1 Elementary enumeration

1. How many ways are there to arrange the seven letters (to make a new nonsense word) in the word SYSTEMS?

2. A car dealership employs five salespeople. A salesperson receives 100 dollars for each car he or she sells. Yesterday the dealership sold seven cars. In how many different ways could this happen? (Let us consider two scenarios different if they result in different bonus payments. Thus what matters is how many cars each salesperson sold.)

3. What is the probability that a 4-digit campus telephone number has one or more repeated digits?

4. A cashier wants to work five days a week, but he wants to have at least one of Saturday and Sunday off. In how many ways can he choose the days he will work?

2 Probability

Here are some finite probability models. Here A is an index set (or a set of particles), and B is a population (or a set of states). The set A has k points and the set B has n points.

- Sampling with replacement (ordered): Each function from A to B has the same probability $\frac{1}{n^k}$.
- Sampling with replacement (unordered); Maxwell-Boltzmann: Each multi-index N on B with sum k has the probability $\frac{\binom{k}{N}}{n^k}$. Here $\binom{k}{N}$ is the multinomial coefficient.
- Sampling without replacement (ordered): Each injective function from A to B has the same probability $\frac{1}{(n)_k}$.
- Sampling without replacement (unordered); Fermi-Dirac: Each subset S of B with k elements has the same probability $\frac{1}{\binom{n}{k}}$.
- Bose-Einstein: Each multi-index N on B with sum k has the same probability $\frac{1}{(\binom{n}{k})}$.

1. Take $k = 2, n = 4$. What is the Maxwell-Boltzmann probability that there exists a point at which N assumes the value 2?
2. Take $k = 2, n = 4$. What is the Fermi-Dirac probability that there exists a point at which N assumes the value 2?
3. Take $k = 2, n = 4$. What is the Bose-Einstein probability that there exists a point at which N assumes the value 2?
4. Consider the case where B consists of s success points and $n - s$ failure points. Find the ordered sampling with replacement probability of exactly ℓ out of k successes.
5. Consider the case where B consists of s success points and $n - s$ failure points. Find the ordered sampling without replacement probability of exactly ℓ out of k successes.
6. Consider the case where B consists of s success points and $n - s$ failure points. Find the Fermi-Dirac probability of exactly ℓ out of k successes.
7. What is the relation between the answers to the two previous questions?
8. Consider the case where B consists of s success points and $n - s$ failure points. Find the Bose-Einstein probability of exactly ℓ out of k successes.

3 Pigeonhole principle

1. A busy airport sees 1500 takeoffs per day. Prove that there are two planes that must take off within a minute of each other.

2. One hundred cards, numbered 1 to 100, are distributed among six people. Show that there must be at least one person on whose cards there appear at least four 3s.

3. Consider an ordered sample with replacement of size n from a population of size n. One member of the population is already infected. Each time a member of the population is chosen, it also becomes infected. Let p_k be the probability that k is the first time that an already infected member is chosen. Show that
$$p_k = \frac{(n-1)_{k-1} k n^{n-k}}{n^n}.$$

4. Show that the sum of the p_k as k ranges from 1 to n is equal to 1. (Hint: Use the pigeonhole principle.)

5. Derive the identity
$$n^n = \sum_{k=1}^{n} \binom{n-1}{k-1} n^{n-k} k!.$$
Hint: Use the result of the previous problem. Note: The left hand side n^n is the number of functions from a set to itself. It may be shown that the summands on the right hand side represent the number of functions from a set to itself that have k periodic points.

4 Set partitions

1. The 29-word sentence "Sphinx of black quartz, judge my vow" uses all 26 letters of the alphabet. Consider the permutation that sends the sequence (a b c d e f g h i j k l m n o p q r s t u v w x y z) into the corresponding sequence (s p h i n x o f b l a c k q u r t z j d g e m y v w). Find the decomposition into cycles.

2. Consider the function that sends (a b c d e f g h i j k l m n o p r s w) into (a b l e w a s i e r e i s a w e l b a). Give the partition whose blocks consist of trees feeding into cycles. How many blocks are there? How many points in each block.

3. Consider the function that sends (a b c d e f g h i j k l m n o p q r s t u v w x y z) into (g r e g o r s a m s a a w o k e o n e m o r n i n g). Give the partition whose blocks consist of trees feeding into cycles. For each block of this partition, give the partition into blocks that are trees. How many trees are there? How many points in each tree?

4. The average size of a subset is

$$\mu_n = \frac{\sum_{k=0}^{n} k\binom{n}{k}}{\sum_{k=0}^{n} \binom{n}{k}}.$$

   Find a simple expression for this quantity. Hint: Use the identity for $k\binom{n}{k}$.

5. The average size of a set in a partition is

$$\mu_n = \frac{\sum_{k=0}^{n} kS(n,k)}{\sum_{k=0}^{n} S(n,k)}.$$

   Find an expression for this quantity in terms of Bell numbers B_m.

6. Consider a set with 26 points. What is the average size of a set in a partition? Give an answer accurate to four decimal places.

5 Inclusion-exclusion principle

These are inclusion-exclusion problems from the text. It should be not so much work, because three of the four problems are essentially the same.

1. Mazur, Section 3.1 Exercise 8
2. Mazur, Section 3.1 Exercise 9
3. Mazur, Section 3.1 Exercise 11
4. Mazur, Section 3.1 Exercise 12

6 Ordinary generating functions

1. Find the number h_n of bags of fruit that can be made out of apples, bananas, oranges, and pears, where, in each bag, the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1.

2. How many ways are there to select 25 toys from seven types of toys with between two and six of each type?

3. Show that
$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\!\binom{n}{k}\!\right)x^k y^n = \frac{1}{1-\frac{1}{1-x}y}.$$

4. Use the preceding result to find the ordinary generating function
$$\sum_{n=0}^{\infty}\left(\!\binom{n}{k}\!\right)y^n.$$

7 Exponential generating functions

1. Determine the number h_n of ways to color n squares with the colors red, white, and blue, where the number of red squares is even and there is at least one blue square.

2. Find the number of ways to place 25 people in three rooms, with at least one person in each room.

3. Since $e^x - 1$ generates non-empty sets, it follows that

$$(e^x - 1)^n = \sum_{k=0}^{\infty} n!S(k,n)\frac{x^k}{k!}$$

   generates ordered partitions into n blocks, that is, surjections onto an n element set. Use this exponential generating function to get a formula for $n!S(k,n)$ that may be compared with the formula given by inclusion-exclusion. Make the comparison.

4. Show that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n!S(k,n)\frac{x^k}{k!}\frac{y^n}{n!} = e^{(e^x-1)y}.$$

5. Since $e^x - 1$ generates non-empty sets, one would expect that e^{e^x-1} should generate partitions. Verify this from the result of the preceding problem. In other words, prove that

$$\sum_{k=0}^{\infty} B_k\frac{x^k}{k!} = e^{e^x-1}.$$

6. Take $y > 0$. A mean y Poisson expectation is a sum of the form

$$\sum_{m=0}^{\infty} f(m)\frac{y^m}{m!}e^{-y}.$$

   Show that if $f(m) = 1$ this sum is one, while if $f(m) = m$ this sum is y.

7. Show that B_k is a mean 1 Poisson expectation of $f(m) = m^k$. In other words, the number of partitions of a k-set is the mean number of functions from the k-set to a set of random size. Hint: Write $e^{e^x-1} = e^{e^x}e^{-1}$.

8. Find a similar formula for the mean y Poisson expectation of $f(m) = m^k$.

8 Groups

1. Mazur, Section 5.2 Exercise 1 Show work.

2. Mazur, Section 5.2 Exercise 2 Show work.

3. Mazur, Section 5.2 Exercise 3 Show work.

4. Mazur, Section 5.2 Exercise 7 Give a complete proof.

5. Mazur, Section 5.2 Exercise 9 Write the group elements given in the text solutions in terms of two generators r, s with $r^4 = 1$ and $s^2 = 1$ and $(rs)^2 = 1$ (which is equivalent to $sr = r^{-1}s$). Be explicit about the correspondence between group elements in the text solutions and the group elements $1, r, r^2, r^3, s, rs, r^2s, r^3s$.

6. Mazur, Section 5.2 Exercise 12 What is the order of the cyclic subgroup of S_5 generated by (134)(25).

   Consider a group G and subgroup H. There is a partition of G into blocks called left cosets. Group elements a, b in G belong to the same left coset if there exists h in H with $a = bh$.

   The practical way to generate a left coset is to pick a group element b in G and write all products bh with h in the subgroup H.

7. Consider the group D_4 of Section 5.2 Exercise 9. Find the left cosets for the cyclic subgroup generated by s with elements $1, s$. How many are there? What is the size of each left coset?

8. Consider the group D_4 of Section 5.2 Exercise 9. Find the left cosets for the cyclic subgroup generated by r with elements $1, r, r^2, r^3$. How many are there? What is the size of each left coset?

9. Consider the group S_5 of Section 5.2 Exercise 12. Consider the cyclic subgroup generated by (134)(25). How many left cosets are there? What is the size of each left coset?

9 Group actions

1. Mazur, Section 5.3 Exercise 2 Show work.
2. Mazur, Section 5.4 Exercise 4 (tetrahedron) Show work.

10 Block designs

1. Consider varieties numbered 0 through 12. Construct a cyclic balanced incomplete block design with starter block consisting of varieties 0,1,3,9. Find all five parameters of the design.

2. Consider a block design with v varieties and b blocks. Each block has size k. Let the incidence matrix A_{im} have entry 1 if variety i is in block m, otherwise have entry 0. Show that each column sum

$$\sum_{i=1}^{v} A_{im} = k$$

while the row sums are

$$\sum_{m=1}^{b} A_{im} = r_i,$$

where r_i is the number of times variety i occurs in a block.

3. Define the concurrence matrix $\Lambda_{ij} = \sum_{m=1}^{b} A_{im} A_{jm}$. For $i \neq j$ the entry Λ_{ij} is the number of times the pair i, j occurs in a block, while $\Lambda_{ii} = r_i$. Show that the row sum

$$\sum_{j=1}^{v} \Lambda_{ij} = k r_i$$

and the column sum

$$\sum_{i=1}^{v} \Lambda_{ij} = k r_j.$$

4. The concurrence matrix defines the concurrence multigraph. This is a multigraph with no loops but with Λ_{ij} edges between vertices $i \neq j$. Thus it depends only on the off-diagonal elements of Λ. Define the corresponding Laplacian matrix L to be the matrix L_{ij} such that for $i \neq j$ we have $L_{ij} = -\Lambda_{ij}$ and such that each row or column sum is zero. Find the values of L_{ii} in terms of k and r_i.

5. Consider varieties numbered 0 through 6. Construct a cyclic balanced incomplete block design with starter block consisting of varieties 0,1,3. Find the incidence matrix, the concurrence matrix, and the Laplacian matrix. Sketch the concurrence graph.

6. Consider varieties numbered 0 through 6. Construct a cyclic block design with starter block consisting of varieties 0,1,4. Find the incidence matrix, the concurrence matrix, and the Laplacian matrix. Sketch the concurrence graph.