6

Introduction to Analytic Methods of Wave Analysis

6.1 Introduction

In the last chapter we introduced the concept of a wave and presented two approaches to the mathematical description of waves. We also illustrated how canonical wave eqautions can be derived by asymptotic methods from more complicated mathematical models.

In this chapter we will introduce several methods for analyzing the *solutions* of wave equations. The first of these methods is *the method of charactereistics*. We introduced this method in the last chapter in order to describe solutions of linear hyperbolic wave equations. The method still works in the nonlinear setting but it may produce solutions which develop singularities in their derivatives in finite time, even though the initial data was smooth. This tendency to produce steep gradients is a characteristic effect of nonlinearity in hyperbolic equations.

In physical systems one hardly ever sees actual blowup of derivatives. This is because such growth is usually arrested by diffusive effects which, mathematically, are modelled by adding a diffusive term to the hyperbolic equation. This can no longer be solved by the method of characteristics; however, the asymptotic behaviour of solutions for large time can often be described by the second method we introduce: *Laplace's method* which is a special case of the *method of steepest descent*. We will illustrate how this works in the famous example of Burgers' equation.

Finally we return to the description of waves through their dispersive character. We introduce *the method of stationary phase* to make precise the subtle concept of a slowly varying wavetrain. Although we develop this only in the linear setting it is a fundamental paradigm in the asymptotic description of nonlinear waves as well.

In the final section we introduce Hamiltonian structures and their relation to the description of nonlinear waves. This will form the basis for a more detailed discussion of nonlinear dispersive waves and conservation laws in Chapter 9.

6.2 Nonlinear Waves: The Method of Characteristics and Singularities

Let us compare the solutions to the two problems:

$$(a)\ u_t + cu_x = 0, \quad c = \text{constant} \tag{6.1}$$

$$(b)\ u_t + uu_x = 0; \tag{6.2}$$

subject to the identical set of initial conditions

$$\begin{aligned} u(x,0) &= a^2 - x^2, \quad |x| \le a \\ &= 0 \qquad\qquad |x| > a \end{aligned} \tag{6.3}$$

which describes the propagation of a parabolic pulse, initially confined between $-a \le x \le a$.

Problem (a)

We already know that the general solution to (a) is $f(x - ct)$ and that it describes the propagation, without change in shape of an arbitrary initial disturbance. The solution to (a) for the specified initial data becomes

$$\begin{aligned} u(x,t) &= a^2 - (x - ct)^2, \quad |x - ct| \le a \\ &= 0 \qquad\qquad\qquad |x - ct| > a \end{aligned} \tag{6.4}$$

or in terms of the moving coordinate $\zeta = x - ct$

$$\begin{aligned} u(x,t) &= a^2 - \zeta^2 \quad |\zeta| \le a \\ &= 0 \qquad\quad |\zeta| > a \end{aligned} \tag{6.5}$$

In this coordinate the solution has no explicit dependence on t (steady pulse) and represents the same pulse as the initial one whose center is shifted by ct along the positive x-axis (see Figure 6.2).

Problem (b)

The problem is nonlinear due to the presence of the term uu_x. Nevertheless the method of characteristics, introduced in chapter 5 for linear hyperbolic waves, can be extended to solve this nonlinear equation.

6.2.1 *The Method of Characteristics*

We consider a generalization of Problem (b)

$$u_t + c(u)u_x = 0 \tag{6.6}$$

in which the speed is taken to be $c(u)$ where c is any differentiable function. We recover the original problem by taking c to be the identity function. Equation (6.6) is subject to the initial condition

$$u(x, 0) = f(x). \tag{6.7}$$

The total derivative of u is given by

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx \tag{6.8}$$

so that if the point $P = (t, x)$ is constrained to lie on a curve C (see Figure 6.1), then at any such point P on C we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \left(\frac{dx}{dt}\right)\frac{\partial u}{\partial x} \tag{6.9}$$

where $(1, dx/dt)$ is the tangent of the curve C in the (t, x) plane at the point P.

Comparison of (6.6) and (6.9) shows that we may interpret (6.6) as an ode,

$$\frac{du}{dt} = 0, \tag{6.10}$$

along any member of a family of curves C which are the solution curves of

$$\frac{dx}{dt} = c(u) \tag{6.11}$$

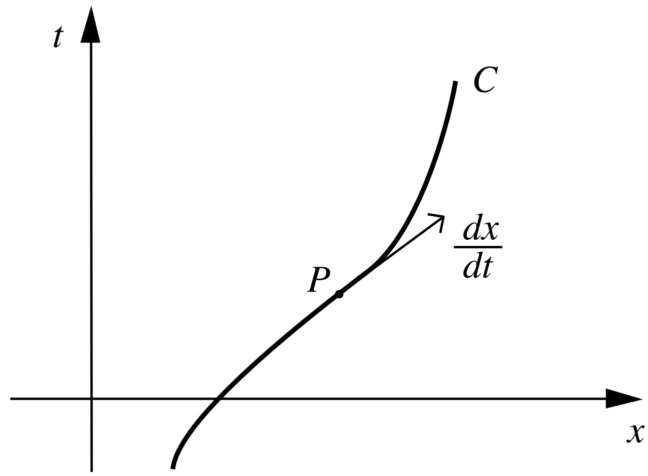

FIGURE 6.1. Tangent to Characteristic

These curves C are called the characteristic curves of equation (6.6).

The solution to the pde (6.6) has then been reduced to the solution of a pair of simultaneous ode's (6.10-6.11). Integrating these two equations respectively we have

$$\begin{aligned} u &= a_1 \\ x &= c(a_1)t + a_2 \end{aligned} \tag{6.12}$$

where a_1 and a_2 are constants along C. The initial data, along $t = 0$ can be parametrized by ζ as

$$\begin{aligned} u &= f(\zeta) \\ x &= \zeta \end{aligned} \tag{6.13}$$

Comparing (6.13) with (6.12) we can eliminate the constants a_i in terms of initial data as $a_2 = \zeta$ and $a_1 = f(\zeta)$ so that

$$x = \zeta + tc(f(\zeta)) \tag{6.14}$$

Thus each characteristic is a straight line whose slope, $c(f(\zeta))$, is determined by the value of the initial data, $f(\zeta)$, at the point $(\zeta, 0)$. Equation (6.14) is the equation of the family of characteristic curves C. The solution may be written in implicit form by eliminating ζ between $u = f(\zeta)$ which is valid along each characteristic and equation (6.14) for the characteristic to give

$$u(x,t) = f(x - c(u)t).$$

6.2.2 *The Formation of Singularities*

Returning to problem (b) we can now write down its exact solution in the implicit form

$$u(x,t) = f(x - ut)$$

where f is an arbitrary function.

Using the initial condition (6.3) we obtain

$$\begin{aligned} u(x,t) &= a^2 - \zeta^2 \quad \text{for } |\zeta| \le a \\ &= 0 \qquad\quad\;\; \text{for } |\zeta| > a \end{aligned} \tag{6.15}$$

where $\zeta = x - ut$.

We can solve (6.15) explicitly in the present case and we obtain

$$\begin{aligned} u(x,t) &= \tfrac{1}{2t^2}\left[(2xt-1) \pm \sqrt{1 - 4xt + 4a^2t^2}\right] \qquad |\zeta| \le a \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; |\zeta| > a \end{aligned} \tag{6.16}$$

We observe a fundamental difference between the linear problem (a) and the nonlinear one (b). In the linear problem the characteristics are parallel straight lines of slope $\frac{dx}{dt} = c$, a constant, whereas in the nonlinear problem, while they are still straight lines, their slope $\left(\frac{dx}{dt} = u\right)$ depends on the value, $u(\zeta, 0)$, at each point ζ along the initial line, $t = 0$.

The characteristic lines can be drawn directly from equation (6.16) for the nonlinear problem (b) (see Figure 6.3).

Where the characteristics intersect, $u(x,t)$ is no longer unique and the solution $u(x,t)$ is physically untenable at the point of intersection. If we are interested in a unique, bounded, single-valued solution in such a situation, we have to introduce the concept of a weak solution which permits moving jump discontinuities. These discontinuities are called shocks in fluid dynamics. In practice, some additional "physics" comes into play when the physical quantities of interest show rapid variation in some local region - we shall see shortly that the addition of a weak dissipative (diffusive) term will prevent the jump discontinuity from developing while maintaining the rapid variation across the shock front. We can understand this from ideas

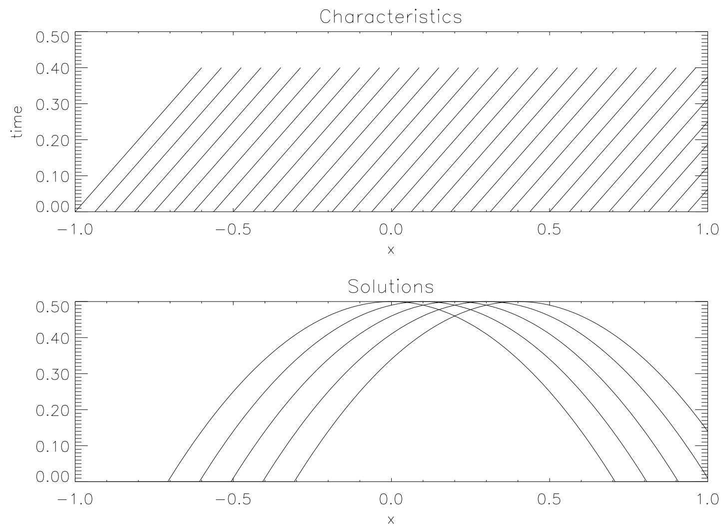

FIGURE 6.2. The Linear Problem: characteristics and solutions at sequential times

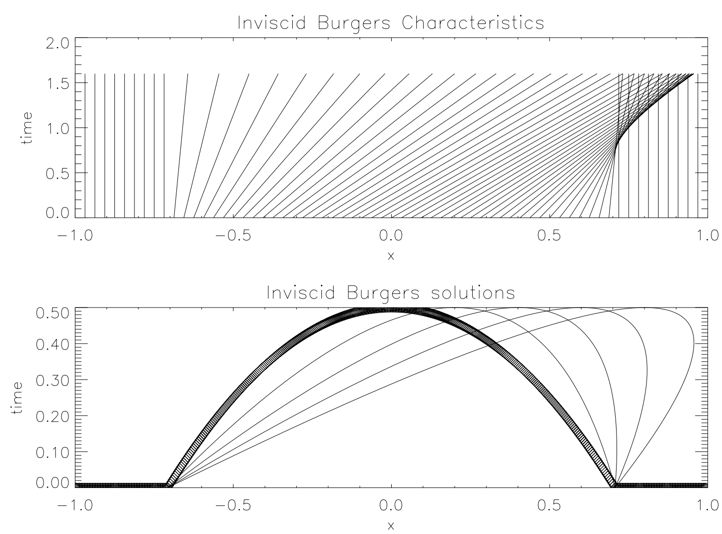

FIGURE 6.3. The Nonlinear Problem: characteristics and solutions at sequential times

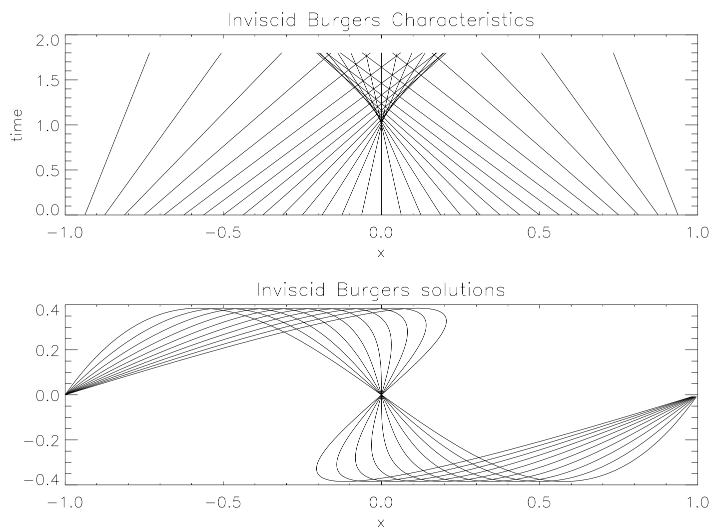

FIGURE 6.4. Characteristics and solutions for cubic initial data

developed earlier in this course if we think of the steepening front as generating higher spatial Fourier modes in k-space; we know that a diffusive term damps out high k-modes most rapidly!

Note: The values at the points $x = \pm a$ remain fixed for all times in this nonlinear problem because $u = 0$; i.e., the velocity is always zero at these points.

In the nonlinear problem, the leading edge (negative sloped) region of the pulse continuously steepens and eventually becomes multi-valued, indicating a nonphysical behavior. Note that the solution remote from this multi-valued region is still physically acceptable.

Figure 6.3 displays sucessive snapshots, with increasing time, of the profile of the explicit solution to (6.16). Notice that these solution profiles eventually become 3-valued. The space-time region where this occurs is the region where the characteristics overlap. One of these solution branches is just the x-axis which we will not consider in the subsequent discussion of this example.

Observe that when t is sufficiently small, only the upper sign before the radical in (6.16) is admissible, in order that the initial data (6.3), which is continuous, be recovered as $t \to 0$ [use L' Hospital's rule twice]. The representation (6.16) also shows that the solution constructed by characteristics can have *at most* two values (here we disregard the x-axis branch). For a fixed t these two values "meet" over a *branch point* x determined by the vanishing of the radical; i.e., $1 - 4xt + 4a^2t^2 = 0$.

We claim there must be a time at which the slope of the profile of the solution at a becomes infinite; i.e., u_x "blows up". ¿From (6.16) one directly calculates that

$$u_x(a,t) = \frac{1}{2t^2}\left[2t \mp \frac{2t}{|1-2at|}\right]. \tag{6.17}$$

At $x = a$, the slope of the initial profile is negative; using the upper branch of the radiacal one has:

$$u_x(a,0) = \lim_{t\to 0} u_x(a,t) = -2a < 0. \tag{6.18}$$

On the other hand, for t sufficiently large one sees that the expression for $u_x(a,t)$ is positive no matter what sign of the radical one takes. In order for $u_x(a,t)$ to change from negative to positive it must pass through infinity; i.e., the derivative of the velocity must blow up. This occurs at the time

$$\tau_s = \frac{1}{2a}.$$

In fact one sees that this is the minimum value of t for which the slope can change sign and so τ_s is the shock formation time. As a consistency check we finally observe that when $t = \tau_s$ and $x = a$ the radical has a zero; thus, for appropriate $x > a$ and $t > \tau_s$ both signs in (6.16) are admissible.

For more general initial data

$$u(x,0) \quad = \quad f(x) \quad -\infty < x < \infty$$

where $f(x)$ is a continuously differentiable function, $\tau_s = (-1/f_x)_{min}$.

In summary, we have seen that nonlinearity brings about a progressive deformation of the initial wave profile. Singularities can form independent of how smooth the initial data is. We can explicitly determine the time τ_s needed for the shock to form.

6.2.3 Exercise

Consider again the more general initial value problem (6.6, 6.7)

$$u_t + c(u)u_x = 0$$

where c is some given function of u and

$$u = f(x)\ , t = 0\ , -\infty < x < \infty.$$

a) We have seen that in the case of a general $c(u)$ the characteristic curves are solutions of

$$\frac{dx}{dt} = c(u).$$

b) Argue that the solution of this problem is given by

$$u = f(\xi), \tag{6.19}$$

for f a continuously differentiable function where $\xi = \xi(x,t)$ is implicitly defined by

$$x = \xi + F(\xi)t \tag{6.20}$$

where $F(\xi) = c(f(\xi))$; in other words, u is again constant along characteristic curves.

c) Show that the evolving profile attains a vertical slope (i.e. u_x becomes infinite) at time

$$\tau_s = \left(-\frac{1}{F'(\xi)}\right)_{min}.$$

(Hint: $u_x = f'(\xi)\xi_x$; now, implicitly differentiate (6.20) to find ξ_x.)

Thus we see that in order for a vertical steepening or "shock" to be possible two things are necessary:

(1) $F'(\xi)$ must be negative (the shock occurs in the future);

(2) $F'(\xi)$ must have a minimum in the region where $F'(\xi)$ is negative (or if F is twice differentiable, $F_{\xi\xi} = 0$ in this region).

This means that the (first) shock emerges from a point where the initial data has an inflection point and negative slope.

If these conditions are not met, there is still the possibility of nonlinear spreading, or *rarefaction*. We will not discuss that here; however, for a complete and unified treatment of shock formation and rarefaction in the inviscid Burgers equation and its application to modelling interesting phenomena such as traffic flow, we refer to the excellent text by G.B. Whitham: LINEAR AND NONLINEAR WAVES.

6.2.4 Quasilinear First Order Partial Differential Equations

In this section we will present a generalization of the above method of characteristics to a general class of first order partial differential equations.

Consider first order partial differential equations of the form

$$f(x,y,u)u_x + g(x,y,u)u_y = h(x,y,u) \tag{6.21}$$

with initial data given parametrically as $x = x(\alpha), y = y(\alpha), u = u(\alpha)$. This differs from the linear hyperbolic equation we considered in chapter 5 in that the coefficient functions here, f, g, h, can depend on u so this equation is nonlinear. Such an equation is called *quasilinear* because, although it is nonlinear, it is linear in derivatives. The *characteristic curves* for this equation are defined to be the solutions of the ODE system

$$\begin{aligned} dx/ds &= f(x,y,u) \\ dy/ds &= g(x,y,u) \\ du/ds &= h(x,y,u). \end{aligned} \tag{6.22}$$

We will need to make some assumptions about (6.21) in order to give a complete description of its solutions. We assume that the system (6.22) has no fixed points; i.e., $(f,g,h) \neq (0,0,0)$ anywhere in the domain of consideration. We also assume that the initial data curve is *non-characteristic*; i.e., it is nowhere tangent to a solution of the ode system (6.22).

Notice that if $f = 1, g = u$ and $h = 0$, then (6.21) reduces to $u_x + uu_y = 0$ which is just the nonlinear equation (6.2) we considered in the previous sections and (6.22) reduces to the definition of characteristics for that equation.

We are going to show how to construct, explicitly and geometrically, a possibly multivalued solution of the initial value problem (6.21).

First, suppose that $u = \phi(x,y)$ is a single-valued solution of (6.21) whose graph is a surface S in (x,y,u)-space. We claim that if $(x_0,y_0,u_0) \in S$, then the characterisitic through (x_0,y_0,u_0) must lie on the surface S. This follows because the curve C determined by

$$\begin{aligned} dx/dt &= f(x,y,\phi(x,y)) \\ dy/dt &= g(x,y,\phi(x,y)) \\ u &= \phi(x,y) \end{aligned}$$

lies on S (by the last equation), and along C

$$\begin{aligned} du/dt &= \phi_x dx/dt + \phi_y dy/dt \\ &= \phi_x f + \phi_y g \\ &= h. \end{aligned}$$

Thus C is an integral curve of the characteristic equations (6.22) and passes through (x_0,y_0,u_0); hence, C coincides with the characteristic through this point by existence and uniqueness for ODE's.

Conversely, suppose we start with some (non-characteristic) curve defined parametrically by $x = x(\alpha), y = y(\alpha), u = u(\alpha)$ and consider the family of characteristics emanating from this curve. The union of this family of characteristics is a surface which, globally, is not necessarily the graph of a function of (x,y). However it can be regarded as a possibly multivalued solution of the PDE, since solving the PDE is equivalent to saying that $du/dt = h$ along any curve defined by $dx/dt = f, dy/dt = g$. This is the condition we used to construct the above family of characteristics.

Exercise: Solve the initial value problem

$$u_t - u_x = e^{-u}, \;\; u(x,0) = h(x).$$

For a classical treatment of the method of characteristics and an example of its application (to the study of shock formation in compressible fluid flow) we refer the reader to SUPERSONIC FLOW AND SHOCK WAVES by Courant and Friedrichs. For a modern mathematical treatment we recommend SHOCK WAVES AND REACTION DIFFUSION EQUATIONS by J. Smoller.

6.3 Nonlinearity and Diffusion: Burgers Equation, the Method of Laplace, and Shocks

6.3.1 Burgers equation

Compare the following evolution equations:

$$\begin{array}{llll} a) \; u_t + cu_x - \mu u_{xx} = 0; & \text{c}, \mu \text{ const}, \mu > 0 & \text{(linear)} \\ b) \; u_t + uu_x - \mu u_{xx} = 0; & \text{Burgers equation} & \text{(nonlinear)} \end{array}$$

Setting $u(x,t) = ae^{i(kx-\omega t)}$ in (a) we get the dispersion relation for a linear diffusive wave:

$$\omega = ck - i\mu k^2 \tag{6.23}$$

and so

$$u(x,t) = ae^{-t/t_0}e^{ik(x-ct)} \tag{6.24}$$

with $t_0 = 1/\mu k^2$ i.e. t_0 becomes smaller as k increases and high k modes are rapidly damped out. The parameter "μ" can then be interpreted as a diffusion coefficient.

Consider now problem (b) and look for traveling wave solutions $u(x,t) = u(\zeta),\ \zeta = x - ct$ where now "c" plays the role of a parameter to be determined.

This transformation yields the second order ode

$$-cu_\zeta + uu_\zeta - \mu u_{\zeta\zeta} = 0 \tag{6.25}$$

which on integration yields,

$$-cu + \frac{1}{2}u^2 - \mu u_\zeta = A \tag{6.26}$$

where A is a constant of integration.
Rewriting (6.26) as

$$u_\zeta = \frac{1}{2\mu}(u^2 - 2cu - 2A) = \frac{1}{2\mu}(u - u_\infty^-)(u - u_\infty^+) \tag{6.27}$$

where

$$\begin{aligned} u_\infty^+ &= c - \sqrt{c^2 + 2A} \\ u_\infty^- &= c + \sqrt{c^2 + 2A} \end{aligned}$$

are roots of

$$u^2 - 2cu - 2A = 0. \tag{6.28}$$

To ensure that the roots $u_\infty^\pm$ are real, we shall assume that $c^2 + 2A > 0$ and then $u_\infty^- > u_\infty^+$

A solution of(6.27) is

$$u(x,t) = c - \sqrt{c^2 + 2A}\tanh\left(\frac{\sqrt{c^2+2A}}{2\mu}\zeta\right) \tag{6.29}$$
$$= \frac{1}{2}\left\{(u_\infty^- + u_\infty^+) - (u_\infty^- - u_\infty^+)\tanh\left[\frac{u_\infty^- - u_\infty^+}{4\mu}(x - \frac{1}{2}(u_\infty^- + u_\infty^+)t)\right]\right\}.$$

Therefore, the solution (6.29) joins the two asymptotic states u_∞^- at $\zeta = -\infty$ and u_∞^+ at $\zeta = +\infty$ through continuously varying states (see figure 6.5). Note that

$$c = \frac{1}{2}(u_\infty^- + u_\infty^+). \tag{6.30}$$

The solution (6.29) gives the structure of the shock wave. In fact, as $\mu \to 0$ the characteristic width of the shock narrows and in the limit we recover the jump discontinuity associated with a weak solution of the equation $u_t + uu_x = 0$.

On the other hand, the traveling wave (steady) solution to the linearized Burgers equation

$$u_t - \mu u_{xx} = 0 \tag{6.31}$$

is

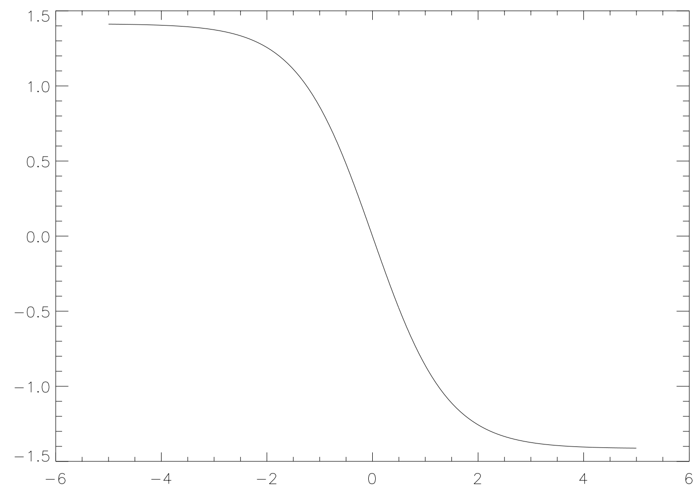

FIGURE 6.5. Traveling Front

$$u(x,t) = u(\zeta) = A + Be^{-\frac{c}{\mu}\zeta} \qquad \zeta = x - ct \tag{6.32}$$

When

$$\begin{aligned} \zeta &\to +\infty, & u &\to A \\ \zeta &\to -\infty, & u &\to \infty \end{aligned}$$

Consequently, the only bounded solution to (6.31) is the constant state. The linearized form of Burgers equation does not admit a solution joining two uniform states through continuously varying states.

We conclude that the nonlinearity of the Burgers equation achieves a smooth joining of two asymptotic uniform states through continuously varying states. We have already seen that if there is a region of negative slope on a pulse profile, then the solution of

$$u_t + uu_x = 0 \tag{6.33}$$

develops a very steep slope and eventually a discontinuity. The term "μu_{xx}" in the Burgers equation smooths out such a discontinuity.

6.3.2 The Cole-Hopf Transformation

We will now present a method to construct more general classes of solutions to Burgers equation than just travelling waves. This method is based on a remarkable observation. Let us apply the following nonlinear transformation

$$u = -2\mu(\ln \varphi)_x \tag{6.34}$$

to Burgers equation

$$u_t + uu_x - \mu u_{xx} = 0. \tag{6.35}$$

We will carry out the transformation in two steps

1. Rewrite the equation in terms of a velocity potential ψ defined by $u = \psi_x$,

$$\psi_{xt} + \psi_x\psi_{xx} - \mu\psi_{xxx} = 0, \tag{6.36}$$

and integrate once w.r.t. x, with zero integration constant, to get

$$\psi_t + \frac{1}{2}\psi_x^2 - \mu\psi_{xx} = 0.$$

2. Now set $\psi = -2\mu \log \varphi$ and substitute into this equation which yields

$$-2\mu\frac{\varphi_t}{\varphi} + 2\mu^2\frac{\varphi_x^2}{\varphi^2} + 2\mu^2\frac{\left[\varphi\varphi_{xx} - \varphi_x^2\right]}{\varphi^2} = 0\,,$$

which then reduces to

$$\varphi_t = \mu\varphi_{xx}. \tag{6.37}$$

The Cole-Hopf transformation transforms the nonlinear Burgers equation (6.3.1) into a linear diffusion equation (6.37)!

Now we can take solutions to the linear diffusion equation (which are simple to obtain by standard means) and work backwards to construct solutions to the nonlinear Burgers equation.

Exercise: We want to consider a multidimensional version of the velocity potential form of Burgers equation:

$$\begin{aligned} \psi_t - \mu\Delta\psi + c|\nabla\psi|^2 &= 0 \ (x,t) \in \mathbb{R}^n \times (0,\infty) \\ \psi &= g \text{ on } \mathbb{R}^n \times \{t=0\}, \end{aligned}$$

where $\mu > 0$. Show that if a transformation $w = \Gamma(\psi)$ is to convert the above equation to a linear equation in w, then Γ must be of the form $w = \exp(-c\psi/\mu)$.

6.3.3 The Lax-Oleinik Representation

Suppose that we are given an initial condition of the Burgers equation:

$$u(x,0) = f(x) \qquad -\infty < x < \infty. \tag{6.38}$$

The Cole-Hopf transformation

$$u = -2\mu(\log\varphi)_x$$

allows us to write

$$\varphi(x,t) = e^{-\frac{1}{2\mu}\int^x u(\zeta,t)d\zeta}$$

so that the corresponding initial conditions for the heat equation

$$\varphi_t = \mu\varphi_{xx}$$

is

$$\begin{aligned}\varphi(x,0) &= e^{-\frac{1}{2\mu}\int_0^x u(\eta,0)d\eta} &(6.39)\\ &= e^{-\frac{1}{2\mu}\int_0^x f(\eta)d\eta} &(6.40)\\ &= \Phi(x)\end{aligned}$$

(Note that we have taken the lower bound in the integral to be zero. However, we could have chosen it to be any value without affecting $u(x,0)$ since the Cole-Hopf map is a logarithmic derivative and is therefore insensitive to changes by a scalar factor.)

The explicit solution of the heat (diffusion) equation (6.37), subject to initial condition (6.39) is

$$\begin{aligned}\varphi(x,t) &= \frac{1}{(\pi\mu t)^{1/2}}\int_{-\infty}^{\infty}\Phi(\xi)e^{\left[-\frac{(x-\xi)^2}{4\mu t}\right]}d\xi\\ &= \frac{1}{(\pi\mu t)^{1/2}}\int_{-\infty}^{\infty}e^{\left\{-\frac{1}{2\mu}\left[\frac{(x-\xi)^2}{2t}+\int_0^\xi f(\eta)d\eta\right]\right\}}d\xi\end{aligned}$$

Substituting into the Cole-Hopf transformation, we obtain the solution to the initial value problem for Burgers equation

$$u^\mu(x,t) = \frac{\int_{-\infty}^{\infty}\frac{x-\xi}{t}e^{-\frac{1}{2\mu}I(x,\xi,t)}d\xi}{\int_{-\infty}^{\infty}e^{-\frac{1}{2\mu}I(x,\xi,t)}d\xi} \tag{6.41}$$

where

$$I(x,\xi,t) = \frac{(x-\xi)^2}{2t} + \int_0^\xi f(\eta)d\eta.$$

This is called the Lax-Oleinik representation.

We can view the Cole-Hopf transformation as a "nonlinear superposition principle" whereby relatively simple solutions to a linear diffusive pde initial value problem can be used to construct fully nonlinear solutions to Burgers equation. The latter solution can exhibit self-steepening and shock formation while the linear problem from which it is derived cannot!

This idea of associating a linear problem with a nonlinear evolution equation through some nonlinear transformation is the key to identifying and solving various "soliton" equations. The latter refer to nonlinear "dispersive" rather than nonlinear "diffusive" equations such as the Burgers equation discussed here.

6.3.4 Laplace's Method

One can view equation (6.41) as giving a sequence of solutions $u^\mu(x,t)$ to the sequence of Burgers equations indexed by μ in which the initial data is held fixed, independent of μ: $u^\mu(x,0) = f(x)$. A natural question to ask is what is the limiting behaviour of $u^\mu(x,t)$ as $\mu \to 0$? Does the limit exist? What is the relation of the limit to the solutions of the inviscid Burgers equation, $u_t + uu_x = 0$, constructed by the method of characteristics? The answer to these questions can be determined by using Laplace's method which enables the asymptotic evaluation as $\mu \to 0$ of integrals containing expressions of the form $e^{-I/\mu}$ where I is some function.

Consider a class of integrals of the form

$$J(s) = \int_0^\infty e^{-sf(x)} g(x) dx \,,$$

generalizing the Laplace transform, where one assumes that $f(x) \to \infty$ as $x \to \infty$. One can in fact try to write this as a Laplace transform by making the substitution $u = f(x)$ so that

$$J(s) = \int_{f(0)}^\infty e^{-su} h(u) du \,,$$

where $h(u) = g(x)/f'(x)$. This makes sense as long as $f' \neq 0$ anywhere. In this case it can be shown (Watson's Lemma) that

$$J(s) = O(1/s).$$

On the other hand, if $f'(x) = 0$ somewhere, then this is not valid. Suppose there is only one critical point, x_0, and that it is a *simple* critical point, i.e., $f'(x_0) = 0$ but $f''(x_0) \neq 0$. The following exercise develops a classical result about normal forms of simple critical points.

Exercise: Morse Lemma in $\mathbb{R}$. Suppose that a twice differentiable function $f(x)$ on $\mathbb{R}$ has a critical point at $x = a$ and that $f''(a) \neq 0$. (Such a critical point is called non-degenerate.) Show that there is a local change of coordinates $x \to y$ near a such that in these new coordinates

$$f = f(a) + f''(a)y^2.$$

Hint: Use the elementary result from calculus:
For any twice differentiable function $f(x)$ on $\mathbb{R}^1$ there is another function $k(x)$ such that

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 k(x).$$

With this, one may approximate f by a parabola in the vicinity of x_0. The part of the integral *outside* the interval where this approximation is valid gives an $O(1/s)$ contribution to the total integral, by Watson's lemma. The dominant contribution comes from within this interval. There, using the Morse lemma, one makes a change of variables $x = x(\sigma)$ such that

$$\sigma \sim (x - x_0)\sqrt{\frac{f''(x_0)}{2}}.$$

near x_0. The final asymptotic result is

$$I(s) = \sqrt{\frac{2\pi}{sf''(x_0)}}e^{-sf(x_0)}g(x_0) + O(1/s).$$

6.3.5 Inviscid Limit of Solutions to Burgers Equation

In the integral

$$\int_{-\infty}^{\infty} e^{-I(\xi)/\mu} d\xi,$$

it is plausible that the leading order (in μ) contribution to this integral should come from the vicinity of points where I has a minimum. In fact it is not hard to show that

Lemma 6.3.1 *Suppose that $k, \ell : \mathbb{R} \to \mathbb{R}$ are continuous, ℓ grows at most linearly and k grows at least quadratically. Assume further that there is a unique point $\xi_0 \in \mathbb{R}$ such that*

$$k(\xi_0) = \min_{y \in \mathbb{R}} \ k(\xi);$$

then

$$\lim_{\mu \to 0} \frac{\int_{-\infty}^{\infty} \ell(\xi) e^{-\frac{1}{\mu}k(\xi)} d\xi}{\int_{-\infty}^{\infty} e^{-\frac{1}{\mu}k(\xi)} d\xi} = \ell(\xi_0).$$

Proof. The growth conditions on k and ℓ insure convergence of the integrals. Set $k_0 = k(\xi_0)$. Then

$$\chi^\mu(y) =_{df} \frac{e^{\frac{k_0 - k(y)}{\mu}}}{\int_{-\infty}^{\infty} e^{\frac{k_0 - k(z)}{\mu}} dz},$$

for $y \in \mathbb{R}$, satisfies

i) $\chi^\mu \geq 0, \quad \forall \mu;$

ii) $\int_{-\infty}^{\infty} \chi^\mu(y) dy = 1, \quad \forall \mu;$

iii) and $\chi^\mu(y) \to 0$ exponentially fast as $\mu \to 0$ for $y \neq \xi_0$.

Thus χ^μ is a delta sequence and so one may conclude that

$$\lim_{\mu \to 0} \int_{-\infty}^{\infty} \ell(y) \chi^\mu(y) dy = \ell(\xi_0).$$

It is immediate that this integral is equal to the ratio of integrals in the statement of the lemma. ∎

For most (x, t), $I(x, \xi, t)$ in (6.41) will have a unique minimum which we will denote by $\xi_0(x, t)$. This minimum is a critical point of I and therefore satisfies the equation

$$0 = I' = -\frac{x - \xi_0}{t} + f(\xi_0).$$

By the lemma we also see that

$$\lim_{\mu \to 0} u^\mu(x, t) = \frac{x - \xi_0(x, t)}{t}.$$

Combining these two observations we find that

$$\lim_{\mu \to 0} u^\mu(x, t) = f(\xi)$$

along the line where $x = \xi + t f(\xi)$ which is precisely the characteristic solution of the conservation law $u_t + u u_x = 0$ which we looked at earlier. The difference arises when for some (x, t), $I(x, \xi, t)$ does not have a unique minimum. This coincidence of minima typically occurs along curves in the (x, t) plane as in Figure 6.6.

One minimum point will be to the left of this curve, denoted $\xi_l(x, t)$, and one will be to the right, denoted $\xi_r(x, t)$. The characteristic lines emerging from these two points meet at (x, t) and typically "carry" different values;

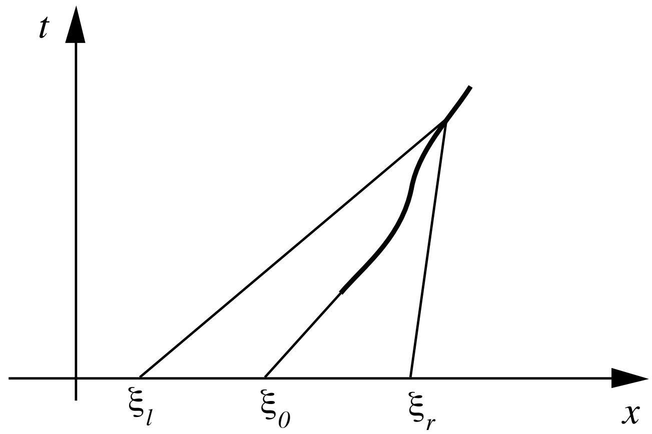

FIGURE 6.6. Coincidence of minima

i.e. $f(\xi_l) \neq f(\xi_r)$. Thus $\lim_{\mu \to 0} u^\mu$ will have a jump discontinuity along this curve corresponding to coincident minima. If one could find an apriroi way to determine the curves of discontinuity in the (x, t) plane one would have a complete description of the inviscid limit. In fact there is such a construction due to the following clever argument originally formulated by Maxwell. A solution of the inviscid Burgers equation necessarily conserves its total integral (physically, its total momentum) in x (we assume that u and u_x vanish at $\pm\infty$) under the time evolution:

$$
\begin{aligned}
& d/dt \int_{-\infty}^{\infty} u(x,t)dx \\
= & \int_{-\infty}^{\infty} u_t dx \\
= & \int_{-\infty}^{\infty} \frac{1}{2}(u^2)_x dx \\
= & \ 0.
\end{aligned}
$$

Similarly, one can show that for solutions of the viscous equation

$$d/dt \int_{-\infty}^{\infty} u^\mu(x,t)dx = 0$$

for all μ. (Show this!) These conservation laws suffice to establish the Maxwell Equal Area Rule.

This rule states that at each time, the jump discontinuity must be placed at a location x_s such that the vertical axis through it in the (x, u)-plane cuts off 'lobes" of equal area on the multi-valued solution. This is illustrated in Figure 6.7.

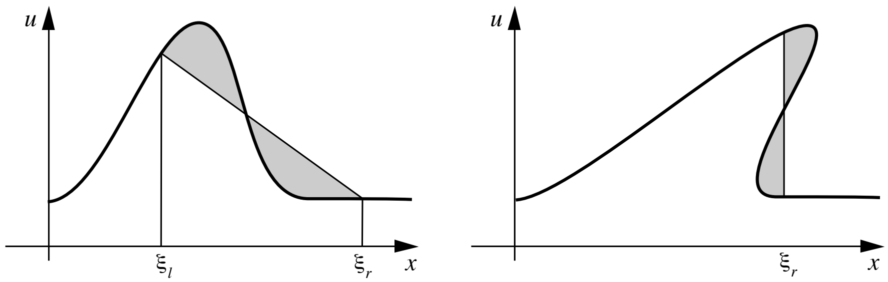

FIGURE 6.7. Maxwell Rule

Momentum conservation also holds for the multi-valued solution as well as the inviscid limit of the Burgers solution. To see this we can argue as follows (below u^0 is just the solution of the inviscid Burgers equation by the method of characteristics, which may be multivalued, and $\bar{u} = \lim_{\mu\to 0} u^\mu$):

$$\begin{array}{ccll} \int u^\mu(x,0) & = & \int u^\mu(x,t) & \text{viscous area conservation} \\ \| & & \| & \\ \int \bar{u}(x,0) & & \int \bar{u}(x,t) & \text{holds } \forall\mu \Rightarrow \text{holds in limit } \mu\to 0 \\ \| & & & \\ \int u^\circ(x,0) & = & \int u^\circ(x,t) & \text{inviscid area conservation} \end{array}$$

$$\Rightarrow \int u^\circ(x,t) = \int \bar{u}(x,t) \quad \forall t$$

$\Rightarrow$ shock cuts off equal area lobes.

On the left hand side, the vertical equalities hold because we take the inital data to be the same for all μ (including $\mu = 0$ which is the inviscid case).

Note that in calculating the integral of u^0 one must explain what this means when this solution becomes multi-valued. In this case one interprets the integral as the sum of the *signed* areas under each of its branches. Using the linearity of the integral it is not hard to check that the argument for momentum conservation extends to this more general setting.

We want to introduce one more picture which helps to understand the minimization of I. It is equivalent to think of this in terms of finding the maximal separation between $-\int_{-\infty}^{\xi} f(\eta)d\eta$ and the parabola $\frac{(x-\xi)^2}{2t}$ centered at x.

Alternatively one can find $\xi_0(x,t)$ by sliding the parabola in the positive vertical direction and then bringing it back down until it first becomes tangent to the graph of the negative velocity potential. The point over which the tangency occurs will be $\xi_0(x,t)$ (see Figure 6.8). Therefore (x,t) will lie on a shock curve if the parabola becomes simultaneously tangent at two different points (see Figure 6.9). This picture shows clearly that for sufficently small time, since the parabola will be very narrow, a shock will

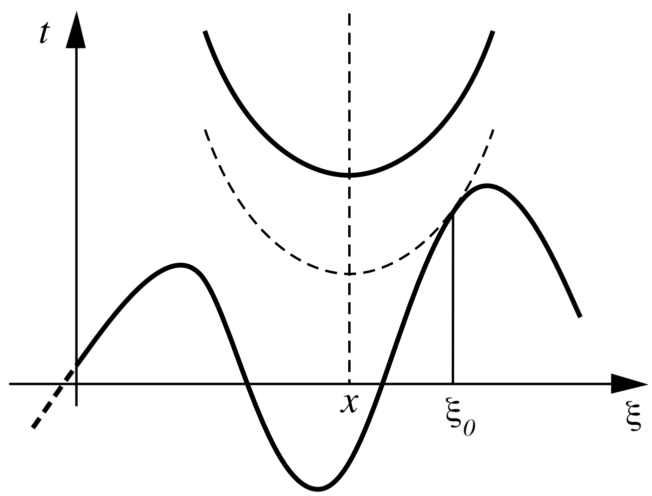

FIGURE 6.8. Sliding Parabola

not occur. However, for large time the parabola widens and the likelihood of a shock forming increases.

One appealing feature of the representation (6.41) for solutions of Burgers equation is that it allows one to consider solutions for rough initial data. All that is required is that the integral of the data make sense. For instance one could consider random white noise as initial data. One can then consider the inviscid limit of the resulting "solution" and ask questions about the statistics of shocks. Using the sliding parabola picture above, do you think one could draw any qualitative conclusions about the distribution of shocks in this model?

In the remaining subsections we will look at some special initial data and consider the behaviour of u^μ as μ gets very small but is not zero.

6.3.6 Single Hump Solution:

Consider a displacement of the general form of Figure 6.10, where $u \to 0$ as $x \to \pm\infty$. This single hump solution can be constructed from the following similarity solution to the linear diffusion equation $\phi_t = \mu\phi_{xx}$:

$$\varphi(x,t) = c_1 + c_2 \int_{\frac{x}{2\sqrt{\mu t}}}^{\infty} e^{-y^2/2} dy.$$

The integrated term in this expression is the error function $\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{x}{\sqrt{4\mu t}}\right)$. The ratio $x/2\sqrt{\mu t}$ is called a *similarity variable* and this solution, which is a function of this ratio, is called a similarity solution. The constants c_1

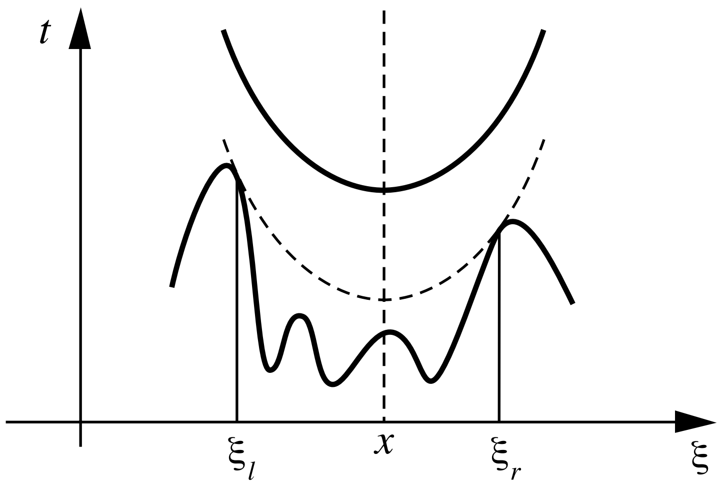

FIGURE 6.9. Shock formation corresponds to double tangencies of parabola

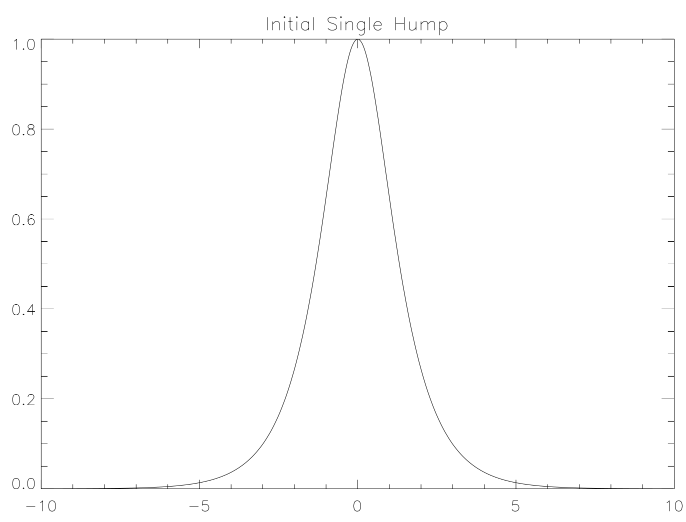

FIGURE 6.10. A Single Hump

and c_2 are chosen such that $u \to 0$ as $x \to \pm\infty$ consistent with the above picture.

Using the Cole-Hopf transformation

$$\varphi(x,t) = e^{\frac{1}{2\mu}\int_x^{\infty} u dx} \tag{6.42}$$

we find that

as $x \to \infty$,

$$\varphi(x,t) \to e^0 = c_1 = 1;$$

as $x \to -\infty$

$$\varphi(x,t) \to e^{\frac{1}{2\mu}\int_{-\infty}^{\infty} u dx} = c_1 + c_2 \int_{-\infty}^{\infty} e^{-y^2/2} dy.$$

Now the quantity $\int_{-\infty}^{\infty} u dx$ is constant, as observed earlier in this section. We define a non-dimensional number $R = \frac{1}{2\mu}\int_{-\infty}^{\infty} u dx$ and call it the Reynold's number. This can be interpreted as the area under the pulse (product of velocity and length scales for such a pulse) divided by the coefficient of viscous diffusion

$$\Rightarrow c_1 = 1, \quad c_2 = \frac{e^R - 1}{\sqrt{2\pi}}.$$

The Cole-Hopf transformation now gives

$$\begin{aligned} u(x,t) &= -2\mu(\log\varphi)_x = -2\mu\frac{\varphi_x}{\varphi} \\ &= \sqrt{\mu/t}\frac{e^{-x^2/4\mu t}}{\frac{\sqrt{2\pi}}{e^R - 1} + \int_{\frac{x}{2\sqrt{\mu t}}}^{\infty} e^{-y^2/2} dy}. \end{aligned}$$

The form of this solution depends on the magnitude of the Reynold's number R of the initial profile. (Remember that R remains constant for all time). Its temporal evolution is shown in Figure 6.11.

When $R \approx 0$ the first term in the denominator dominates over the integral term and we recover the usual characteristic diffusion.

For R large, the solution exhibits the shock structure discussed earlier. However, the viscous term $\mu\varphi_{xx}$ prevents the solutions from becoming multi-valued.

6.3.7 Planar N- Wave

This example has a physical realization as a spherical explosion in the context of gas dynamics. Look for a solution $\varphi(x,t)$ of the linear diffusion

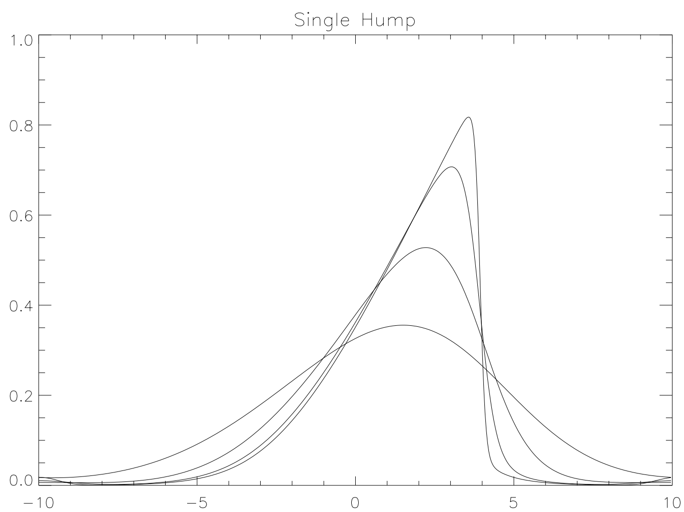

FIGURE 6.11. Evolved Single Hump for Increasing Values of R

equation which is even about $x = 0$. Then the corresponding solution to the Burgers equation, proportional to φ_x, is odd about $x = 0$. We also require that the Burgers solution vanish at $x = \pm\infty$ for all time to accord with the model of an explosion. Such a solution to the diffusion equation is

$$\varphi(x,t) = 1 + (t_0/t)^{1/2}\, e^{-x^2/4\mu t}$$

where t_0 is a constant.

The corresponding solution to the Burgers equation is

$$u(x,t) = -2\mu\frac{\varphi_x}{\varphi} = \frac{x/t}{1 + (t/t_0)^{1/2} e^{x^2/4\mu t}}.$$

We define the Reynold's number in this case as the area, A, under one lobe divided by 2μ (since the area under the full wave is equal to zero);

$$\text{i.e. } R = \frac{A}{2\mu} = \frac{1}{2\mu}\int_0^\infty u\,dx = \log\varphi(0,t) = \log\left[1 + (t_0/t)^{1/2}\right]$$

Note that R is no longer constant and decays to zero with time.

With this definition of R, the planar N wave solution may be written as

$$u(x,t) = \frac{x/t}{1 + (e^{x^2/4\mu t})/(e^R - 1)}$$

For $R << 1$ the exponential term $(e^R - 1) << 1$ so that, approximately, $u(x,t)$ is simply a time dependent multiple of a differentiated Gaussian,

$$u(x,t) \simeq \frac{t_0 x}{t^{3/2}} e^{-x^2/4\mu t}$$

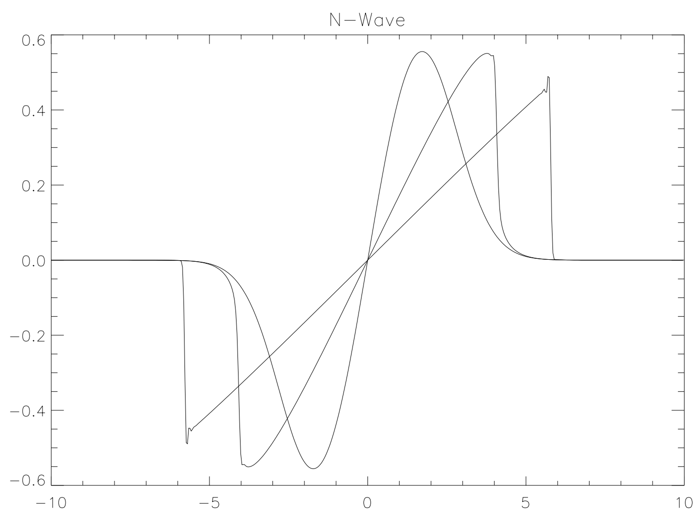

FIGURE 6.12. Evolution of N-Wave

(use $e^R = 1 + (\frac{t_0}{t})^{1/2}$). This occurs for μ very large or, later in time, at the stage of the solution where R has decayed sufficiently from its initial large value, so that the advection has died out, and the flow is essentially diffusive.

For μ very small, when $R >> 1$,

$$\begin{aligned} u(x,t) &\simeq \frac{x}{t}\{1 + e^{(x^2/4\mu t - R)}\}^{-1} \\ &= \frac{x}{t}\{1 + e^{\frac{1}{2\mu}(x^2/2t - A)}\}^{-1} \end{aligned}$$

so that

$$u \sim \begin{array}{lll} x/t, |x| & < & (2tA)^{1/2} - \delta \\ 0, |x| & > & (2tA)^{1/2} + \delta \end{array}$$

where $(-\delta, \delta)$ is a small window about the front whose size depends on the smallness of μ.

When $\mu \to 0$, the characteristic width of the front narrows (i.e. $\delta \to 0$) and this becomes the inviscid solution to Burgers equation (see Figure 6.12).

The shock center x_s may be found by locating the outer inflection points: for example, the right one which occurs, approximately, where u is midway between the maximum of the discontinuous profile and zero; i.e., where $u \simeq \frac{1}{2}x_s/t$ or

$$1 + e^{x_s^2/4\mu t}/(e^R - 1) \quad \simeq \quad 2 \Rightarrow e^{x_s^2/4\mu t} \simeq e^R - 1$$

$$x_s \simeq \sqrt{4\mu t \ln(e^R - 1)} \quad = \quad \sqrt{4\mu t \ln(t_0/t)^{1/2}}.$$

The last expression describes the evolution of the shock front.
Exercise: Estimate the rate at which the maximum of the N-wave profile decays.

Digression: Decay Estimates. A natural question to ask about the solution of a differential equation is whether and how that solution grows or decays asymptotically in time (i.e. as $t \to \infty$). For the solution of a PDE there are several ways to measure growth or decay corresponding to the several ways one has of measuring the size of the function. Different measures can lead to different answers to the above questions. One way to measure the size of a function is in terms of its *sup norm.* Precisely, if $f(x)$ is a function on $\mathbb{R}$ then it's sup or L^∞ norm is denoted and defined as

$$||f||_\infty = Sup\{f(x)|x \in \mathbb{R}\}.$$

Another natural measure is the *mean-square* or L^2 norm of $f(x)$:

$$||f||_2 = \int_{-\infty}^{\infty} f(x)^2 dx.$$

The Lax-Oleinik formula can be used to describe the asymptotic behaviour of the inviscid limit ($\bar{u} = \lim_{\mu \to 0} u^\mu$) of a solution to Burgers equation as $t \to \infty$ in terms of either of these norms. Precisely, one has the following

Theorem 6.3.1 *(L^∞ estimates) For initial data $f(x)$ which is bounded and integrable on $\mathbb{R}$, there exists a constant C such that the invisicd limit of the solution to Burgers equation with this data satisfies*

$$|\bar{u}(x,t)| \leq \frac{C}{t^{1/2}}$$

for all $x \in \mathbb{R}$, $t > 0$.

Thus $\bar{u} \to 0$ as $t \to \infty$ in L^∞ as one would expect for a diffusive equation. On the other hand one also has the following

Theorem 6.3.2 *(L^1 estimates) If $f(x)$ has compact support containing 0, then there exists a constant C such that*

$$\int_{-\infty}^{\infty} |\bar{u}(x,t) - N(x,t)| dx \leq \frac{C}{t^{1/2}}$$

where

$$N(x,t) = \begin{matrix} \frac{x}{t} & \textit{if} \quad -(pt)^{1/2} < x < (qt)^{1/2} \\ 0 & \textit{otherwise}, \end{matrix}$$

for all $t > 0$ and with

$$p \equiv -2 \min_{y \in \mathbb{R}} \int_{-\infty}^{y} f(x)dx$$
$$q \equiv 2 \max_{y \in \mathbb{R}} \int_{y}^{-\infty} f(x)dx.$$

So u does *not* decay to zero in L^1 unlike the case of the linear heat equation. In fact the momentum conservation law already implies that the L^1-norm of u remains constant in time. This theorm shows that what the solution does in fact evolve towards an N-wave of the form described in section 5.3.7.

For proofs of these decay estimates we refer the reader to PARTIAL DIFFERENTIAL EQUATIONS by Lawrence C. Evans.

6.3.8 Exercise: Periodic Initial Conditions

Consider initial values and boundary conditions

$$\begin{aligned} u(x,0) &= u_0 \sin \frac{\pi x}{l} \quad 0 \le x \le l \\ u(0,t) &= u(l,t) = 0 \quad t > 0 \end{aligned}$$

The Cole-Hopf transform of the initial data is

$$\begin{aligned} \varphi(x,0) &= e^{\frac{u_0}{2\mu} \int_0^x \sin \frac{\pi x}{l} dx} \\ &= e^{\frac{u_0 l}{2\mu\pi}(1-\cos \frac{\pi x}{l})} \end{aligned}$$

The boundary conditions will be satisfied if the corresponding solution of the heat equation is periodic and even:

$$\varphi(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\mu \frac{n^2\pi^2 t}{l^2}} \cos \frac{n\pi x}{l}$$

where

$$\begin{aligned} A_0 &= \frac{1}{l} \int_0^l e^{\frac{-u_0 l}{2\mu\pi}(1-\cos \pi x/l)} dx = e^{-\frac{u_0 l}{2\mu\pi}} I_0(\frac{u_0 l}{2\mu\pi}) \\ A_n &= \frac{2}{l} \int_0^l e^{-\frac{u_0 l}{2\mu\pi}(1-\cos \pi x/l)} \cos \frac{n\pi x}{l} dx \end{aligned}$$

1. Evaluate $u(x,t)$ for this data using Cole-Hopf.
2. Show that this solution recovers the initial data when $t \to 0$.

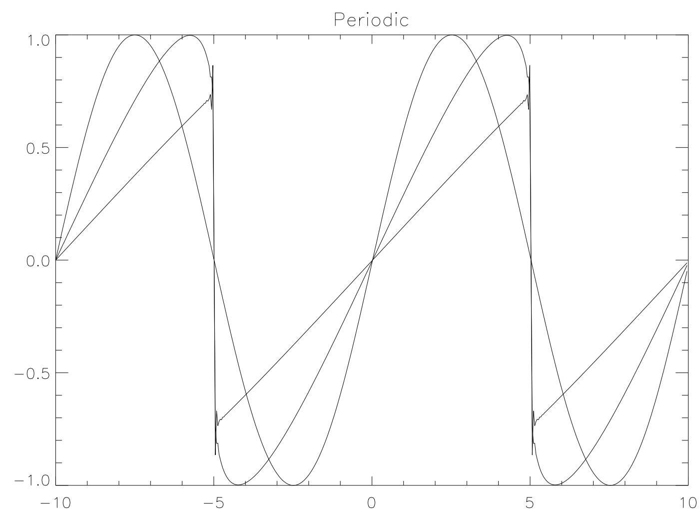

FIGURE 6.13. Evolution of Periodic Wave

3. Take the Reynold's number here, associated to one lobe, to be $R = \frac{u_0}{2\mu}$. Show that if R is small, then for large time u approaches

$$u_0 e^{-\mu\pi^2/l^2 t} \sin(\pi x/l).$$

On the other hand, using the method of Laplace, show that for R large this solution approaches the inviscid solution, a sawtooth, shown in figure 6.13.

4. From this example one learns that i) nonlinearity generates an infinity of higher harmonics with diminishing amplitudes ii) the nonlinear problem depends on the Reynold's number R rather than only u_0 as it would in the linear case. Explain and justify these statements.

6.4 Dispersion: Slowly Varying Wavetrains and the Method of Stationary Phase

The *Fourier Transform* provides an effective tool for analyzing the asymptotic (e.g. long time) behaviour of linearly dispersive waves. It also provides a framework for discussing dispersion in nonlinear wave equations. Therefore we will spend some time in this section developing a linear dispersion analysis.

Recall that the Fourier transform of a function $f(x)$ on $\mathbf{R}$ is

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

which is certainly well-defined if f is in $L^1(\mathbf{R})$. The inverse transform is given by

$$f(x) = \hat{f}^{\vee}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk .$$

We will employ that fact that the inverse Fourier transform carries multiplication to convolution:

$$(f * g) = \frac{1}{\sqrt{2\pi}} (\hat{f}\hat{g})^{\vee} ,$$

where $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ whenever all terms make sense.

A fundamental property of the Fourier transform is the *Riemann-Lebesgue lemma* which states that if f is continuous, then $|\hat{f}(k)| \to 0$ as $|k| \to \infty$. Intuitively, one can believe that a sufficiently nice function is locally constant and therefore, for k sufficiently large, the transform is locally effectively like integrating a constant against a sinusoid which certainly vanishes. The proof is accomplished by a little trick:

By a translation substitution one sees that

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x + \frac{\pi}{2k}) dx .$$

Hence,

$$\begin{aligned} \sqrt{2\pi}\, |\hat{f}(k)| &= \left| \frac{1}{2} \int_{-\infty}^{\infty} e^{-ikx} \{ f(x) - f(x + \frac{\pi}{2k}) \} dx \right| \\ &\leq \frac{1}{2} \int_{-\infty}^{\infty} \left| f(x) - f(x + \frac{\pi}{2k}) \right| dx , \end{aligned}$$

which tends to 0 as $|k| \to \infty$ by continuity.

More regularity gives stronger decay: If f is n-times differentiable then $\hat{f}(k) = o(1/k^n)$.

Exercise:

(a) Show that this decay estimate follows directly from the Riemann-Lebegue lemma.

(b) Calculate the Fourier transform of the following continuous but non-differentiable function: $f(x) =$

$$\begin{array}{rl} 0 & x < -1 \\ x + 1 & -1 < x < 0 \\ -x + 1 & 0 < x < 1 \\ 0 & x > 1. \end{array}$$

How does $\hat{f}(k)$ decay with large $|k|$? Is this consistent with Riemann-Lebesgue and part (a) of this exercise? Explain.

6.4.1 Oscillatory Integrals

An integral depending on a parameter t which is of the form

$$I(t) = \int_{-\infty}^{\infty} e^{itS(y)} g(y) dy$$

may be regarded as a generalization of the Fourier transform in which the *phase* function $tS(y)$ replaces $-kx$. Such an integral is called an *oscillatory integral*. A natural question to ask is whether or when the correlation between the regularity of f and the decay of $\hat{f}$ also holds for a more general oscillatory integral, such as between g and I. An answer is that, with some mild assumptions, if the phase $S(t)$ is differentiable and has no stationary, or critical, points, then the regularity-decay relation is the same. Since $S'(y) \neq 0$ anywhere, one can rewrite the integral as

$$I(t) = \frac{1}{it} \int_{-\infty}^{\infty} \frac{d}{dy} \left(e^{itS(y)} \right) \frac{g(y)}{S'(y)} dy.$$

Integrating by parts gives

$$I(t) = \frac{-1}{it} \int_{-\infty}^{\infty} \frac{d}{dy} \left(\frac{g(y)}{S'(y)} \right) e^{itS(y)} dy + \frac{g(y)}{S'(y)} e^{itS(y)} |_{-\infty}^{\infty}.$$

Thus, as long as $\frac{g(y)}{S'(y)}$ vanishes at $y = \pm\infty$ and $\frac{d}{dy} \left(\frac{g(y)}{S'(y)} \right)$ is integrable, $I(t) = O(1/t)$.

6.4.2 Linear Dispersive Waves

Earlier (in Chapter 5) we discussed two model soliton equations

KdV:

$$u_t - 6uu_x + u_{xxx} = 0\,;$$

and

NLS:

$$q_t - i/2 q_{xx} + i|q|^2 q = 0.$$

The linearization of these equations around the zero solution was used to introduce the dispersion relation. In this section we will consider these as models of linear dispersive wave motion: **Linearized KdV:**

$$u_t + u_{xxx} = 0\,;$$

and

Linearized NLS:

$$q_t - i/2 q_{xx} = 0.$$

If $u(x,0) = f(x)$ and $f, f_x, f_{xx} \to 0$ as $|x| \to \infty$, the solution of linearized KdV may be represented through the Fourier transform as an oscillatory integral

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(kx+k^3 t)} \hat{f}(k) dk$$

Exercise: Show that the solution of Linearized KdV is given by

$$u(x,t) = (3t)^{-1/3} \int_{-\infty}^{\infty} f(y) \mathrm{Ai}\left(\frac{x-y}{3t^{1/3}}\right) dy$$

where

$$\mathrm{Ai}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{i(kz + \frac{1}{3}k^3)\} dk$$

is the Airy function of z. *Remark:* $\mathrm{Ai}(z)$ solves $w_{zz} - zw = 0$.

Similarly, the solution of linearized NLS is given by the oscillatory integral

$$q(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(kx - \frac{1}{2}k^2 t)} \hat{f}(k) dk.$$

where now f and q are complex-valued.

The phases for these integrals are not monotonic; therefore we may expect some new behaviour corresponding to places where the phase has a critical point.

6.4.3 Slowly Varying Wavetrains

Consider the oscillatory integral representation of a solution of linearized NLS in a region where $\frac{x}{t} \approx v$, a fixed constant, and where t (and therefore x) are very large. In this case we can effectively take the integrand to be of the form $e^{it(kv - \frac{1}{2}k^2)} \hat{f}(k)$ so that the phase function $S(k) = (kv - \frac{1}{2}k^2)$ has a critical point at $k = v$. We want to see if this makes a difference in the long time decay rate of the oscillatory integral.

One can evaluate this integral directly using a little bit of complex function theory. The integral representation of q is the inverse Fourier transform of a product

$$q(x,t) = (\hat{f} e^{-itk^2/2})^{\vee} = \sqrt{2\pi} f * (e^{-itk^2/2})^{\vee}.$$

Exercise:

a) Make a change of variables in $(e^{-itk^2/2})^\vee = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(kx-\frac{1}{2}k^2t)}dk$ which reduces this integral to

$$\frac{1}{\sqrt{2\pi}} e^{itv^2/2} \int_{-\infty}^{\infty} e^{-i(t/2)(k-v)^2} dk.$$

b) Show that $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$

c) Use (b) and Cauchy's theorem to show that $\int_{-\infty}^{\infty} e^{-iz^2} dz = e^{i\pi/4}\sqrt{\pi}$.

d) From (a) and (c) conclude that $(e^{-itk^2/2})^\vee = e^{-i\pi/4} e^{itv^2/2}/\sqrt{t}$.

Thus

$$q(x,t) = e^{i\pi/4}\sqrt{\frac{2\pi}{t}} \int_{-\infty}^{\infty} f(y) e^{i(x-y)^2/2t}$$

which is $O(1/\sqrt{t})$ and therefore dominates over the decay we found in the non-stationary case.

Now take $\hat{f} = 1$. Recall that near the stationary point one has

$$k_0(x,t) = v = x/t.$$

Observe that one can write

$$tv^2/2 = (x/t)x - \frac{1}{2}(x^2/t^2)t = k_0 x - \frac{1}{2}k_0^2 t\,.$$

Thus the solution may be asymptotically represented as

$$\sqrt{\frac{1}{t}} e^{-i\pi/4} e^{i(k_0(x,t)x - \frac{1}{2}k_0^2(x,t)t)},$$

when one is in a region where $\frac{x}{t} \approx v$, a fixed constant, and where t (and therefore x) are very large.

Note that

$$\frac{1}{k_0}\frac{dk_0}{dx} = O(1/x).$$

This quantity is a local wavelength times the spatial rate of change of the wavenumber. Since x and t are very large this says that over a few wavelengths, the change in the wavenumber is effectively zero. Similarly one can examine the spatial variation of the frequency:

$$\frac{2\pi}{\omega_0}\frac{d\omega_0}{dx} = O(1/x)\,,$$

where $\omega_0 = \frac{1}{2}k_0^2$. This quantity is a local period times the spatial rate of change of the frequency and once again one sees that over a few periods this remains quite small. One can make a similar assessment for temporal variations. The upshot is that the scale $\frac{1}{k_0}$ is the order of distances over which k_0 and ω_0 are effectively constant and $\frac{1}{\omega_0}$ is the order of times over which these quantities are effectively constant. Thus we see that asymptotically the wave looks locally like a harmonic oscillation but in fact the wavenumbers and frequencies vary slowly in space and time.

Exercise:

1. What is the group velocity v_g in this region?
2. Is the amplitude of the wave big or small in this region? How big or how small?
3. Sketch a snapshot in time of what the wave looks like when v_g is large; when v_g is small.

6.4.4 The Method of Stationary Phase

The asymptotic representation we developed in the last section for solutions of linear Schrödinger extends to general oscillatory integrals having a simple stationary point.

If $S(x)$ has one stationary point at $x = \alpha$ and if $S''(\alpha) > 0$, respectively < 0, then

$$I(t) = \int_{-\infty}^{\infty} e^{itS(y)} g(y) dy = \sqrt{\frac{\pi}{\pm 2tS''(\alpha)}} g(\alpha) e^{itS(\alpha) \pm i\pi/4} + O(1/t)$$

If there is more than one simple critical point, then the asymptotic behaviour is a superposition of contributions of this form, one from each critical point. The representation may also be extended to critical points of higher multiplicity

Exercise:

1. Use the stationary phase formula to reproduce the representation of solutions to linear NLS from the last section.
2. Use the stationary phase formula to describe the asymptotic behaviour of solutions to the linearized KdV equation for large time t.

Exercise: Geometric Optics and the WKB method. There is another method, closely related to stationary phase, called the WKB method, which is also useful in describing the behaviour of oscillatory integrals. This method grew out of the classical theory of geometric optics and provides insight into the formation of caustics and their regularization by diffraction..

We want to develop a rudimentary understanding of the WKB method in this exercise.

Travelling wave solutions of Maxwell's equations in a medium solve the wave equation

$$\Psi_{tt} = c^2/n^2\nabla^2\Psi$$

where $c =$ speed of light in vacuum and $n(x) =$ refractive index. Ignoring polarization, we may remove the dependence on t by taking

$$\Psi = \psi(x)\exp(i\omega t)$$

with $\psi(x)$ a complex scalar wave function and so reduce to the Helmholtz equation

$$\nabla^2\psi + \kappa^2 n^2(x)\psi = 0$$

in which $\kappa = \omega/c$ is the wavenumber in vacuum and $\omega =$ frequency.

We want to understand the behavior of the wave function as $\kappa \to \infty$. Look for a solution in the form of a slowly varying plane wave

$$\psi(x) = a(x)\exp(i\kappa\chi(x))$$

where we take a to have an expansion in κ: $a(x) = \sum_{n=0}^{\infty}\kappa^{-n}a_n(x)$. a is the amplitude and χ is the phase of the wave function.

1) Derive the following equations for the amplitude and phase:

$$\text{REAL} : |\nabla\chi|^2 = n^2 + \frac{1}{\kappa^2}\frac{\nabla^2 a}{a}$$

$$\text{IMAG} : \nabla\cdot(a^2\nabla\chi) = 0.$$

If one neglects the order $\geq \frac{1}{\kappa^2}$ terms the resulting leading order equations are the well known

$$\text{eikonal equation} : |\nabla\chi_0|^2 = n^2$$

and

$$\text{transport equation} : \nabla\cdot(a_0^2\nabla\chi) = 0.$$

Given χ, the level curves $\chi(x) =$ constant are called the *wavefronts* and the integral curves of $\nabla\chi$ are called the *rays*. The formation of caustics can be understood through an analysis of the transport equation.

Consider a tube Ω consisting of rays and cut by two wavefronts S_0, S_1: Let $\hat{\sigma}$ be the normal to this tube.

2) Show that on the sides $\hat{\sigma}\cdot\nabla\chi = 0$ and on the ends $\hat{\sigma}\cdot\nabla\chi = \pm|\nabla\chi| = \pm n$ (+ on S_1, - on S_0).

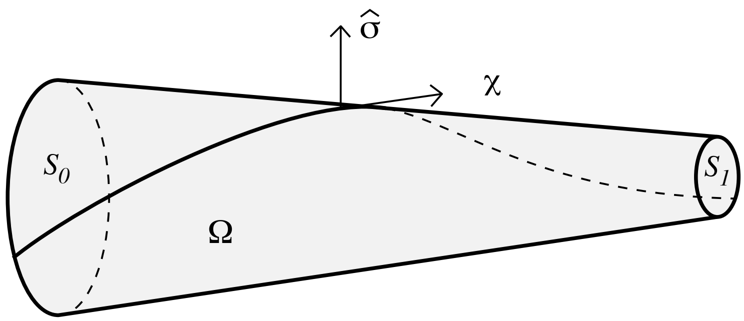

FIGURE 6.14. Ray Tube

3) Use the transport equation and the divergence theorem to show that

$$\int_{S_1} a^2 n ds = \int_{S_0} a^2 n ds.$$

Thus the area integral of $a^2 n$ is constant. If the areas of these cross-sections are taken sufficiently small so that a^2 and n are nearly constant then

$$a^2(x) = \frac{n(x_0)}{n(x)} a^2(x_0) \left(\frac{dA(x)}{dA(x_0)} \right)^{-1}.$$

Thus if $\frac{dA(x)}{dA(x_0)} \to 0$, then $a(x) \to \infty$. This is precisely the condition for an envelope of rays, i.e. a caustic, to form. So our theory fails on caustics and we must bring back the $\frac{1}{\kappa^2}$ terms. This requires one to calculate diffraction integrals which we will mention later and for which our geometric singularity theory can be applied again to approximate the structure of the diffraction fringes.

6.4.5 Nonlinearity and Dispersion: Solitons

In 1965 Zabusky and Kruskal solved the KdV equation numerically with the value 0.022 replacing 6 as the coefficent of the nonlinear term and using the initial condition

$$u(x, 0) = \sin \pi x \qquad 0 \le x \le 2,$$

with u, u_x and u_{xx} required to be periodic on [0,2] for all t. The evolution of their solution is depicted in Figure 6.15. The first figure shows the initial profile at $t = 0$; the second figure represents the profile at $t \approx 1/\pi$; and the final figure is the profile at $t \approx 3.6\pi$. What one observes is that the wave initially steepens, reminiscent of the solution to the inviscid Burgers equation, until a local balance between nonlinearity and dispersion is achieved. At later times, unlike the Burgers equation, a train of eight well defined solitary waves, each (locally) like sech^2 solutions above, develop with the taller

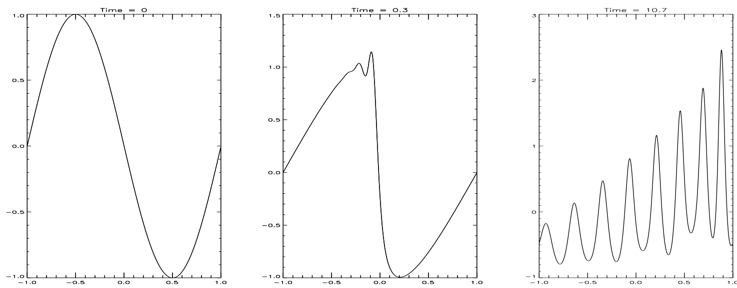

FIGURE 6.15. Kruskal-Zabusky Experiment

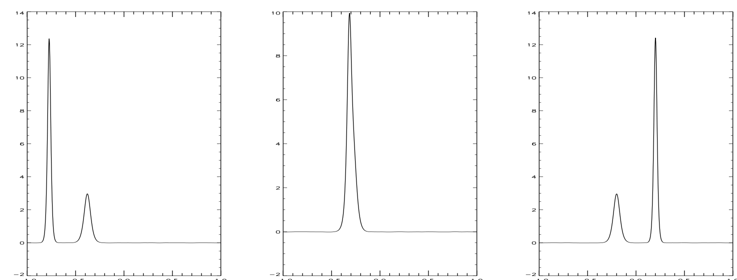

FIGURE 6.16. Two-Soliton Evolution

(faster) ones overtaking the smaller ones. After a long time, the original profile reappears. This is called Fermi-Pasta-Ulam (FPU) recurrence.

The crucial observation in this and many other KdV numerical simulations that were done at that time, is that even though two such solitary waves in the profile interact strongly on collision, they retain their original identities when they emerge from the collision. This is what one would expect for linear waves but these waves are not linear. The only indication that this interaction is in fact *not* linear is that the two emerging waves are phase shifted; that is, they are not in the positions after the collision which they would occupy if they had moved at constant speed through the collision. This behavior is reminiscent of the elastic collision of two particles and led Zabusky and Kruskal to coin the term *soliton* for a solution which appears to consist of an ensemble (i.e., more than one) of these solitary sech^2-like pulses. Figure 6.16 depicts the evolution of a *two-soliton*.

Although it might be tempting to believe that the asymptotic wavetrain obtained for the linearized KdV by the method of stationary phase is a reasonable first approximation for small amplitude solutions of KdV, in fact the asymptotic form of a wavetrain solution to the nonlinear equation is fundamentally different as the discussion above may suggest. Partially this is because although the nonlinearity has a small effect on the wave pattern for $O(1)$ times, the effect over long times is not negligible. However, one important aspect of our linear analysis does have the potential to carry over to the nonlinear setting. This is the idea of representing dispersive waves, linear or not, by an ansatz of slowly varying wavetrains. One can then try to employ a variety of asymptotic methods to describe how the wavenumbers and frequencies depend on slow space and time scales. In the nonlinear but near-integrable setting this will lead us to averaging methods and modulation theory.

6.5 Hamiltonian Systems and Conservation Laws

In chapter 5 we mentioned that certain pde's, such as the KdV and NLS equations were completely integrable. We mentioned that, as a consequence, these equations had many associated constants of motion and also that they had solitary wave solutions. In this section we will begin to explain the structures that underlie this integrability. The most basic of these is that of a *Hamiltonian system*. It is within the framework of Hamiltonian systems that complete integrability is defined. We will explain what it means for a dynamical system to be Hamiltonian first in one degree of freedom (the case of phase plane analysis), then in two degrees of freedom, and finally in infinitely many degrees of freedom. The last is the setting for Hamiltonian pde's of which KdV and NLS are examples.

6.5.1 The Pendulum

We consider again the example of the ode which models the pendulum:

$$u_{tt} + \sin(u) = 0. \tag{6.43}$$

Multiplying this equation by u_t gives

$$u_t u_{tt} + (\sin(u))u_t = 0$$

which is a perfect derivative of

$$\frac{1}{2}(u_t)^2 - \cos(u) = \text{constant}.$$

If one rewrites equation (6.43) as the system

$$\begin{aligned} u_t &= v \\ v_t &= -\sin(u) \end{aligned} \tag{6.44}$$

then the above integral can be written

$$\frac{v^2}{2} + (1 - \cos(u)) = H$$

for constant H. However, since the form of this integral is the same no matter what orbit of the system (6.44) we are considering, we can regard it as a function on the (u, v) phase plane:

$$H(u, v) = \frac{v^2}{2} + (1 - \cos(u)).$$

Thus the level curves of this function (see Figure **??** are just the orbits of the system (6.44). The minima of H which occur at $u = 2n\pi$, $v = 0$ where $H = 0$ are the stable fixed points of the system while the saddle points at $u = (2n+1)\pi$, $v = 0$ where $H = 2$ are the unstable fixed points. The remaining orbits are, topologically, either circles (inside the separatrices) or lines (outside the separatrices). If one takes u to be a periodic variable of period 2π then all the orbits which are not fixed points are circles.

The gradient

$$\nabla H = \begin{pmatrix} \sin(u) \\ v \end{pmatrix}$$

is normal to these level curves. Rotating these gradients 90° counterclockwise they become tangent vectors to the level curves with counterclockwise orientation inside the separatrix and oriented right to left above the separatrix and left to right below the separatrix. This operation can be represented via rotation by the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sin(u) \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\sin(u) \end{pmatrix}$$

which is the RHS of (6.44); i.e., it is the phase vector field for the pendulum. If one sets $\vec{x} = \begin{pmatrix} u \\ v \end{pmatrix}$ then we can summarize all this in the equation

$$\vec{x}_t = J\nabla H(\vec{x}). \tag{6.45}$$

More generally, given *any* differentiable function on the phase plane, one can associate to it a dynamical system given by equation (6.45). For any dynamical system having this form, it is guaranteed that the function H remains invariant under under the phase flow; i.e., $H(\vec{x}(t))$ is constant for all t. To see this, observe first that in coordinates (6.45) reads

$$\begin{aligned} u_t &= \frac{\partial H}{\partial v} \\ v_t &= -\frac{\partial H}{\partial u}. \end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dt}H(\vec{x}(t)) &= \frac{\partial H}{\partial u}u_t + \frac{\partial H}{\partial v}v_t \\ &= \frac{\partial H}{\partial u}\frac{\partial H}{\partial v} - \frac{\partial H}{\partial v}\frac{\partial H}{\partial u} \\ &= 0.\end{aligned}$$

A dynamical system of the form (6.45) is called a *Hamiltonian system* and H is called the *Hamiltonian* of the system.

6.5.2 Higher Dimensional Hamiltonian Systems and Integrability

Consider the example of the two degree of freedom harmonic oscillator whose dynamical system on a four dimensional phase space with coordinates $\vec{x} = (u_1, v_1, u_2, v_2)$ is

$$\begin{aligned}u_{it} &= v_i \\ v_{it} &= -u_i\end{aligned}$$

This is a Hamiltonian system on $\mathbb{R}^4$; i.e., it has the form (6.45) if we now define

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

and take $H = \frac{1}{2}(v_1^2 + v_2^2 + u_1^2 + u_2^2)$. More generally an N degree of freedom Hamiltonian system is a dynamical system of the form (6.45) where J is a $2N \times 2N$ block diagonal matrix 2×2 diagonal blocks each a copy of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The phase flow of the harmonic oscillator is rather simple to describe since there are two obvious quantities that are invariant under this flow: $r_1^2 = u_1^2 + v_1^2$ and $r_2^2 = u_2^2 + v_2^2$. Geometrically this says that the projection of this flow into the coordinate 2-plane (u_i, v_i) maintains a constant radius r_i. In $\mathbb{R}^4$ the product of these two circles is a 2-torus of bi-radius (r_1, r_2). Thus, a phase orbit starting on such a 2-torus will remain on it for all time. The projected flow into each of the coordinate 2-planes is a rotation of constant angular velocity 1. In the identification space representation of the fundamental domain of the torus in the plane this flow is given as a linear flow of constant slope.

The above description is almost completely typical of the topological dynamical structure of a *completely integrable* two degree of freedom Hamiltonian system. The only thing that is not typical is that generally the angular velocities depend on the torus and the linear flow on the fundamental domain has irrational slope, and so is ergodic, on a set of tori of full measure.

There is a concise description of completely integrable Hamiltonian systems. To state it, we first define the *Poisson bracket* of two Hamiltonian functions, F and G by

$$\{F, G\} = \left(\frac{\partial F}{\partial u_1}\frac{\partial G}{\partial v_1} - \frac{\partial F}{\partial v_1}\frac{\partial G}{\partial u_1}\right) + \left(\frac{\partial F}{\partial u_2}\frac{\partial G}{\partial v_2} - \frac{\partial F}{\partial v_2}\frac{\partial G}{\partial u_2}\right).$$

Note that the Poisson bracket of two functions is another function. This bracket may be expressed more compactly as $\{F, G\} = < \nabla F, J\nabla G >$ where $< \cdot, \cdot >$ is the standard Euclidean inner product. In this latter form the bracket generalizes to N degree of freedom Hamiltonian systems. One says that F and G are in involution if $\{F, G\} = 0$. The criterion for complete integrability of a Hamiltonian system is called the *Arnold-Liouville Theorem*:

Theorem 6.5.1 *A two degree of freedom Hamiltonian system with Hamiltonian H is said to be completely integrable on a region of $\mathbb{R}^4$ if there is another function, G, independent of H on this region (meaning that the 2×4 matrix $(\nabla H, \nabla G)$ has a non-vanishing 2×2 minor at every point of the region) such that $\{H, G\} \equiv 0$. In this case there is a canonical (meaning "preserving J") change of coordinates such that in these coordinates the flow is linear on the 2-dimensional common level sets $H =$ constant and $G =$ constant. If the region is bounded, then these level sets are 2-tori.*

The harmonic oscillator satisfies these conditions if one takes $G = r_1^2$. (Check that the hypotheses of the theorem are satisfied in this example on the region where $r_1 \neq 0$ and $r_2 \neq 0$.) This theorem generalizes to N degree of freedom systems by replacing "two" by "N" and requiring that there be N-1 functions, G_i such that $H, G_1, \ldots, G_N$ are independent and pairwise in involution.

6.5.3 Infinite Dimensional Hamiltonian Systems

In the previous section we introduced the Poisson bracket in terms of which we defined what it means for a Hamiltonian system to be completely integrable. The notion of a Poisson bracket can be abstracted to more general settings including infinite dimensional Hilbert spaces. The properties that the bracket on a Hilbert space, $\mathcal{H}$, must have in order to carry out the usual constructions of Hamiltonian systems theory are that it should be a bilinear pairing on the space of differentiable functionals on $\mathcal{H}$ to itself which satisfies

(i) antisymmetry: $\{F, G\} = -\{G, F\}$,
(ii) the product rule: $\{F, GH\} = \{F, G\}H + \{F, H\}G$,

(iii) the Jacobi identity: $\{F,\{G,H\}\}+\{H,\{F,G\}\}+\{G,\{H,F\}\}=0$.

We will illustrate this for the case of the KdV equation which we recall here:

$$u_t - 6uu_x + u_{xxx} = 0.$$

First notice that this can be written in the form of a conservation law:

$$\mathcal{T}_t + \mathcal{X}_x = 0$$

where $\mathcal{T} = u$ and $\mathcal{X} = -3u^2 + u_{xx}$. If we work with either periodic boundary conditions or vanishing boundary conditions at $\pm\infty$ on $\mathbb{R}$, then, as in the case of Burgers' equation, we find that $\frac{d}{dt}\int \mathcal{T} dx = \int \mathcal{X}_x dx = 0$. Thus, we see that $\int \mathcal{T} dx$ is an invariant of the KdV flow. In chapter 9 we will see that there are other invariant functionals of the flow. One might naturally ask if there is a Hamiltonian structure underlying these invariances.

In the early '70's Gardner, Greene, Kruskal and Miura answered this question affirmatively. The underlying Hilbert space is taken to be the space of square-integrable functions on $\mathbb{R}$, $L^2(\mathbb{R})$ or, in the case of periodic boundary conditions, $L^2(S^1)$, square-integrable functions on the circle. The correct choice for the J operator is not obvious, but it turns out that taking $J = -\partial_x$, differentiation, is a very natural choice. Thus the Poisson bracket for KdV is

$$\{F, G\} =< \nabla F, -\partial_x \nabla G >$$

where $< \cdot, \cdot >$ is the standard L^2-inner product on functions

$$< f, g >= \int f(x)g(x)dx\,,$$

and ∇F is the Frechet derivative. We briefly recall that the Frechet derivative generalizes the gradient in $\mathbb{R}^N$ to general Hilbert spaces. Operationally one defines this by saying that for any $v \in \mathcal{H}$,

$$\frac{d}{d\epsilon} F(u + \epsilon v)|_{\epsilon=0} =< \nabla F(u), v > .$$

Now, if one wants to write KdV in the form

$$u_t = J\nabla H$$

with $J = -\partial_x$, then one must have $\nabla H = -3u^2 + u_{xx}$. It is straightforward to check that if one takes the Hamiltonian functional to be

$$H(u) = -\int u^3 + \frac{1}{2}u_x^2 dx\,,$$

then KdV is a Hamiltonian system with this Hamiltonian. (Check this. It is an exercise using integration by parts.) Technically, one needs to restrict to a domain in L^2 on which H is defined (for example, H involves derivatives of u which need not be in L^2) but we will not worry about that here. In chapter 9 we will see that this KdV system has infinitely many functionals in addition to H which are pairwise in involution with respect to the above Poisson structure. KdV is in fact an infinite dimensional completely integrable system.

The NLS equation is also a completely integrable Hamiltonian system. The appropriate Hilbert space for NLS is square-integrable *complex-valued* functions on $\mathbb{R}$ or S^1. In this case the J-operator is just multiplication by $-i$ ($i = \sqrt{-1}$). Hamilton's equations then take the form

$$\begin{aligned} \frac{\partial q}{\partial t} &= -i\frac{\delta H}{\delta \bar{q}} \\ \frac{\partial \bar{q}}{\partial t} &= i\frac{\delta H}{\delta q} \end{aligned} \tag{6.46}$$

with Hamiltonian function

$$H(q, \bar{q}) = \int \frac{1}{2}(|q_z|^2 - |q|^4)dz.$$

Exercise

Show that the system (6.46) is in fact equivalent to the NLS equation.