Chapter Ten

APPROXIMATING FUNCTIONS

In Section ??, for $|x| < 1$, we found the sum of a geometric series as a function of the common ratio x:

$$1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

Viewed from another perspective, this formula gives us a series whose sum is the function $f(x) = 1/(1-x)$. We now work in the opposite direction, starting with a function and looking for a series of simpler functions whose sum is that function. We first use polynomials, which lead us to Taylor series, and then trigonometric functions, which lead us to Fourier series.

Taylor approximations use polynomials, which may be considered the simplest functions. They are easy to use because they can be evaluated by simple arithmetic, unlike transcendental functions, such as e^x and $\ln x$.

Fourier approximations use sines and cosines, the simplest periodic functions, instead of polynomials. Taylor approximations are generally good approximations to the function locally (that is, near a specific point), whereas Fourier approximations are generally good approximations over an interval.

10.1 TAYLOR POLYNOMIALS

In this section, we see how to approximate a function by polynomials.

Linear Approximations

We already know how to approximate a function using a degree 1 polynomial, namely the tangent line approximation given in Section 4.8 :

$$f(x) \approx f(a) + f'(a)(x-a).$$

The tangent line and the curve share the same slope at $x = a$. As Figure 10.1 suggests, the tangent line approximation to the function is generally more accurate for values of x close to a.

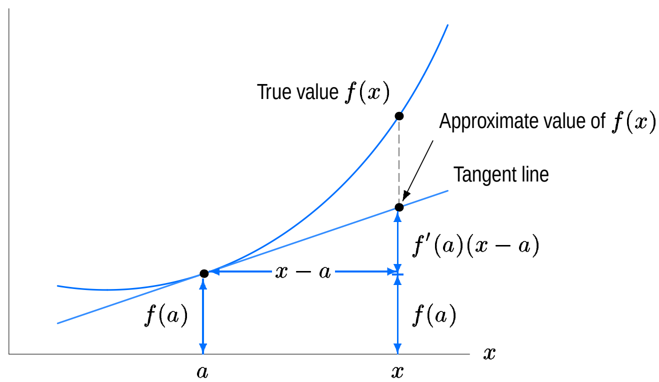


Figure 10.1: Tangent line approximation of $f(x)$ for x near a

We first focus on $a = 0$. The tangent line approximation at $x = 0$ is referred to as the *first Taylor approximation* at $x = 0$, or as follows:

Taylor Polynomial of Degree 1 Approximating $f(x)$ for x near 0

$$f(x) \approx P_1(x) = f(0) + f'(0)x$$

We now look at approximations by polynomials of higher degree. Let us compare approximations $g(x) = \cos x$ near $x = 0$ by linear and quadratic functions.

Example 1 Approximate $g(x) = \cos x$, with x in radians, by its tangent line at $x = 0$.

Solution The tangent line at $x = 0$ is just the horizontal line $y = 1$, as shown in Figure 10.2, so

$$g(x) = \cos x \approx 1, \quad \text{for } x \text{ near } 0.$$

If we take $x = 0.05$, then

$$g(0.05) = \cos(0.05) = 0.998\ldots,$$

which is quite close to the approximation $\cos x \approx 1$. Similarly, if $x = -0.1$, then

$$g(-0.1) = \cos(-0.1) = 0.995\ldots$$

is close to the approximation $\cos x \approx 1$. However, if $x = 0.4$, then

$$g(0.4) = \cos(0.4) = 0.921\ldots,$$

so the approximation $\cos x \approx 1$ is less accurate. The graph suggests that the farther a point x is away from 0, the worse the approximation is likely to be.

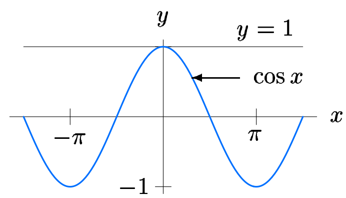


Figure 10.2: Graph of $\cos x$ and its tangent line at $x = 0$

Quadratic Approximations

To get a more accurate approximation, we use a quadratic function instead of a linear function.

Example 2 Find the quadratic approximation to $g(x) = \cos x$ for x near 0.

Solution To ensure that the quadratic, $P_2(x)$, is a good approximation to $g(x) = \cos x$ at $x = 0$, we require that $\cos x$ and the quadratic have the same value, the same slope, and the same second derivative at $x = 0$. That is, we require $P_2(0) = g(0)$, $P_2'(0) = g'(0)$, and $P_2''(0) = g''(0)$. We take the quadratic polynomial

$$P_2(x) = C_0 + C_1 x + C_2 x^2,$$

and determine C_0, C_1, and C_2. Since

$$\begin{aligned} P_2(x) &= C_0 + C_1 x + C_2 x^2 \qquad \text{and} \qquad & g(x) &= \cos x \\ P_2'(x) &= C_1 + 2C_2 x & g'(x) &= -\sin x \\ P_2''(x) &= 2C_2 & g''(x) &= -\cos x, \end{aligned}$$

we have

$$\begin{aligned} C_0 &= P_2(0) = g(0) = \cos 0 = 1 \qquad \text{so} \qquad & C_0 &= 1 \\ C_1 &= P_2'(0) = g'(0) = -\sin 0 = 0 & C_1 &= 0 \\ 2C_2 &= P_2''(0) = g''(0) = -\cos 0 = -1, & C_2 &= -\tfrac{1}{2}. \end{aligned}$$

Consequently, the quadratic approximation is

$$\cos x \approx P_2(x) = 1 + 0 \cdot x - \frac{1}{2}x^2 = 1 - \frac{x^2}{2}, \qquad \text{for } x \text{ near } 0.$$

Figure 10.3 suggests that the quadratic approximation $\cos x \approx P_2(x)$ is better than the linear approximation $\cos x \approx P_1(x)$ for x near 0. Let's compare the accuracy of the two approximations. Recall that $P_1(x) = 1$ for all x. At $x = 0.4$, we have $\cos(0.4) = 0.921\ldots$ and $P_2(0.4) = 0.920$, so the quadratic approximation is a significant improvement over the linear approximation. The magnitude of the error is about 0.001 instead of 0.08.

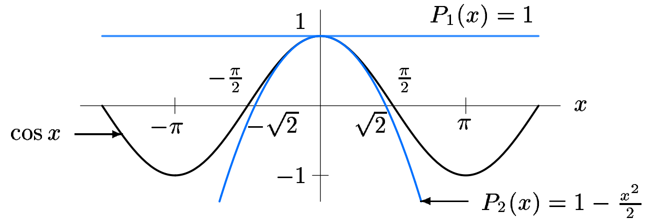


Figure 10.3: Graph of $\cos x$ and its linear, $P_1(x)$, and quadratic, $P_2(x)$, approximations for x near 0

Generalizing the computations in Example 2, we define the *second Taylor approximation* at $x = 0$.

Taylor Polynomial of Degree 2 Approximating $f(x)$ for x near 0

$$f(x) \approx P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

Higher-Degree Polynomials

In a small interval around $x = 0$, the quadratic approximation to a function is usually a better approximation than the linear (tangent line) approximation. However, Figure 10.3 shows that the quadratic can still bend away from the original function for large x. We can attempt to fix this by using an approximating polynomial of higher degree. Suppose that we approximate a function $f(x)$ for x near 0 by a polynomial of degree n:

$$f(x) \approx P_n(x) = C_0 + C_1x + C_2x^2 + \cdots + C_{n-1}x^{n-1} + C_nx^n.$$

We need to find the values of the constants: $C_0,\ C_1,\ C_2, \ldots,\ C_n$. To do this, we require that the function $f(x)$ and each of its first n derivatives agree with those of the polynomial $P_n(x)$ at the point $x = 0$. In general, the more derivatives there are that agree at $x = 0$, the larger the interval on which the function and the polynomial remain close to each other.

To see how to find the constants, let's take $n = 3$ as an example

$$f(x) \approx P_3(x) = C_0 + C_1x + C_2x^2 + C_3x^3.$$

Substituting $x = 0$ gives

$$f(0) = P_3(0) = C_0.$$

Differentiating $P_3(x)$ yields

$$P_3'(x) = C_1 + 2C_2x + 3C_3x^2,$$

so substituting $x = 0$ shows that

$$f'(0) = P_3'(0) = C_1.$$

Differentiating and substituting again, we get

$$P_3''(x) = 2 \cdot 1C_2 + 3 \cdot 2 \cdot 1C_3x,$$

which gives

$$f''(0) = P_3''(0) = 2 \cdot 1C_2,$$

so that

$$C_2 = \frac{f''(0)}{2 \cdot 1}.$$

The third derivative, denoted by P_3''', is

$$P_3'''(x) = 3 \cdot 2 \cdot 1 C_3,$$

so

$$f'''(0) = P_3'''(0) = 3 \cdot 2 \cdot 1 C_3,$$

and then

$$C_3 = \frac{f'''(0)}{3 \cdot 2 \cdot 1}.$$

You can imagine a similar calculation starting with $P_4(x)$, using the fourth derivative $f^{(4)}$, which would give

$$C_4 = \frac{f^{(4)}(0)}{4 \cdot 3 \cdot 2 \cdot 1},$$

and so on. Using factorial notation,[1] we write these expressions as

$$C_3 = \frac{f'''(0)}{3!}, \quad C_4 = \frac{f^{(4)}(0)}{4!}.$$

Writing $f^{(n)}$ for the n^{th} derivative of f, we have, for any positive integer n

$$C_n = \frac{f^{(n)}(0)}{n!}.$$

So we define the n^{th} *Taylor approximation* at $x = 0$:

Taylor Polynomial of Degree n Approximating $f(x)$ for x near 0

$$\begin{aligned} f(x) &\approx P_n(x) \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \end{aligned}$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = 0$, or the Taylor polynomial about $x = 0$.

Example 3 Construct the Taylor polynomial of degree 7 approximating the function $f(x) = \sin x$ for x near 0. Compare the value of the Taylor approximation with the true value of f at $x = \pi/3$.

Solution We have

$$\begin{array}{rrcrr} f(x) = & \sin x & \text{giving} & f(0) = & 0 \\ f'(x) = & \cos x & & f'(0) = & 1 \\ f''(x) = & -\sin x & & f''(0) = & 0 \\ f'''(x) = & -\cos x & & f'''(0) = & -1 \\ f^{(4)}(x) = & \sin x & & f^{(4)}(0) = & 0 \\ f^{(5)}(x) = & \cos x & & f^{(5)}(0) = & 1 \\ f^{(6)}(x) = & -\sin x & & f^{(6)}(0) = & 0 \\ f^{(7)}(x) = & -\cos x, & & f^{(7)}(0) = & -1. \end{array}$$

[1] Recall that $k! = k(k-1)\cdots 2 \cdot 1$. In addition, $1! = 1$, and $0! = 1$.

Using these values, we see that the Taylor polynomial approximation of degree 7 is

$$\sin x \approx P_7(x) = 0 + 1\cdot x + 0\cdot\frac{x^2}{2!} - 1\cdot\frac{x^3}{3!} + 0\cdot\frac{x^4}{4!} + 1\cdot\frac{x^5}{5!} + 0\cdot\frac{x^6}{6!} - 1\cdot\frac{x^7}{7!}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \quad \text{for } x \text{ near } 0.$$

Notice that since $f^{(8)}(0) = 0$, the seventh and eighth Taylor approximations to $\sin x$ are the same.

In Figure 10.4 we show the graphs of the sine function and the approximating polynomial of degree 7 for x near 0. They are indistinguishable where x is close to 0. However, as we look at values of x farther away from 0 in either direction, the two graphs move apart. To check the accuracy of this approximation numerically, we see how well it approximates $\sin(\pi/3) = \sqrt{3}/2 = 0.8660254\ldots$.

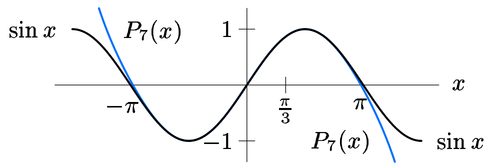


Figure 10.4: Graph of $\sin x$ and its seventh degree Taylor polynomial, $P_7(x)$, for x near 0

When we substitute $\pi/3 = 1.0471976\ldots$ into the polynomial approximation, we obtain $P_7(\pi/3) = 0.8660213\ldots$, which is extremely accurate—to about four parts in a million.

Example 4 Graph the Taylor polynomial of degree 8 approximating $g(x) = \cos x$ for x near 0.

Solution We find the coefficients of the Taylor polynomial by the method of the preceding example, giving

$$\cos x \approx P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.$$

Figure 10.5 shows that $P_8(x)$ is close to the cosine function for a larger interval of x-values than the quadratic approximation $P_2(x) = 1 - x^2/2$ in Example 2 on page 447.

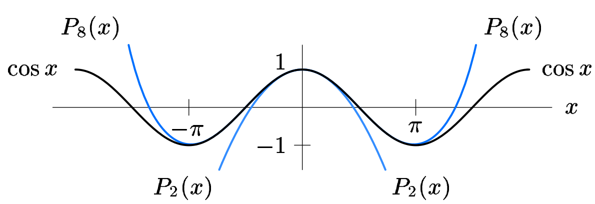


Figure 10.5: $P_8(x)$ approximates $\cos x$ better than $P_2(x)$ for x near 0

Example 5 Construct the Taylor polynomial of degree 10 about $x = 0$ for the function $f(x) = e^x$.

Solution We have $f(0) = 1$. Since the derivative of e^x is equal to e^x, all the higher-order derivatives are equal to e^x. Consequently, for any $k = 1, 2, \ldots, 10$, $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = e^0 = 1$. Therefore, the Taylor polynomial approximation of degree 10 is given by

$$e^x \approx P_{10}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{10}}{10!}, \quad \text{for } x \text{ near } 0.$$

To check the accuracy of this approximation, we use it to approximate $e = e^1 = 2.718281828\ldots$. Substituting $x = 1$ gives $P_{10}(1) = 2.718281801$. Thus, P_{10} yields the first seven decimal places for e. For large values of x, however, the accuracy diminishes because e^x grows faster than any polynomial as $x \to \infty$. Figure 10.6 shows graphs of $f(x) = e^x$ and the Taylor polynomials of degree $n = 0, 1, 2, 3, 4$. Notice that each successive approximation remains close to the exponential curve for a larger interval of x-values.

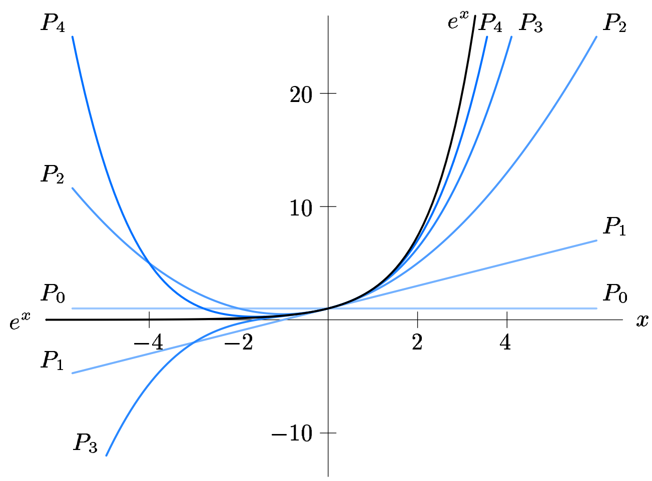


Figure 10.6: For x near 0, the value of e^x is more closely approximated by higher-degree Taylor polynomials

Example 6 Construct the Taylor polynomial of degree n approximating $f(x) = \dfrac{1}{1-x}$ for x near 0.

Solution Differentiating gives $f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 3!$, $f^{(4)}(0) = 4!$, and so on. This means

$$\frac{1}{1-x} \approx P_n(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n, \quad \text{for } x \text{ near } 0,$$

Let us compare the Taylor polynomial with the formula obtained on page **??** for the sum of a finite geometric series:

$$\frac{1 - x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n.$$

If x is close to 0 and x^{n+1} is small enough to neglect, the formula for the sum of a finite geometric series gives us the Taylor approximation of degree n:

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4 + \cdots + x^n.$$

Taylor Polynomials Around $x = a$

Suppose we want to approximate $f(x) = \ln x$ by a Taylor polynomial. This function has no Taylor polynomial about $x = 0$ because the function is not defined for $x \leq 0$. However, it turns out that we can construct a polynomial centered about some other point, $x = a$.

First, let's look at the equation of the tangent line at $x = a$:

$$y = f(a) + f'(a)(x - a).$$

This gives the first Taylor approximation

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{for } x \text{ near } a.$$

The $f'(a)(x - a)$ term is a correction term which approximates the change in f as x moves away from a.

Similarly, the Taylor polynomial $P_n(x)$ centered at $x = a$ is set up as $f(a)$ plus correction terms which are zero for $x = a$. This is achieved by writing the polynomial in powers of $(x - a)$ instead of powers of x:

$$f(x) \approx P_n(x) = C_0 + C_1(x - a) + C_2(x - a)^2 + \cdots + C_n(x - a)^n.$$

If we require n derivatives of the approximating polynomial $P_n(x)$ and the original function $f(x)$ to agree at $x = a$, we get the following result for the n^{th} Taylor approximations at $x = a$:

Taylor Polynomial of Degree n Approximating $f(x)$ for x near a

$$\begin{aligned} f(x) &\approx P_n(x) \\ &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n \end{aligned}$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = a$, or the Taylor polynomial about $x = a$.

You can derive the formula for these coefficients in the same way that we did for $a = 0$. (See Problem 28, page 454.)

Example 7 Construct the Taylor polynomial of degree 4 approximating the function $f(x) = \ln x$ for x near 1.

Solution We have

$$\begin{aligned} f(x) &= \ln x & \text{so} \qquad f(1) &= \ln(1) = 0 \\ f'(x) &= 1/x & f'(1) &= 1 \\ f''(x) &= -1/x^2 & f''(1) &= -1 \\ f'''(x) &= 2/x^3 & f'''(1) &= 2 \\ f^{(4)}(x) &= -6/x^4, & f^{(4)}(1) &= -6. \end{aligned}$$

The Taylor polynomial is therefore

$$\begin{aligned} \ln x \approx P_4(x) &= 0 + (x - 1) - \frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} - 6\frac{(x-1)^4}{4!} \\ &= (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}, \qquad \text{for } x \text{ near } 1. \end{aligned}$$

Graphs of $\ln x$ and several of its Taylor polynomials are shown in Figure 10.7. Notice that $P_4(x)$ stays reasonably close to $\ln x$ for x near 1, but bends away as x gets farther from 1. Also, note that the Taylor polynomials are defined for $x \leq 0$, but $\ln x$ is not.

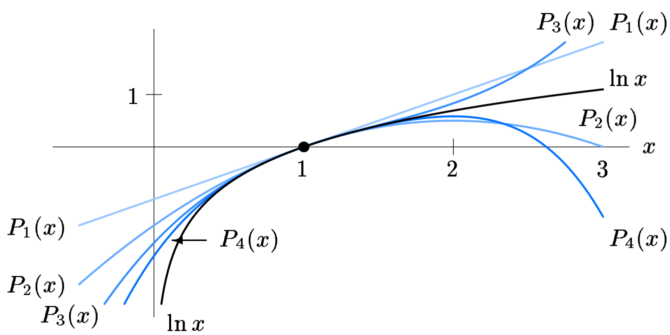


Figure 10.7: Taylor polynomials approximate $\ln x$ closely for x near 1, but not necessarily farther away

The examples in this section suggest that the following results are true for common functions:

- Taylor polynomials centered at $x = a$ give good approximations to $f(x)$ for x near a. Farther away, they may or may not be good.
- The higher the degree of the Taylor polynomial, the larger the interval over which it fits the function closely.

Exercises and Problems for Section 10.1

Exercises

For Problems 1–10, find the Taylor polynomials of degree n approximating the given functions for x near 0. (Assume p is a constant.)

1. $\dfrac{1}{1+x}$, $n = 4, 6, 8$

2. $\dfrac{1}{1-x}$, $n = 3, 5, 7$

3. $\sqrt{1+x}$, $n = 2, 3, 4$

4. $\cos x$, $n = 2, 4, 6$

5. $\arctan x$, $n = 3, 4$

6. $\tan x$, $n = 3, 4$

7. $\sqrt[3]{1-x}$, $n = 2, 3, 4$

8. $\ln(1+x)$, $n = 5, 7, 9$

9. $\dfrac{1}{\sqrt{1+x}}$, $n = 2, 3, 4$

10. $(1+x)^p$, $n = 2, 3, 4$

For Problems 11–14, find the Taylor polynomial of degree n for x near the given point a.

11. $\sin x$, $a = \pi/2$, $n = 4$

12. $\cos x$, $a = \pi/4$, $n = 3$

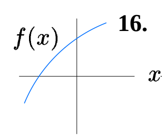
13. e^x, $a = 1$, $n = 4$

14. $\sqrt{1+x}$, $a = 1$, $n = 3$

Problems

For Problems 15–18, suppose $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function f about $x = 0$. What can you say about the signs of a, b, c if f has the graph given below?

15.



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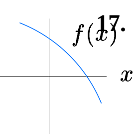

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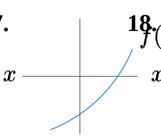

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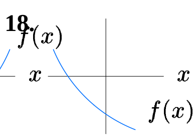

19. Suppose the function $f(x)$ is approximated near $x = 0$

by a sixth degree Taylor polynomial

$$P_6(x) = 3x - 4x^3 + 5x^6.$$

Give the value of

(a) $f(0)$ (b) $f'(0)$ (c) $f'''(0)$ (d) $f^{(5)}(0)$ (e) $f^{(6)}(0)$

20. Suppose g is a function which has continuous derivatives, and that $g(5) = 3, g'(5) = -2, g''(5) = 1, g'''(5) = -3$.

(a) What is the Taylor polynomial of degree 2 for g near 5? What is the Taylor polynomial of degree 3 for g near 5?

(b) Use the two polynomials that you found in part (a) to approximate $g(4.9)$.

21. Find the second-degree Taylor polynomial for $f(x) = 4x^2 - 7x + 2$ about $x = 0$. What do you notice?

22. Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ about $x = 0$. What do you notice?

23. **(a)** Based on your observations in Problems 21–22, make a conjecture about Taylor approximations in the case when f is itself a polynomial.

(b) Show that your conjecture is true.

24. Show how you can use the Taylor approximation $\sin x \approx x - \frac{x^3}{3!}$, for x near 0, to explain why $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

25. Use the fourth-degree Taylor approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for x near 0 to explain why $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

26. Use a fourth degree Taylor approximation for e^h, for h near 0, to evaluate the following limits. Would your answer be different if you used a Taylor polynomial of higher degree?

(a) $\lim_{h\to 0} \frac{e^h - 1 - h}{h^2}$

(b) $\lim_{h\to 0} \frac{e^h - 1 - h - \frac{h^2}{2}}{h^3}$

27. If $f(2) = g(2) = h(2) = 0$, and $f'(2) = h'(2) = 0$, $g'(2) = 22$, and $f''(2) = 3$, $g''(2) = 5$, $h''(2) = 7$, calculate the following limits. Explain your reasoning.

(a) $\lim_{x\to 2} \frac{f(x)}{h(x)}$ (b) $\lim_{x\to 2} \frac{f(x)}{g(x)}$

28. Derive the formulas given in the box on page 452 for the coefficients of the Taylor polynomial approximating a function f for x near a.

29. **(a)** Find the Taylor polynomial approximation of degree 4 about $x = 0$ for the function $f(x) = e^{x^2}$.

(b) Compare this result to the Taylor polynomial approximation of degree 2 for the function $f(x) = e^x$ about $x = 0$. What do you notice?

(c) Use your observation in part (b) to write out the Taylor polynomial approximation of degree 20 for the function in part (a).

(d) What is the Taylor polynomial approximation of degree 5 for the function $f(x) = e^{-2x}$?

30. Consider the equations $\sin x = 0.2$ and $x - \frac{x^3}{3!} = 0.2$

(a) How many zeros does each equation have?

(b) Which of the zeros of the two equations are approximately equal? Explain.

31. The integral $\int_0^1 (\sin t/t)\,dt$ is difficult to approximate using, for example, left Riemann sums or the trapezoid rule because the integrand $(\sin t)/t$ is not defined at $t = 0$. However, this integral converges; its value is $0.94608\ldots$. Estimate the integral using Taylor polynomials for $\sin t$ about $t = 0$ of

(a) Degree 3 (b) Degree 5

32. One of the two sets of functions, f_1, f_2, f_3, or g_1, g_2, g_3, is graphed in Figure 10.8; the other set is graphed in Figure 10.9. Points A and B each have $x = 0$. Taylor polynomials of degree 2 approximating these functions near $x = 0$ are as follows:

$f_1(x) \approx 2 + x + 2x^2$	$g_1(x) \approx 1 + x + 2x^2$
$f_2(x) \approx 2 + x - x^2$	$g_2(x) \approx 1 + x + x^2$
$f_3(x) \approx 2 + x + x^2$	$g_3(x) \approx 1 - x + x^2$.

(a) Which group of functions, the fs or the gs, is represented by each figure?

(b) What are the coordinates of the points A and B?

(c) Match the functions with the graphs (I)-(III) in each figure.

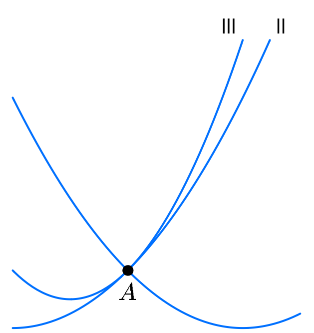

Figure 10.8

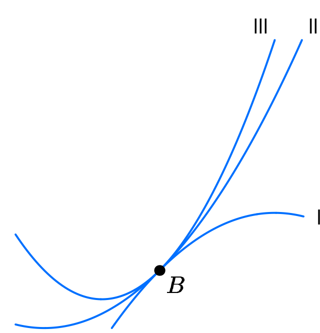

Figure 10.9

10.2 TAYLOR SERIES

In the previous section we saw how to approximate a function near a point by Taylor polynomials. Now we define a Taylor series, which is a power series that can be thought of as a Taylor polynomial that goes on forever.

Taylor Series for $\cos x$, $\sin x$, e^x

We have the following Taylor polynomials centered at $x = 0$ for $\cos x$:

$$\begin{aligned}
\cos x &\approx P_0(x) = 1 \\
\cos x &\approx P_2(x) = 1 - \frac{x^2}{2!} \\
\cos x &\approx P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\
\cos x &\approx P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\
\cos x &\approx P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.
\end{aligned}$$

Here we have a sequence of polynomials, $P_0(x)$, $P_2(x)$, $P_4(x)$, $P_6(x)$, $P_8(x)$, ..., each of which is a better approximation to $\cos x$ than the last, for x near 0. When we go to a higher-degree polynomial (say from P_6 to P_8), we add more terms ($x^8/8!$, for example), but the terms of lower degree don't change. Thus, each polynomial includes the information from all the previous ones. We represent the whole sequence of Taylor polynomials by writing the *Taylor series* for $\cos x$:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots.$$

Notice that the partial sums of this series are the Taylor polynomials, $P_n(x)$.

We define the Taylor series for $\sin x$ and e^x similarly. It turns out that, for these functions, the Taylor series converges to the function for all x, so we can write the following:

$$\boxed{\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\end{aligned}}$$

These series are also called *Taylor expansions* of the functions $\sin x$, $\cos x$, and e^x about $x = 0$. The *general term* of a Taylor series is a formula which gives any term in the series. For example, $x^n/n!$ is the general term in the Taylor expansion for e^x, and $(-1)^k x^{2k}/(2k)!$ is the general term in the expansion for $\cos x$. We call n or k the *index*.

Taylor Series in General

Any function f, all of whose derivatives exist at 0, has a Taylor series. However, the Taylor series for f does not necessarily converge to $f(x)$ for all values of x. For the values of x for which the series does converge to $f(x)$, we have the following formula:

Taylor Series for $f(x)$ about $x = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

In addition, just as we have Taylor polynomials centered at points other than 0, we can also have a Taylor series centered at $x = a$ (provided all the derivatives of f exist at $x = a$). For the values of x for which the series converges to $f(x)$, we have the following formula:

Taylor Series for $f(x)$ about $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series is a power series whose partial sums are the Taylor polynomials. As we saw in Section **??**, power series generally converge on an interval centered at $x = a$. The Taylor series for such a function can be interpreted when x is replaced by a complex number. This extends the domain of the function. See Problem 41.

For a given function f and a given x, even if the Taylor series converges, it might not converge to $f(x)$, though it does for most commonly encountered functions. Functions for which this is true for the Taylor series about every point $x = a$ in their domain are called *analytic functions*. Fortunately, the Taylor series for e^x, $\cos x$, and $\sin x$ converge to the original function for all x, so they are analytic. See Section 10.4.

Intervals of Convergence of Taylor Series

Let us look again at the Taylor polynomial for $\ln x$ about $x = 1$ that we derived in Example 7 on page **??**. A similar calculation gives the Taylor Series

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots.$$

Example 2 on page 449 and Example 5 on page 451 show that this power series has interval of convergence $0 < x \leq 2$. However, although we know that the series converges in this interval, we do not yet know that its sum is $\ln x$. The fact that in Figure 10.10 the polynomials fit the curve well for $0 < x < 2$ suggests that the Taylor series does converge to $\ln x$ for $0 < x \leq 2$. For such x-values, a higher-degree polynomial gives, in general, a better approximation.

However, when $x > 2$, the polynomials move away from the curve and the approximations get worse as the degree of the polynomial increases. Thus, the Taylor polynomials are effective only as approximations to $\ln x$ for values of x between 0 and 2; outside that interval, they should not be used. Inside the interval, but near the ends, 0 or 2, the polynomials converge very slowly. This means we might have to take a polynomial of very high degree to get an accurate value for $\ln x$.

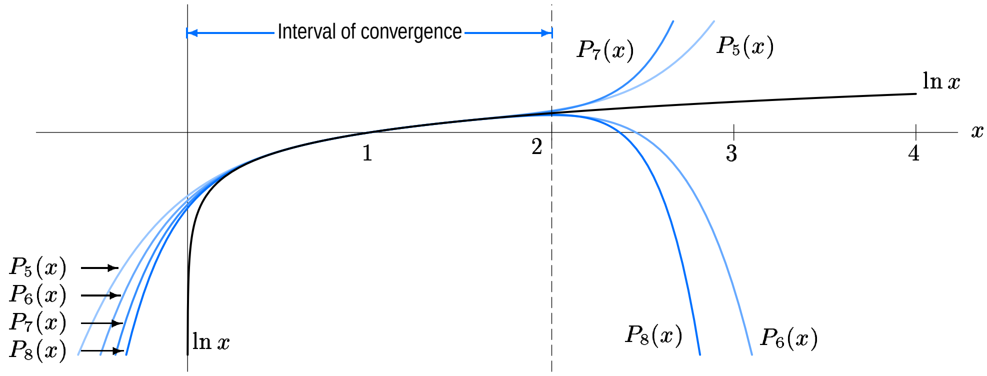

Figure 10.10: Taylor polynomials $P_5(x), P_6(x), P_7(x), P_8(x), \ldots$ converge to $\ln x$ for $0 < x \le 2$ and diverge outside that interval

To compute the interval of convergence exactly, we first compute the radius of convergence using the method on page **??**. Convergence at the endpoints, $x = 0$ and $x = 2$, has to be determined separately. However, proving that the series converges to $\ln x$ on its interval of convergence, as Figure 10.10 suggests, requires the error term introduced in Section 10.4.

Example 1 Find the Taylor series for $\ln(1 + x)$ about $x = 0$, and calculate its interval of convergence.

Solution Taking derivatives of $\ln(1 + x)$ and substituting $x = 0$ leads to the Taylor series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Notice that this is the same series that we get by substituting $(1 + x)$ for x in the series for $\ln x$:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \qquad \text{for } 0 < x \le 2.$$

Since the series for $\ln x$ about $x = 1$ converges for $0 < x \le 2$, the interval of convergence for the Taylor series for $\ln(1 + x)$ about $x = 0$ is $-1 < x \le 1$. Thus we write

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \le 1.$$

Notice that the series could not possibly converge to $\ln(1 + x)$ for $x \le -1$ since $\ln(1 + x)$ is not defined there.

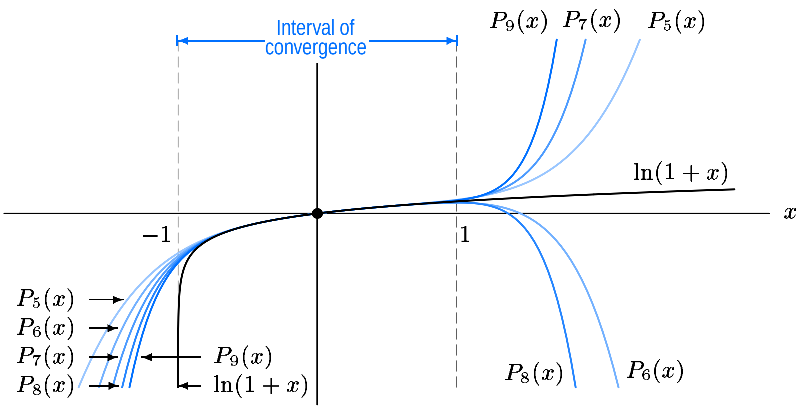

Figure 10.11: Interval of convergence for the Taylor series for $\ln(1 + x)$ is $-1 < x \le 1$

The Binomial Series Expansion

We now find the Taylor series about $x = 0$ for the function $f(x) = (1 + x)^p$, with p a constant, but not necessarily a positive integer. Taking derivatives:

$$\begin{aligned} f(x) &= (1+x)^p & \text{so} \qquad f(0) &= 1 \\ f'(x) &= p(1+x)^{p-1} & f'(0) &= p \\ f''(x) &= p(p-1)(1+x)^{p-2} & f''(0) &= p(p-1) \\ f'''(x) &= p(p-1)(p-2)(1+x)^{p-3}, & f'''(0) &= p(p-1)(p-2). \end{aligned}$$

Thus, the third-degree Taylor polynomial for x near 0 is

$$(1+x)^p \approx P_3(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3.$$

Graphing $P_3(x), P_4(x), \ldots$ for various specific values of p suggests that the Taylor polynomials converge to $f(x)$ for $-1 < x < 1$. (See Problems 24–23, page 459.) This can be confirmed using the radius of convergence test. The Taylor series for $f(x) = (1 + x)^p$ about $x = 0$ is as follows:

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad \text{for } -1 < x < 1.$$

In fact the binomial series gives the same result as multiplying $(1+x)^p$ out when p is a positive integer. (Newton discovered that the binomial series can be used for noninteger exponents.)

Example 2 Use the binomial series with $p = 3$ to expand $(1 + x)^3$.

Solution The series is

$$(1+x)^3 = 1 + 3x + \frac{3\cdot 2}{2!}x^2 + \frac{3\cdot 2\cdot 1}{3!}x^3 + \frac{3\cdot 2\cdot 1\cdot 0}{4!}x^4 + \cdots.$$

The x^4 term and all terms beyond it turn out to be zero, because each coefficient contains a factor of 0. Simplifying gives

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$

which is the usual expansion obtained by multiplying out $(1 + x)^3$.

Example 3 Find the Taylor series about $x = 0$ for $\dfrac{1}{1+x}$.

Solution Since $\dfrac{1}{1+x} = (1+x)^{-1}$, use the binomial series with $p = -1$. Then

$$\begin{aligned} \frac{1}{1+x} = (1+x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \quad \text{for } -1 < x < 1. \end{aligned}$$

This series is both a special case of the binomial series and an example of a geometric series. It converges for $-1 < x < 1$.

Exercises and Problems for Section 10.2

Exercises

For Problems 1–4, find the first four terms of the Taylor series for the given function about 0.

1. $\dfrac{1}{1-x}$ **2.** $\sqrt{1+x}$ **3.** $\dfrac{1}{\sqrt{1+x}}$ **4.** $\sqrt[3]{1-y}$

For Problems 5–11, find the first four terms of the Taylor series for the function about the point a.

5. $\sin x$, $a = \pi/4$

6. $\cos\theta$, $a = \pi/4$

7. $\sin\theta$, $a = -\pi/4$

8. $\tan x$, $a = \pi/4$

9. $1/x$, $a = 1$

10. $1/x$, $a = 2$

11. $1/x$, $a = -1$

Find an expression for the general term of the series in Problems 12–19 and give the starting value of the index (n or k, for example).

12. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

13. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$

14. $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$

15. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \cdots$

16. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

17. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$

18. $e^{x^2} = 1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dfrac{x^8}{4!} + \cdots$

19. $x^2\cos x^2 = x^2 - \dfrac{x^4}{2!} + \dfrac{x^6}{4!} - \dfrac{x^8}{6!} + \cdots$

Problems

20. **(a)** Find the terms up to degree 6 of the Taylor series for $f(x) = \sin(x^2)$ about $x = 0$ by taking derivatives.
(b) Compare your result in part (a) to the series for $\sin x$. How could you have obtained your answer to part (a) from the series for $\sin x$?

21. **(a)** Find the Taylor series for $f(x) = \ln(1+2x)$ about $x = 0$ by taking derivatives.
(b) Compare your result in part (a) to the series for $\ln(1+x)$. How could you have obtained your answer to part (a) from the series for $\ln(1+x)$?
(c) What do you expect the interval of convergence for the series for $\ln(1+2x)$ to be?

22. By graphing the function $f(x) = \sqrt{1+x}$ and several of its Taylor polynomials, estimate the interval of convergence of the series you found in Problem 2.

23. By graphing the function $f(x) = \dfrac{1}{\sqrt{1+x}}$ and several of its Taylor polynomials, estimate the interval of convergence of the series you found in Problem 3.

24. By graphing the function $f(x) = \dfrac{1}{1-x}$ and several of its Taylor polynomials, estimate the interval of convergence of the series you found in Problem 1. Compute the radius of convergence analytically.

25. Find the radius of convergence of the Taylor series around zero for $\ln(1-x)$.

26. **(a)** Write the general term of the binomial series for $(1+x)^p$ about $x = 0$.
(b) Find the radius of convergence of this series.

By recognizing each series in Problems 27–35 as a Taylor series evaluated at a particular value of x, find the sum of each of the following convergent series.

27. $1 + \dfrac{2}{1!} + \dfrac{4}{2!} + \dfrac{8}{3!} + \cdots + \dfrac{2^n}{n!} + \cdots$

28. $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \cdots + \dfrac{(-1)^n}{(2n+1)!} + \cdots$

29. $1 + \dfrac{1}{4} + \left(\dfrac{1}{4}\right)^2 + \left(\dfrac{1}{4}\right)^3 + \cdots$

30. $1 - \dfrac{100}{2!} + \dfrac{10000}{4!} + \cdots + \dfrac{(-1)^n \cdot 10^{2n}}{(2n)!} + \cdots$

31. $\dfrac{1}{2} - \dfrac{(\frac{1}{2})^2}{2} + \dfrac{(\frac{1}{2})^3}{3} - \dfrac{(\frac{1}{2})^4}{4} + \cdots + \dfrac{(-1)^n \cdot (\frac{1}{2})^{n+1}}{(n+1)} + \cdots$

32. $1 - 0.1 + 0.1^2 - 0.1^3 + \cdots$

33. $1 + 3 + \dfrac{9}{2!} + \dfrac{27}{3!} + \dfrac{81}{4!} + \cdots$

34. $1 - \dfrac{1}{2!} + \dfrac{1}{4!} - \dfrac{1}{6!} + \cdots$

35. $1 - 0.1 + \dfrac{0.01}{2!} - \dfrac{0.001}{3!} + \cdots$

In Problems 36–37 solve exactly for the variable.

36. $1 + x + x^2 + x^3 + \cdots = 5$

37. $x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \cdots = 0.2$

38. One of the two sets of functions, f_1, f_2, f_3, or g_1, g_2, g_3 is graphed in Figure 10.12; the other set is graphed in Figure 10.13. Taylor series for the functions about a point corresponding to either A or B are as follows:

$$f_1(x) = 3 + (x-1) - (x-1)^2 \cdots \qquad g_1(x) = 5 - (x-4) - (x-4)^2 \cdots$$
$$f_2(x) = 3 - (x-1) + (x-1)^2 \cdots \qquad g_2(x) = 5 - (x-4) + (x-4)^2 \cdots$$
$$f_3(x) = 3 - 2(x-1) + (x-1)^2 \cdots \qquad g_3(x) = 5 + (x-4) + (x-4)^2 \cdots$$

(a) Which group of functions is represented in each figure?

(b) What are the coordinates of the points A and B?

(c) Match the functions with the graphs (I)-(III) in each figure.

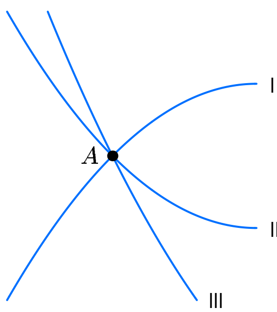

Figure 10.12

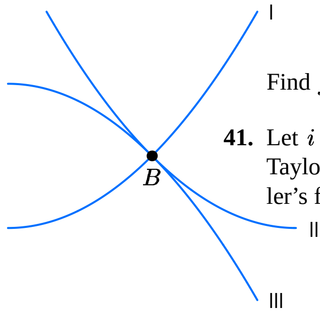

Figure 10.13

39. Suppose that you are told that the Taylor series of $f(x) = x^2 e^{x^2}$ about $x = 0$ is

$$x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \frac{x^{10}}{4!} + \cdots.$$

Find $\left.\frac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0}$ and $\left.\frac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

40. Suppose you know that all the derivatives of some function f exist at 0, and that Taylor series for f about $x = 0$ is

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots.$$

Find $f'(0)$, $f''(0)$, $f'''(0)$, and $f^{(10)}(0)$.

41. Let $i = \sqrt{-1}$. We define $e^{i\theta}$ by substituting $i\theta$ in the Taylor series for e^x. Use this definition[2] to explain Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

10.3 FINDING AND USING TAYLOR SERIES

Finding a Taylor series for a function means finding the coefficients. Assuming the function has all its derivatives defined, finding the coefficients can always be done, in theory at least, by differentiation. That is how we derived the four most important Taylor series, those for the functions e^x, $\sin x$, $\cos x$, and $(1+x)^p$. For many functions, however, computing Taylor series coefficients by differentiation can be a very laborious business. We now introduce easier ways of finding Taylor series, if the series we want is closely related to a series that we already know.

New Series by Substitution

Suppose we want to find the Taylor series for e^{-x^2} about $x = 0$. We could find the coefficients by differentiation. Differentiating e^{-x^2} by the chain rule gives $-2xe^{-x^2}$, and differentiating again gives $-2e^{-x^2} + 4x^2e^{-x^2}$. Each time we differentiate we use the product rule, and the number of terms grows. Finding the tenth or twentieth derivative of e^{-x^2}, and thus the series for e^{-x^2} up to the x^{10} or x^{20} terms, by this method is tiresome (at least without a computer or calculator that can differentiate).

Fortunately, there's a quicker way. Recall that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots \quad \text{for all } y.$$

Substituting $y = -x^2$ tells us that

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots \quad \text{for all } x.$$

[2] Complex numbers are discussed in Appendix B.

Simplifying shows that the Taylor series for e^{-x^2} is

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots \quad \text{for all } x.$$

Using this method, it is easy to find the series up to the x^{10} or x^{20} terms.

Example 1 Find the Taylor series about $x = 0$ for $f(x) = \dfrac{1}{1+x^2}$.

Solution The binomial series tells us that

$$\frac{1}{1+y} = (1+y)^{-1} = 1 - y + y^2 - y^3 + y^4 + \cdots \quad \text{for } -1 < y < 1.$$

Substituting $y = x^2$ gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots \quad \text{for } -1 < x < 1,$$

which is the Taylor series for $\dfrac{1}{1+x^2}$.

These examples demonstrate that we can get new series from old ones by substitution. More advanced texts show that series obtained by this method are indeed correct.

In Example 1, we made the substitution $y = x^2$. We can also substitute an entire series into another one, as in the next example.

Example 2 Find the Taylor series about $\theta = 0$ for $g(\theta) = e^{\sin\theta}$.

Solution For all y and θ, we know that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots.$$

Let's substitute the series for $\sin\theta$ for y:

$$e^{\sin\theta} = 1 + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + \frac{1}{2!}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^2 + \frac{1}{3!}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^3 + \cdots.$$

To simplify, we multiply out and collect terms. The only constant term is the 1, and there's only one θ term. The only θ^2 term is the first term we get by multiplying out the square, namely $\theta^2/2!$. There are two contributors to the θ^3 term: the $-\theta^3/3!$ from within the first parentheses, and the first term we get from multiplying out the cube, which is $\theta^3/3!$. Thus the series starts

$$\begin{aligned} e^{\sin\theta} &= 1 + \theta + \frac{\theta^2}{2!} + \left(-\frac{\theta^3}{3!} + \frac{\theta^3}{3!}\right) + \cdots \\ &= 1 + \theta + \frac{\theta^2}{2!} + 0\cdot\theta^3 + \cdots \quad \text{for all } \theta. \end{aligned}$$

New Series by Differentiation and Integration

Just as we can get new series by substitution, we can also get new series by differentiation and integration. Here again, proof that this method gives the correct series and that the new series has the same interval of convergence as the original series, can be found in more advanced texts.

Example 3 Find the Taylor Series about $x = 0$ for $\frac{1}{(1-x)^2}$ from the series for $\frac{1}{1-x}$.

Solution We know that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$, so we start with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \text{ for } -1 < x < 1.$$

Differentiation term by term gives the binomial series

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots \text{ for } -1 < x < 1.$$

Example 4 Find the Taylor series about $x = 0$ for $\arctan x$ from the series for $\frac{1}{1+x^2}$.

Solution We know that $\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$, so we use the series from Example 1 on page 461:

$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad \text{for } -1 < x < 1.$$

Antidifferentiating term by term gives

$$\arctan x = \int \frac{1}{1+x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad \text{for } -1 < x < 1,$$

where C is the constant of integration. Since $\arctan 0 = 0$, we have $C = 0$, so

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad \text{for } -1 < x < 1.$$

The series for $\arctan x$ was discovered by James Gregory (1638–1675).

Applications of Taylor Series

Example 5 Use the series for $\arctan x$ to estimate the numerical value of π.

Solution Since $\arctan 1 = \pi/4$, we use the series for $\arctan x$ from Example 4. We assume—as is the case—that the series does converge to $\pi/4$ at $x = 1$, the endpoint of its interval of convergence. Substituting $x = 1$ into the series for $\arctan x$ gives

$$\pi = 4\arctan 1 = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right).$$

Table 10.1 *Approximating π using the series for $\arctan x$*

n	4	5	25	100	500	1000	10,000
S_n	2.895	3.340	3.182	3.132	3.140	3.141	3.141

Table 10.1 shows the value of the n^{th} partial sum, S_n, obtained by summing the nonzero terms from 1 through n. The values of S_n do seem to converge to $\pi = 3.141\ldots$. However, this series converges very slowly, meaning that we have to take a large number of terms to get an accurate estimate for π. So this way of calculating π is not particularly practical. (A better one is given in Problem 1, page 475.) However, the expression for π given by this series is surprising and elegant.

A basic question we can ask about two functions is which one gives larger values. Taylor series can often be used to answer this question over a small interval. If the constant terms of the series for two functions are the same, compare the linear terms; if the linear terms are the same, compare the quadratic terms, and so on.

Example 6 By looking at their Taylor series, decide which of the following functions is largest, and which is smallest, for θ near 0. (a) $1 + \sin\theta$ (b) e^θ (c) $\dfrac{1}{\sqrt{1-2\theta}}$

Solution The Taylor expansion about $\theta = 0$ for $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

So

$$1 + \sin\theta = 1 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

The Taylor expansion about $\theta = 0$ for e^θ is

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots.$$

The Taylor expansion about $\theta = 0$ for $1/\sqrt{1+\theta}$ is

$$\begin{aligned}\frac{1}{\sqrt{1+\theta}} = (1+\theta)^{-1/2} &= 1 - \frac{1}{2}\theta + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\theta^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\theta^3 + \cdots \\ &= 1 - \frac{1}{2}\theta + \frac{3}{8}\theta^2 - \frac{5}{16}\theta^3 + \cdots.\end{aligned}$$

So, substituting -2θ for θ:

$$\begin{aligned}\frac{1}{\sqrt{1-2\theta}} &= 1 - \frac{1}{2}(-2\theta) + \frac{3}{8}(-2\theta)^2 - \frac{5}{16}(-2\theta)^3 + \cdots \\ &= 1 + \theta + \frac{3}{2}\theta^2 + \frac{5}{2}\theta^3 + \cdots.\end{aligned}$$

For θ near 0, we can neglect terms beyond the second degree. We are left with the approximations:

$$\begin{aligned} 1+\sin\theta &\approx 1+\theta \\ e^{\theta} &\approx 1+\theta+\frac{\theta^2}{2} \\ \frac{1}{\sqrt{1-2\theta}} &\approx 1+\theta+\frac{3}{2}\theta^2. \end{aligned}$$

Since

$$1+\theta < 1+\theta+\frac{1}{2}\theta^2 < 1+\theta+\frac{3}{2}\theta^2,$$

and since the approximations are valid for θ near 0, we conclude that, for θ near 0,

$$1+\sin\theta < e^{\theta} < \frac{1}{\sqrt{1-2\theta}}.$$

Example 7 Two electrical charges of equal magnitude and opposite signs located near one another are called an electrical dipole. The charges Q and $-Q$ are a distance r apart. (See Figure 10.14.) The electric field, E, at the point P is given by

$$E=\frac{Q}{R^2}-\frac{Q}{(R+r)^2}.$$

Use series to investigate the behavior of the electric field far away from the dipole. Show that when R is large in comparison to r, the electric field is approximately proportional to $1/R^3$.

Figure 10.14: Approximating the electric field at P due to a dipole consisting of charges Q and $-Q$ a distance r apart

Solution In order to use a series approximation, we need a variable whose value is small. Although we know that r is much smaller than R, we do not know that r itself is small. The quantity r/R is, however, very small. Hence we expand $1/(R+r)^2$ in powers of r/R so that we can safely use only the first few terms of the Taylor series. First we rewrite using algebra:

$$\frac{1}{(R+r)^2}=\frac{1}{R^2(1+r/R)^2}=\frac{1}{R^2}\left(1+\frac{r}{R}\right)^{-2}.$$

Now we use the binomial expansion for $(1+x)^p$ with $x=r/R$ and $p=-2$:

$$\begin{aligned} \frac{1}{R^2}\left(1+\frac{r}{R}\right)^{-2} &= \frac{1}{R^2}\left(1+(-2)\left(\frac{r}{R}\right)+\frac{(-2)(-3)}{2!}\left(\frac{r}{R}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{r}{R}\right)^3+\cdots\right) \\ &= \frac{1}{R^2}\left(1-2\frac{r}{R}+3\frac{r^2}{R^2}-4\frac{r^3}{R^3}+\cdots\right). \end{aligned}$$

So, substituting the series into the expression for E, we have

$$\begin{aligned} E=\frac{Q}{R^2}-\frac{Q}{(R+r)^2} &= Q\left[\frac{1}{R^2}-\frac{1}{R^2}\left(1-2\frac{r}{R}+3\frac{r^2}{R^2}-4\frac{r^3}{R^3}+\cdots\right)\right] \\ &= \frac{Q}{R^2}\left(2\frac{r}{R}-3\frac{r^2}{R^2}+4\frac{r^3}{R^3}-\cdots\right). \end{aligned}$$

Since r/R is smaller than 1, the binomial expansion for $(1+r/R)^{-2}$ converges. We are interested in the electric field far away from the dipole. The quantity r/R is small there, and $(r/R)^2$ and higher powers are smaller still. Thus, we approximate by disregarding all terms except the first, giving

$$E \approx \frac{Q}{R^2}\left(\frac{2r}{R}\right), \quad \text{so} \quad E \approx \frac{2Qr}{R^3}.$$

Since Q and r are constants, this means that E is approximately proportional to $1/R^3$.

In the previous example, we say that E is *expanded in terms of* r/R, meaning that the variable in the expansion is r/R.

Exercises and Problems for Section 10.3

Exercises

Find the first four nonzero terms of the Taylor series about 0 for the functions in Problems 1–12.

1. $\sqrt{1-2x}$ **2.** $\cos(\theta^2)$ **3.** e^{-x} **4.** $\dfrac{t}{1+t}$

5. $\ln(1-2y)$ **6.** $\arcsin x$ **7.** $\dfrac{1}{\sqrt{1-z^2}}$ **8.** $\phi^3\cos(\phi^2)$

9. $\dfrac{z}{e^{z^2}}$ **10.** $\sqrt{(1+t)}\sin t$ **11.** $e^t\cos t$ **12.** $\sqrt{1+\sin\theta}$

Find the Taylor series around 0 for the functions in Problems 13–15. Give the general term.

13. $(1+x)^3$

14. $t\sin(t^2)-t^3$

15. $\dfrac{1}{\sqrt{1-y^2}}$

For Problems 16–17, expand the quantity about 0 in terms of the variable given. Give four nonzero terms.

16. $\dfrac{1}{2+x}$ in terms of $\dfrac{x}{2}$

17. $\dfrac{a}{\sqrt{a^2+x^2}}$ in terms of $\dfrac{x}{a}$, where $a>0$

Problems

18. For a, b positive constants, the upper half of an ellipse has equation

$$y=f(x)=b\sqrt{1-\frac{x^2}{a^2}},$$

(a) Find the second degree Taylor polynomial of $f(x)$ about 0.
(b) Explain how you could have predicted the coefficient of x in the Taylor polynomial from a graph of f.
(c) For x near 0, the ellipse is approximated by a parabola. What is the equation of the parabola? What are its x-intercepts?
(d) Taking $a=3$ and $b=2$, estimate the maximum difference between $f(x)$ and the second-degree Taylor polynomial for $-0.1\le x\le 0.1$.

19. By looking at the Taylor series, decide which of the following functions is largest, and which is smallest, for small positive θ.

(a) $1+\sin\theta$ (b) $\cos\theta$ (c) $\dfrac{1}{1-\theta^2}$

20. For values of y near 0, put the following functions in increasing order, using their Taylor expansions.

(a) $\ln(1+y^2)$ (b) $\sin(y^2)$ (c) $1-\cos y$

21. Figure 10.15 shows the graphs of the four functions below for values of x near 0. Use Taylor series to match graphs and formulas.

(a) $\dfrac{1}{1-x^2}$ (b) $(1+x)^{1/4}$ (c) $\sqrt{1+\dfrac{x}{2}}$
(d) $\dfrac{1}{\sqrt{1-x}}$

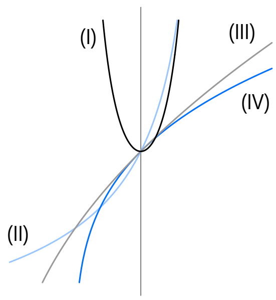

Figure 10.15

22. Consider the functions $y = e^{-x^2}$ and $y = 1/(1+x^2)$.

(a) Write the Taylor expansions for the two functions about $x = 0$. What is similar about the two series? What is different?

(b) Looking at the series, which function do you predict will be greater over the interval $(-1, 1)$? Graph both and see.

(c) Are these functions even or odd? How might you see this by looking at the series expansions?

(d) By looking at the coefficients, explain why it is reasonable that the series for $y = e^{-x^2}$ converges for all values of x, but the series for $y = 1/(1+x^2)$ converges only on $(-1, 1)$.

23. The hyperbolic sine and cosine are differentiable and satisfy the conditions $\cosh 0 = 1$ and $\sinh 0 = 0$, and

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\sinh x) = \cosh x.$$

(a) Using only this information, find the Taylor approximation of degree $n = 8$ about $x = 0$ for $f(x) = \cosh x$.

(b) Estimate the value of $\cosh 1$.

(c) Use the result from part (a) to find a Taylor polynomial approximation of degree $n = 7$ about $x = 0$ for $g(x) = \sinh x$.

24. Padé approximants are rational functions used to approximate more complicated functions. In this problem, you will derive the Padé approximant to the exponential function.

(a) Let $f(x) = (1 + ax)/(1 + bx)$, where a and b are constants. Write down the first three terms of the Taylor series for $f(x)$ about $x = 0$.

(b) By equating the first three terms of the Taylor series about $x = 0$ for $f(x)$ and for e^x, find a and b so that $f(x)$ approximates e^x as closely as possible near $x = 0$.

25. An electric dipole on the x-axis consists of a charge Q at $x = 1$ and a charge $-Q$ at $x = -1$. The electric field, E, at the point $x = R$ on the x-axis is given (for $R > 1$) by

$$E = \frac{kQ}{(R-1)^2} - \frac{kQ}{(R+1)^2}$$

where k is a positive constant whose value depends on the units. Expand E as a series in $1/R$, giving the first two nonzero terms.

26. Assume a is a positive constant. Suppose z is given by the expression

$$z = \sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}.$$

Expand z as a series in x as far as the second nonzero term.

27. The electric potential, V, at a distance R along the axis perpendicular to the center of a charged disc with radius a and constant charge density σ, is given by

$$V = 2\pi\sigma(\sqrt{R^2 + a^2} - R).$$

Show that, for large R,

$$V \approx \frac{\pi a^2 \sigma}{R}.$$

28. One of Einstein's most amazing predictions was that light traveling from distant stars would bend around the sun on the way to earth. His calculations involved solving for ϕ in the equation

$$\sin\phi + b(1 + \cos^2\phi + \cos\phi) = 0$$

where b is a very small positive constant.

(a) Explain why the equation could have a solution for ϕ which is near 0.

(b) Expand the left-hand side of the equation in Taylor series about $\phi = 0$, disregarding terms of order ϕ^2 and higher. Solve for ϕ. (Your answer will involve b.)

29. The Michelson-Morley experiment, which contributed to the formulation of the Theory of Relativity, involved the difference between the two times t_1 and t_2 that light took to travel between two points. If v is the velocity of light; l_1, l_2, and c are constants; and $v < c$, then the times t_1 and t_2 are given by

$$t_1 = \frac{2l_2}{c(1 - v^2/c^2)} - \frac{2l_1}{c\sqrt{1 - v^2/c^2}} \qquad t_2 = \frac{2l_2}{c\sqrt{1 - v^2/c^2}} - \frac{2l_1}{c(1 - v^2/c^2)}$$

(a) Find an expression for $\Delta t = t_1 - t_2$, and give its Taylor expansion in terms of v^2/c^2 up to the second nonzero term.

(b) For small v, to what power of v is Δt proportional? What is the constant of proportionality?

30. A hydrogen atom consists of an electron, of mass m, orbiting a proton, of mass M, where m is much smaller than M. The *reduced mass*, μ, of the hydrogen atom is defined by

$$\mu = \frac{mM}{m+M}.$$

(a) Show that $\mu \approx m$.
(b) To get a more accurate approximation for μ, express μ as m times a series in m/M.
(c) The approximation $\mu \approx m$ is obtained by disregarding all but the constant term in the series. The first-order correction is obtained by including the linear term but no higher terms. If $m \approx M/1836$, by what percentage does including the first-order correction change the estimate $\mu \approx m$?

31. A thin disk of radius a and mass M lies horizontally; a particle of mass m is at a height h directly above the center of the disk. The gravitational force, F, exerted by the disk on the mass m is given by

$$F = \frac{2GMmh}{a^2}\left(\frac{1}{h} - \frac{1}{(a^2+h^2)^{1/2}}\right).$$

Assume $a < h$ and think of F as a function of a, with the other quantities constant.

(a) Expand F as a series in a/h. Give the first two nonzero terms.
(b) Show that the approximation for F obtained by using only the first nonzero term in the series is independent of the radius, a.
(c) If $a = 0.02h$, by what percentage does the approximation in part (a) differ from the approximation in part (b)?

32. When a body is near the surface of the earth, we usually assume that the force due to gravity on it is a constant mg, where m is the mass of the body and g is the acceleration due to gravity at sea level. For a body at a distance h above the surface of the earth, a more accurate expression for the force F is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth. We will consider the situation in which the body is close to the surface of the earth so that h is much smaller than R.

(a) Show that $F \approx mg$.
(b) Express F as mg multiplied by a series in h/R.
(c) The first-order correction to the approximation $F \approx mg$ is obtained by taking the linear term in the series but no higher terms. How far above the surface of the earth can you go before the first-order correction changes the estimate $F \approx mg$ by more than 10%? (Assume $R = 6400$ km.)

33. **(a)** Estimate the value of $\int_0^1 e^{-x^2}dx$ using Riemann sums for both left-hand and right-hand sums with $n = 5$ subdivisions.
(b) Approximate the function $f(x) = e^{-x^2}$ with a Taylor polynomial of degree 6.
(c) Estimate the integral in part (a) by integrating the Taylor polynomial from part (b).
(d) Indicate briefly how you could improve the results in each case.

34. Use Taylor series to explain how the following patterns arise:
(a) $\dfrac{1}{0.98} = 1.020408163264\ldots$ (b) $\left(\dfrac{1}{0.99}\right)^2 = 1.020304050607\ldots$

10.4 THE ERROR IN TAYLOR POLYNOMIAL APPROXIMATIONS

In order to use an approximation intelligently, we need to be able to estimate the size of the error, which is the difference between the exact answer (which we do not know) and the approximate value.

When we use $P_n(x)$, the n^{th} degree Taylor polynomial, to approximate $f(x)$, the error is the difference

$$E_n(x) = f(x) - P_n(x).$$

If E_n is positive, the approximation is smaller than the true value. If E_n is negative, the approximation is too large. Often we are only interested in the magnitude of the error, $|E_n|$.

Recall that we constructed $P_n(x)$ so that its first n derivatives equal the corresponding derivatives of $f(x)$. Therefore, $E_n(0) = 0$, $E_n'(0) = 0$, $E_n''(0) = 0$, $\cdots$, $E_n^{(n)}(0) = 0$. Since $P_n(x)$ is an n^{th} degree polynomial, its $(n+1)^{\text{st}}$ derivative is 0, so $E_n^{(n+1)}(x) = f^{(n+1)}(x)$. In addition, suppose that $\left|f^{(n+1)}(x)\right|$ is bounded by a positive constant M, for all positive values of x near 0, say for $0 \le x \le d$, so that

$$-M \le f^{(n+1)}(x) \le M \quad \text{for } 0 \le x \le d.$$

This means that

$$-M \le E_n^{(n+1)}(x) \le M \quad \text{for } 0 \le x \le d.$$

Writing t for the variable, we integrate this inequality from 0 to x, giving

$$-\int_0^x M\,dt \le \int_0^x E_n^{(n+1)}(t)\,dt \le \int_0^x M\,dt \quad \text{for } 0 \le x \le d,$$

so

$$-Mx \le E_n^{(n)}(x) \le Mx \quad \text{for } 0 \le x \le d.$$

We integrate this inequality again from 0 to x, giving

$$-\int_0^x Mt\,dt \le \int_0^x E_n^{(n)}(t)\,dt \le \int_0^x Mt\,dt \quad \text{for } 0 \le x \le d,$$

so

$$-\frac{1}{2}Mx^2 \le E_n^{(n-1)}(x) \le \frac{1}{2}Mx^2 \quad \text{for } 0 \le x \le d.$$

By repeated integration, we obtain the following estimate:

$$-\frac{1}{(n+1)!}Mx^{n+1} \le E_n(x) \le \frac{1}{(n+1)!}Mx^{n+1} \quad \text{for } 0 \le x \le d,$$

which means that

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{1}{(n+1)!}Mx^{n+1} \quad \text{for } 0 \le x \le d.$$

When x is to the left of 0, so $-d \le x \le 0$, and when the Taylor series is centered at $a \ne 0$, similar calculations lead to the following result:

Theorem 10.1: Bounding the Error in $P_n(x)$

If $P_n(x)$ is the n^{th} Taylor approximation to $f(x)$ about a, then

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1},$$

where $M = \max f^{(n+1)}$ on the interval between a and x.

Using the Error Bound for Taylor Polynomials

Example 1 Give a bound on the error, E_4, when e^x is approximated by its fourth-degree Taylor polynomial for $-0.5 \le x \le 0.5$.

Solution Let $f(x) = e^x$. Then the fifth derivative is $f^{(5)}(x) = e^x$. Since e^x is increasing,

$$|f^{(5)}(x)| \le e^{0.5} = \sqrt{e} < 2 \quad \text{for} \quad -0.5 \le x \le 0.5.$$

Thus

$$|E_4| = |f(x) - P_4(x)| \le \frac{2}{5!}|x|^5.$$

This means, for example, that on $-0.5 \le x \le 0.5$, the approximation

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

has an error of at most $\frac{2}{120}(0.5)^5 < 0.0006$.

The error formula for Taylor polynomials can be used to bound the error in a particular numerical approximation, or to see how the accuracy of the approximation depends on the value of x or the value of n. Observe that the error for a Taylor polynomial of degree n depends on the $(n+1)^{\text{st}}$ power of x. That means, for example, with a Taylor polynomial of degree n centered at 0, if we decrease x by a factor of 2, the error bound decreases by a factor of 2^{n+1}.

Example 2 Compare the errors in the approximations

$$e^{0.1} \approx 1 + 0.1 + \frac{1}{2!}(0.1)^2 \qquad \text{and} \qquad e^{0.05} \approx 1 + (0.05) + \frac{1}{2!}(0.05)^2.$$

Solution We are approximating e^x by its second-degree Taylor polynomial, first at $x = 0.1$, and then at $x = 0.05$. Since we have decreased x by a factor of 2, the error bound decreases by a factor of about $2^3 = 8$. To see what actually happens to the errors, we compute them:

$$e^{0.1} - \left(1 + 0.1 + \frac{1}{2!}(0.1)^2\right) = 1.105171 - 1.105000 = 0.000171$$
$$e^{0.05} - \left(1 + 0.05 + \frac{1}{2!}(0.05)^2\right) = 1.051271 - 1.051250 = 0.000021$$

Since $(0.000171)/(0.000021) = 8.1$, the error has also decreased by a factor of about 8.

Convergence of Taylor Series for cos x

We have already seen that the Taylor polynomials centered at $x = 0$ for $\cos x$ are good approximations for x near 0. (See Figure 10.16.) In fact, for any value of x, if we take a Taylor polynomial centered at $x = 0$ of high enough degree, its graph is nearly indistinguishable from the graph of the cosine near that point.

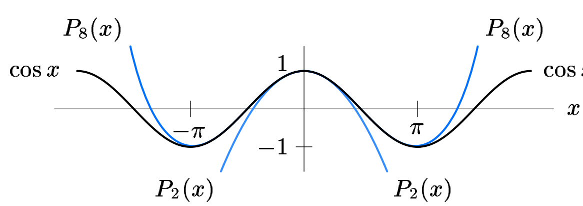

Figure 10.16: Graph of $\cos x$ and two Taylor polynomials for x near 0

Let's see what happens numerically. Let $x = \pi/2$. The successive Taylor polynomial approximations to $\cos(\pi/2) = 0$ about $x = 0$ are

$$\begin{aligned}
P_2(\pi/2) &= 1 - (\pi/2)^2/2! &&= -0.23370\ldots\\
P_4(\pi/2) &= 1 - (\pi/2)^2/2! + (\pi/2)^4/4! &&= 0.01997\ldots\\
P_6(\pi/2) &= \cdots &&= -0.00089\ldots\\
P_8(\pi/2) &= \cdots &&= 0.00002\ldots.
\end{aligned}$$

It appears that the approximations converge to the true value, $\cos(\pi/2) = 0$, very rapidly. Now take a value of x somewhat farther away from 0, say $x = \pi$, then $\cos\pi = -1$ and

$$\begin{aligned}
P_2(\pi) &= 1 - (\pi)^2/2! = -3.93480\ldots \\
P_4(\pi) &= \quad\cdots\quad = 0.12391\ldots \\
P_6(\pi) &= \quad\cdots\quad = -1.21135\ldots \\
P_8(\pi) &= \quad\cdots\quad = -0.97602\ldots \\
P_{10}(\pi) &= \quad\cdots\quad = -1.00183\ldots \\
P_{12}(\pi) &= \quad\cdots\quad = -0.99990\ldots \\
P_{14}(\pi) &= \quad\cdots\quad = -1.000004\ldots
\end{aligned}$$

We see that the rate of convergence is somewhat slower; it takes a 14^{th} degree polynomial to approximate $\cos\pi$ as accurately as an 8^{th} degree polynomial approximates $\cos(\pi/2)$. If x were taken still farther away from 0, then we would need still more terms to obtain as accurate an approximation of $\cos x$.

Using the ratio test, we can show the Taylor series for $\cos x$ converges for all values of x. In addition, we will prove that it converges to $\cos x$ using Theorem 10.1. Thus, we are justified in writing the equality:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \qquad \text{for all } x.$$

Showing the Taylor Series for cos x Converges to cos x

The error bound in Theorem 10.1 allows us to see if the Taylor series for a function converges to that function. In the series for $\cos x$, the odd powers are missing, so we assume n is even and write

$$E_n(x) = \cos x - P_n(x) = \cos x - \left(1 - \frac{x^2}{2!} + \cdots + (-1)^{n/2}\frac{x^n}{(n)!}\right),$$

giving

$$\cos x = 1 + x + \frac{x^2}{2!} + \cdots + (-1)^{n/2}\frac{x^n}{(n)!} + E_n(x).$$

Thus, for the Taylor series to converge to $\cos x$, we must have $E_n(x) \to 0$ as $n \to \infty$.

Showing $E_n(x) \to 0$ as $n \to \infty$

Proof Since $f(x) = \cos x$, the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ is $\pm\cos x$ or $\pm\sin x$, no matter what n is. So for all n, we have $|f^{(n+1)}(x)| \le 1$ on the interval between 0 and x.

By the error bound formula in Theorem 10.1, we have

$$|E_n(x)| = |\cos x - P_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \quad \text{for every } n.$$

To show that the errors go to zero, we must show that for a fixed x,

$$\frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty.$$

To see why this is true, think about what happens when n is much larger than x. Suppose, for example, that $x = 17.3$. Let's look at the value of the sequence for n more than twice as big as 17.3, say $n = 36$, or $n = 37$, or $n = 38$:

$$\begin{aligned}
&\text{For } n = 36: \quad \frac{1}{37!}(17.3)^{37} \\
&\text{For } n = 37: \quad \frac{1}{38!}(17.3)^{38} = \frac{17.3}{38}\cdot\frac{1}{37!}(17.3)^{37}, \\
&\text{For } n = 38: \quad \frac{1}{39!}(17.3)^{39} = \frac{17.3}{39}\cdot\frac{17.3}{38}\cdot\frac{1}{37!}(17.3)^{37}, \quad \ldots
\end{aligned}$$

Since $17.3/36$ is less than $\frac{1}{2}$, each time we increase n by 1, the term is multiplied by a number less than $\frac{1}{2}$. No matter what the value of $\frac{1}{37!}(17.3)^{37}$ is, if we keep on dividing it by two, the result gets closer to zero. Thus $\frac{1}{(n+1)!}(17.3)^{n+1}$ goes to 0 as n goes to infinity.

We can generalize this by replacing 17.3 by an arbitrary $|x|$. For $n > 2|x|$, the following sequence converges to 0 because each term is obtained from its predecessor by multiplying by a number less than $\frac{1}{2}$:

$$\frac{x^{n+1}}{(n+1)!}, \quad \frac{x^{n+2}}{(n+2)!}, \quad \frac{x^{n+3}}{(n+3)!}, \ldots.$$

Therefore, the Taylor series $1 - x^2/2! + x^4/4! - \cdots$ does converge to $\cos x$.

Problems 16 and 15 ask you to show that the Taylor series for $\sin x$ and e^x converge to the original function for all x. In each case, you again need the following limit:

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0.}$$

Exercises and Problems for Section 10.4

Exercises

Use the methods of this section to show how you can estimate the magnitude of the error in approximating the quantities in Problems 1–4 using a third-degree Taylor polynomial about $x = 0$.

1. $0.5^{1/3}$ **2.** $\ln(1.5)$ **3.** $1/\sqrt{3}$ **4.** $\tan 1$

Problems

5. Suppose you approximate $f(t) = e^t$ by a Taylor polynomial of degree 0 about $t = 0$ on the interval $[0, 0.5]$.

(a) Is the approximation an overestimate or an underestimate?

(b) Estimate the magnitude of the largest possible error. Check your answer graphically on a computer or calculator.

6. Repeat Problem 5 using the second-degree Taylor approximation, $P_2(t)$, to e^t.

7. Consider the error in using the approximation $\sin\theta \approx \theta$ on the interval $[-1, 1]$.

(a) Where is the approximation an overestimate, and where is it an underestimate?

(b) Estimate the magnitude of the largest possible error. Check your answer graphically on a computer or calculator.

8. Repeat Problem 7 for the approximation $\sin\theta \approx \theta - \theta^3/3!$.

9. Use the graphs of $y = \cos x$ and its Taylor polynomials, $P_{10}(x)$ and $P_{20}(x)$, in Figure 10.17 to estimate the following quantities.

(a) The error in approximating $\cos 5$ by $P_{10}(5)$ and by $P_{20}(5)$.

(b) The maximum error in approximating $\cos x$ by $P_{20}(x)$ for $|x| \le 10$.

(c) If we want to approximate $\cos x$ by $P_{10}(x)$ to an accuracy of within 0.2, what is the largest interval of x-values on which we can work? Give your answer to the nearest integer.

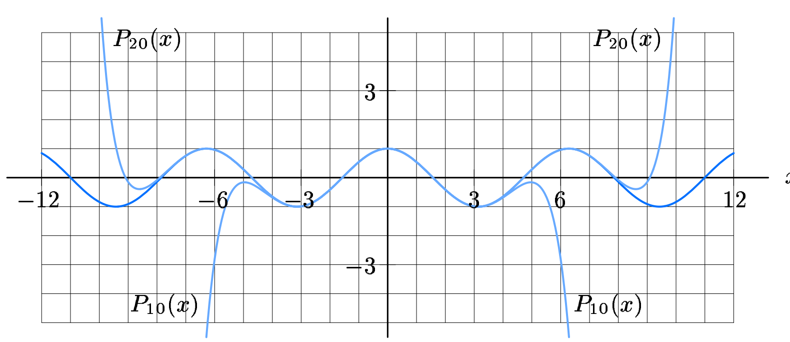

Figure 10.17

10. Give a bound for the maximum possible error for the n^{th} degree Taylor polynomial about $x = 0$ approximating $\cos x$ on the interval $[0, 1]$. What is the bound for $\sin x$?

11. What degree Taylor polynomial about $x = 0$ do you need to calculate $\cos 1$ to four decimal places? To six decimal places? Justify your answer using the results of Problem 10.

12. **(a)** Using a calculator, make a table of the values to four decimal places of $\sin x$ for

$$x = -0.5, -0.4, \dots, -0.1, 0, 0.1, \dots, 0.4, 0.5.$$

(b) Add to your table the values of the error $E_1 = \sin x - x$ for these x-values.

(c) Using a calculator or computer, draw a graph of the quantity $E_1 = \sin x - x$ showing that

$$|E_1| < 0.03 \quad \text{for} \quad -0.5 \le x \le 0.5.$$

13. In this problem, you will investigate the error in the n^{th} degree Taylor approximation to e^x for various values of n.

(a) Let $E_1 = e^x - P_1(x) = e^x - (1 + x)$. Using a calculator or computer, graph E_1 for $-0.1 \le x \le 0.1$. What shape is the graph of E_1? Use the graph to confirm that

$$|E_1| \le x^2 \quad \text{for} \quad -0.1 \le x \le 0.1.$$

(b) Let $E_2 = e^x - P_2(x) = e^x - (1+x+x^2/2)$. Choose a suitable range and graph E_2 for $-0.1 \le x \le 0.1$. What shape is the graph of E_2? Use the graph to confirm that

$$|E_2| \le x^3 \quad \text{for} \quad -0.1 \le x \le 0.1.$$

(c) Explain why the graphs of E_1 and E_2 have the shapes they do.

14. For $|x| \le 0.1$, graph the error

$$E_0 = \cos x - P_0(x) = \cos x - 1.$$

Explain the shape of the graph, using the Taylor expansion of $\cos x$. Find a bound for $|E_0|$ for $|x| \le 0.1$.

15. Show that the Taylor series about 0 for e^x converges to e^x for every x. Do this by showing that the error $E_n(x) \to 0$ as $n \to \infty$.

16. Show that the Taylor series about 0 for $\sin x$ converges to $\sin x$ for every x.

REVIEW PROBLEMS FOR CHAPTER TEN

Exercises

For Problems 1–4, find the second-degree Taylor polynomial about the given point.

1. e^x, $x = 1$ **2.** $\ln x$, $x = 2$ **3.** $\sin x$, $x = -\pi/4$

4. $\tan\theta$, $\theta = \pi/4$

5. Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ at $x = 1$.

In Problems 6–9, find the first four nonzero terms of the Taylor series about the origin of the given functions.

6. $\theta^2 \cos\theta^2$ **7.** $\sin t^2$ **8.** $\dfrac{1}{\sqrt{4-x}}$ **9.** $\dfrac{1}{1-4z^2}$

For Problems 10–11, expand the quantity in a Taylor series around the origin in terms of the variable given. Give the first four nonzero terms.

10. $\dfrac{a}{a+b}$ in terms of $\dfrac{b}{a}$ **11.** $\sqrt{R-r}$ in terms of $\dfrac{r}{R}$

Problems

12. Find an exact value for each of the following sums.

(a) $7(1.02)^3 + 7(1.02)^2 + 7(1.02) + 7 + \dfrac{7}{(1.02)} + \dfrac{7}{(1.02)^2} + \cdots + \dfrac{7}{(1.02)^{100}}.$

(b) $7 + 7(0.1)^2 + \dfrac{7(0.1)^4}{2!} + \dfrac{7(0.1)^6}{3!} + \cdots.$

Find the exact value of the sums of the series in Problems 13–18.

13. $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \dfrac{1}{81} - \cdots$ **14.** $8+4+2+1+\dfrac{1}{2}+\dfrac{1}{4}+\cdots+\dfrac{1}{2^{10}}$

15. $1-2+\dfrac{4}{2!}-\dfrac{8}{3!}+\dfrac{16}{4!}+\cdots$ **16.** $2-\dfrac{8}{3!}+\dfrac{32}{5!}-\dfrac{128}{7!}+\cdots$

17. $3+3+\frac{3}{2!}+\frac{3}{3!}+\frac{3}{4!}+\frac{3}{5!}+\cdots$

18. $(0.1)^2-\frac{(0.1)^4}{3!}+\frac{(0.1)^6}{5!}-\frac{(0.1)^8}{7!}+\cdots$

19. A function f has $f(3)=1$, $f'(3)=5$ and $f''(3)=-10$. Find the best estimate you can for $f(3.1)$.

20. Suppose x is positive but very small. Arrange the following expressions in increasing order:

$$x,\quad \sin x,\quad \ln(1+x),\quad 1-\cos x,\quad e^x-1,\quad \arctan x,\quad x\sqrt{1-x}.$$

21. By plotting several of its Taylor polynomials and the function $f(x)=1/(1+x)$, estimate graphically the interval of convergence of the series expansions for this function about $x=0$. Compute the radius of convergence analytically.

22. Use Taylor series to evaluate $\lim\limits_{x\to 0}\dfrac{\ln(1+x+x^2)-x}{\sin^2 x}$.

23. **(a)** Find $\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)}{\theta}$. Explain your reasoning.

(b) Use series to explain why $f(\theta)=\dfrac{\sin(2\theta)}{\theta}$ looks like a parabola near $\theta=0$. What is the equation of the parabola?

24. **(a)** Find the Taylor series for $f(t)=te^t$ about $t=0$.

(b) Using your answer to part (a), find a Taylor series expansion about $x=0$ for

$$\int_0^x te^t\,dt.$$

(c) Using your answer to part (b), show that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4(2!)}+\frac{1}{5(3!)}+\frac{1}{6(4!)}+\cdots=1.$$

25. **(a)** Find the first two nonzero terms of the Taylor series for $f(x)=\sqrt{4-x^2}$ about $x=0$

(b) Use your answer to part (a) and the Fundamental Theorem of Calculus to calculate an approximate value for $\displaystyle\int_0^1\sqrt{4-x^2}\,dx$.

(c) Use the substitution $x=2\sin t$ to calculate the exact value of the integral in part (b).

26. **(a)** Find the Taylor series expansion of $\arcsin x$.

(b) Use Taylor series to find the limit $\dfrac{\arctan x}{\arcsin x}$ as $x\to 0$.

27. A particle moving along the x-axis has potential energy at the point x given by $V(x)$. The potential energy has a minimum at $x=0$.

(a) Write the Taylor polynomial of degree 2 for V about $x=0$. What can you say about the signs of the coefficients of each of the terms of the Taylor polynomial?

(b) The force on the particle at the point x is given by $-V'(x)$. For small x, show that the force on the particle is proportional to its distance from the origin. What is the sign of the proportionality constant? Describe the direction in which the force points.

28. The theory of relativity predicts that when an object moves at speeds close to the speed of light, the object appears heavier. The apparent, or relativistic, mass, m, of the object when it is moving at speed v is given by the formula

$$m=\frac{m_0}{\sqrt{1-v^2/c^2}}$$

where c is the speed of light and m_0 is the mass of the object when it is at rest.

(a) Use the formula for m to decide what values of v are possible.

(b) Sketch a rough graph of m against v, labeling intercepts and asymptotes.

(c) Write the first three nonzero terms of the Taylor series for m in terms of v.

(d) For what values of v do you expect the series to converge?

29. The potential energy, V, of two gas molecules separated by a distance r is given by

$$V=-V_0\left(2\left(\frac{r_0}{r}\right)^6-\left(\frac{r_0}{r}\right)^{12}\right),$$

where V_0 and r_0 are positive constants.

(a) Show that if $r=r_0$, then V takes on its minimum value, $-V_0$.

(b) Write V as a series in $(r-r_0)$ up through the quadratic term.

(c) For r near r_0, show that the difference between V and its minimum value is approximately proportional to $(r-r_0)^2$. In other words, show that $V-(-V_0)=V+V_0$ is approximately proportional to $(r-r_0)^2$.

(d) The force, F, between the molecules is given by $F=-dV/dr$. What is F when $r=r_0$? For r near r_0, show that F is approximately proportional to $(r-r_0)$.

30. The *gravitational field* at a point in space is the gravitational force that would be exerted on a unit mass placed there. We will assume that the gravitational field strength at a distance d away from a mass M is

$$\frac{GM}{d^2}$$

where G is constant. In this problem you will investigate the gravitational field strength, F, exerted by a system consisting of a large mass M and a small mass m, with a distance r between them. (See Figure 10.18.)

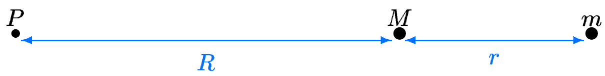

Figure 10.18

(a) Write an expression for the gravitational field strength, F, at the point P.

(b) Assuming r is small in comparison to R, expand F in a series in r/R.

(c) By discarding terms in $(r/R)^2$ and higher powers, explain why you can view the field as resulting from a single particle of mass $M+m$, plus a correction term. What is the position of the particle of mass $M+m$? Explain the sign of the correction term.

31. Expand $f(x+h)$ and $g(x+h)$ in Taylor series and take a limit to confirm the product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

32. Use Taylor expansions for $f(y+k)$ and $g(x+h)$ to confirm the chain rule:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

33. Suppose all the derivatives of g exist at $x=0$ and that g has a critical point at $x=0$.

(a) Write the n^{th} Taylor polynomial for g at $x=0$.

(b) What does the Second Derivative test for local maxima and minima say?

(c) Use the Taylor polynomial to explain why the Second Derivative test works.

34. (Continuation of Problem 33) You may remember that the Second Derivative test tells us nothing when the second derivative is zero at the critical point. In this problem you will investigate that special case.

Assume g has the same properties as in Problem 33, and that, in addition, $g''(0)=0$. What does the Taylor polynomial tell you about whether g has a local maximum or minimum at $x=0$?

35. Use the Fourier polynomials for the square wave

$$f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x \le \pi \end{cases}$$

to explain why the following sum must approach $\pi/4$ as $n \to \infty$:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{2n+1}\frac{1}{2n+1}.$$

36. Find a Fourier polynomial of degree three for $f(x) = e^{2\pi x}$, for $0 \le x < 1$.

37. Suppose that $f(x)$ is a differentiable periodic function of period 2π. Assume the Fourier series of f is differentiable term by term.

(a) If the Fourier coefficients of f are a_k and b_k, show that the Fourier coefficients of its derivative f' are kb_k and $-ka_k$.

(b) How are the amplitudes of the harmonics of f and f' related?

(c) How are the energy spectra of f and f' related?

38. If the Fourier coefficients of f are a_k and b_k, and the Fourier coefficients of g are c_k and d_k, and if A and B are real, show that the Fourier coefficients of $Af+Bg$ are $Aa_k + Bc_k$ and $Ab_k + Bd_k$.

39. Suppose that f is a periodic function of period 2π and that g is a horizontal shift of f, say $g(x) = f(x+c)$. Show that f and g have the same energy.

PROJECTS

Exercises

1. **Machin's Formula and the Value of π**

(a) Use the tangent addition formula

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

with $A = \arctan(120/119)$ and $B = -\arctan(1/239)$ to show that

$$\arctan\left(\frac{120}{119}\right) - \arctan\left(\frac{1}{239}\right) = \arctan 1.$$

(b) Use $A = B = \arctan(1/5)$ to show that

$$2\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{5}{12}\right).$$

Use a similar method to show that

$$4\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{120}{119}\right).$$

(c) Obtain Machin's formula: $\pi/4 = 4\arctan(1/5) - \arctan(1/239)$.

(d) Use the Taylor polynomial approximation of degree 5 to the arctangent function to approximate the value of π. (Note: In 1873 William Shanks used this approach to calculate π to 707 decimal places. Unfortunately, in 1946 it was found that he made an error in the 528^{th} place.)

(e) Why do the two series for arctangent converge so rapidly here while the series used in Example 5 on page 462 converges so slowly?

2. **Approximating the Derivative**[3] In applications, the values of a function $f(x)$ are frequently known only at discrete values $x_0,\ x_0 \pm h,\ x_0 \pm 2h, \ldots$. Suppose we are interested in approximating the derivative $f'(x_0)$. The definition

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h}$$

suggests that for small h we can approximate $f'(x)$ as follows:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}.$$

Such *finite-difference approximations* are used frequently in programming a computer to solve differential equations.

Taylor series can be used to analyze the error in this approximation. Substituting

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \cdots$$

into the approximation for $f'(x_0)$, we find

$$\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + \frac{f''(x_0)}{2}h + \cdots$$

This suggests (and it can be proved) that the error in the approximation is bounded as follows:

$$\left|\frac{f(x_0+h) - f(x_0)}{h} - f'(x_0)\right| \leq \frac{Mh}{2},$$

where

$$|f''(x)| \leq M \qquad \text{for} \qquad |x - x_0| \leq |h|.$$

[3]From Mark Kunka

Notice that as $h \to 0$, the error also goes to zero, provided M is bounded.

As an example, we take $f(x) = e^x$ and $x_0 = 0$, so $f'(x_0) = 1$. The error for various values of h are given in Table 10.2. We see that decreasing h by a factor of 10 decreases the error by a factor of about 10, as predicted by the error bound $Mh/2$.

Table 10.2

h	$(f(x_0+h) - f(x_0))/h$	Error
10^{-1}	1.05171	5.171×10^{-2}
10^{-2}	1.00502	5.02×10^{-3}
10^{-3}	1.00050	5.0×10^{-4}
10^{-4}	1.00005	5.0×10^{-5}

(a) Using Taylor series, suggest an error bound for each of the following finite-difference approximations.

(i) $f'(x_0) \approx \dfrac{f(x_0) - f(x_0-h)}{h}$

(ii) $f'(x_0) \approx \dfrac{f(x_0+h) - f(x_0-h)}{2h}$

(iii) $f'(x_0) \approx \dfrac{-f(x_0+2h) + 8f(x_0+h) - 8f(x_0-h) + f(x_0}{12h}$

(iv) Use each of the formulas in part (a) to approximate the first derivative of e^x at $x = 0$ for $h = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$. As h is decreased by a factor of 10, how does the error decrease? Does this agree with the error bounds found in part (a)? Which is the most accurate formula?

(v) Repeat part (b) using $f(x) = 1/x$ and $x_0 = 10^{-5}$. Why are these formulas not good approximations anymore? Continue to decrease h by factors of 10. How small does h have to be before formula (iii) is the best approximation? At these smaller values of h, what changed to make the formulas accurate again?

3. (a) Use a computer algebra system to find $P_{10}(x)$ and $Q_{10}(x)$, the Taylor polynomials of degree 10 about $x = 0$ for $\sin^2 x$ and $\cos^2 x$.

(b) What similarities do you observe between the two poynomials? Explain your observation in terms of properties of sine and cosine.

4. (a) Use your computer algebra system to find $P_7(x)$ and $Q_7(x)$, the Taylor polynomials of degree 7 about $x = 0$ for $f(x) = \sin x$ and $g(x) = \sin x \cos x$.
   (b) Find the ratio between the coefficient of x^3 in the two polynomials. Do the same for the coefficients of x^5 and x^7.
   (c) Describe the pattern in the ratios that you computed in part (b). Explain it using the identity $\sin(2x) = 2\sin x \cos x$.
5. Let $f(x) = \frac{x}{e^x - 1} + \frac{x}{2}$. Although the formula for f is not defined at $x = 0$, we can make f continuous by setting $f(0) = 1$. If we do this, f has Taylor series around $x = 0$.
   (a) Use a computer algebra system to find $P_{10}(x)$, the Taylor polynomial of degree 10 about $x = 0$ for f.
   (b) What do you notice about the degrees of the terms in the polynomial? What property of f does this suggest?
   (c) Prove that f has the property suggested by part (b).
6. Let $S(x) = \int_0^x \sin(t^2)\, dt$.
   (a) Use a computer algebra system to find $P_{11}(x)$, the Taylor polynomial of degree 11 about $x = 0$, for $S(x)$.
   (b) What is the percentage error in the approximation of $S(1)$ by $P_{11}(1)$? What about the approximation of $S(2)$ by $P_{11}(2)$?