x-1)$ or

$$1 = (A+B)x + (A-B)$$

For this to be true for all x we must have

$$\begin{aligned} A + B &= 0 \\ A - B &= 1 \end{aligned}$$

which can be solved to give $A = 1 \ 2$ and $B = -1 \ 2$. Thus, the partial fraction form of $1 \ \left(x^2 - 1\right)$ is

$$\frac{1}{x^2 - 1} = \frac{1 \ 2}{x-1} - \frac{1 \ 2}{x+1}$$

□

Example 2 :

Write $x \ \ (x-1)^2$ as a partial fraction.

Here $Q(x) = (x-1)^2$ has a linear factor of degree two, so we try

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

This gives $x = A(x-1) + B$ or

$$x = Ax - A + B$$

Thus $A = 1$ and $-A + B = 0$, so $B = 1$, giving

$$\frac{x}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}$$

□

Example 3 :

Write $(x+1)\ [x(x^2+1)]$ as a partial fraction.

The contribution from x will be $A\ x$, and from $\left(x^2+1\right)$ will be $(Bx+C)\ (x^2+1)$. Thus, we try

$$\frac{x+1}{x\,(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

Cross-multiplying by $x(x^2+1)$ gives $x+1=A(x^2+1)+(Bx+C)\,x$ or

$$x+1=(A+B)\,x^2+Cx+A$$

Thus $A=1$, $B=-A=-1$, and $C=1$, so we have

$$\frac{x+1}{x\,(x^2+1)}=\frac{1}{x}+\frac{-x+1}{x^2+1}$$

□

A.3 Infinite Series, Power Series, and Taylor Series

An infinite series, $\sum_{k=0} a_k$ either **converges** or **diverges**.

Convergent series are divided into two groups **absolutely convergent** and **conditionally convergent**.

- If $\sum_{k=0} a_k$ and $\sum_{k=0}\ a_k$ are both convergent, then $\sum_{k=0} a_k$ is absolutely convergent.
- If $\sum_{k=0} a_k$ is convergent, but $\sum_{k=0}\ a_k$ is divergent, then $\sum_{k=0} a_k$ is conditionally convergent.
- The third possibility that $\sum_{k=0} a_k$ is divergent, but $\sum_{k=0}\ a_k$ is convergent cannot occur because if $\sum_{k=0}\ a_k$ is convergent, so is $\sum_{k=0} a_k$.

The reason that it is important to distinguish between absolutely and conditionally convergent series is Riemann s rearrangement theorem.

Theorem 4 *: The terms of an absolutely convergent series may be rearranged in any order without affecting the convergence of the series. In particular, its sum is unchanged. Rearranging the terms of a conditionally convergent series may change its sum.*

There are a variety of tests to decide whether a series converges or diverges, but, for ordinary differential equations, the most important is the RATIO TEST.