1 The Inclusion-Exclusion Principle

We have a universal set $\mathcal{U}$ that consists of all possible objects of interest. Here is some notation. If $A \subseteq \mathcal{U}$, then A^c is the complement $\mathcal{U} \setminus A$. If $A \subseteq \mathcal{U}$, then the indicator function of A is the function 1_A defined on $\mathcal{U}$ with value 1 on A and 0 on A_c. The fundamental equation in what follows is

$$1_{A^c} = 1 - 1_A. \tag{1}$$

We have a given collection P of "bad" properties. For each property p in P there is a corresponding subset A_p of $\mathcal{U}$ consisting of all the "bad" objects in $\mathcal{U}$ that satisfy that property.

Let $J \subseteq P$. This is a collection of some of the bad properties.

Let $A_{\supseteq}(J)$ be the set of all points in $\mathcal{U}$ that have all the bad properties in J. In other words, the collection K of properties satisfied by a point in $A_{\supseteq}(J)$ satisfies $K \supseteq J$. Notice that $A_{\supseteq}(\emptyset) = \mathcal{U}$.

There is a simple formula for this set $A_{\supseteq}(J)$. A point x in $\mathcal{U}$ is in $A_{\supseteq}(J)$ if and only if for each p in J the point x is in A_p. In other words, x belongs to the intersection of the sets A_p for $p \in J$. Thus

$$A_{\supseteq}(J) = \bigcap_{p \in J} A_p. \tag{2}$$

Consider the set $A_{=}(\emptyset)$ of all points in $\mathcal{U}$ that have none of the bad properties. There is a simple formula for this set $A_{=}(\emptyset)$. A point x in $\mathcal{U}$ is in $A_{=}(\emptyset)$ if and only if for each $p \in P$ the point x is not in A_p. In other words, x belongs to the intersection of the sets A_p^c for p in P. Thus

$$A_{=}(\emptyset) = \bigcap_{p \in P} A_p^c. \tag{3}$$

Theorem 1 (Inclusion-Exclusion for indicator functions)

$$1_{A_{=}(\emptyset)} = \sum_{J \subseteq P} (-1)^{|J|} 1_{A_{\supseteq}(J)}. \tag{4}$$

The proof is to use the distributive law of algebra. In this instance it says that

$$\prod_{p \in P} 1_{A_p^c} = \prod_{p \in P} (1 - 1_{A_p}) = \sum_{J \subseteq P} \prod_{p \in J} (-1_{A_p}) = \sum_{J \subseteq P} (-1)^{|J|} \prod_{p \in J} 1_{A_p}. \tag{5}$$

Theorem 2 (Inclusion-Exclusion for counting) *Let $N_{=}(\emptyset)$ be the number of points in $A_{=}(\emptyset)$, and let $N_{\supseteq}(J)$ be the number of points in $A_{\supseteq}(J)$. Then*

$$N_{=}(\emptyset) = \sum_{J \subseteq P} (-1)^{|J|} N_{\supseteq}(J). \tag{6}$$

This says that the number of points $N_=(\emptyset)$ with no bad properties may be found by taking the total number of points $N_\supseteq(\emptyset)$, subtracting the number of points for each bad property, adding the number of points for each pair of bad properties, subtracting the number of points for each triple of bad properties, and so on. The proof follows immediately from the indicator function identity. All that is needed is to sum over all the points in $\mathcal{U}$.

Theorem 3 (Inclusion-Exclusion for probability) *Let P assign probabilities to subsets of $\mathcal{U}$. Then*

$$P(\bigcap_{p\in P} A_p^c) = \sum_{J\subseteq P} (-1)^{|J|} P(\bigcap_{p\in J} A_p). \tag{7}$$

The proof of the probability principle also follows from the indicator function identity. Take the expectation, and use the fact that the expectation of the indicator function 1_A is the probability $P(A)$.

Sometimes the Inclusion-Exclusion Principle is written in a different form. Let $A_{\neq}(\emptyset)$ be the set of points in $\mathcal{U}$ that have some property in P. A point is in $A_{\neq}(\emptyset)$ if the collection K of properties in P that hold at that point satisfies $K \neq \emptyset$. Then $A_{\neq}(\emptyset) = \bigcup_{p\in P} A_p$. Next note that the first term $A_\supseteq(\emptyset) = \mathcal{U}$, and $A_{\neq}(\emptyset)$ is the complement in $\mathcal{U}$ of $A_=(\emptyset)$. The Inclusion-Exclusion principle for indicator functions may thus be written.

$$1_{A_{\neq}(\emptyset)} = \sum_{\emptyset\neq J\subseteq P} (-1)^{|J|-1} 1_{A_\supseteq(J)}. \tag{8}$$

The corresponding counting principle is

$$N_{\neq}(\emptyset) = \sum_{\emptyset\neq J\subseteq P} (-1)^{|J|-1} N_\supseteq(J). \tag{9}$$

This says that the number of points with some bad property may be found by adding the number of points for each bad property, subtracting the number of points for each pair of bad properties, adding the number of points for each triple of bad properties, and so on. The probability principle is

$$P(\bigcup_{p\in P} A_p) = \sum_{\emptyset\neq J\subseteq P} (-1)^{|J|-1} P(\bigcap_{p\in J} A_p). \tag{10}$$

2 The generalized Inclusion-Exclusion Principle

Let $A_=(S)$ be the set of all points in $\mathcal{U}$ that have all the bad properties in S and none of the bad properties in $P \setminus S$. In other words, the collection of properties satisfied by a point in $A_=(S)$ is equal to S.

There is a simple formula for this set $A_=(S)$. A point x in $\mathcal{U}$ is in $A_=(S)$ if and only if for each p in S the point x is in A_p and for each $p \in P \setminus S$ the point

x is not in A_p. In other words, x belongs to the intersection of the sets A_p for $p \in S$ intersected with the intersection of the sets A_p^c for p in $P \setminus S$. Thus

$$A_=(S) = \bigcap_{p \in S} A_p \cap \bigcap_{p \in P \setminus S} A_p^c. \tag{11}$$

Theorem 4 (Generalized Inclusion-Exclusion for indicator functions)

$$1_{A_=(S)} = \sum_{S \subseteq J \subseteq P} (-1)^{|J|-|S|} 1_{A_\supseteq(J)}. \tag{12}$$

The proof is again to use the distributive law of algebra. Start with

$$\prod_{p \in P \setminus S} (1 - 1_{A_p}) = \sum_{S \subseteq J \subseteq P} (-1)^{|J \setminus S|} \prod_{p \in J \setminus S} 1_{A_p}. \tag{13}$$

Then multiply both sides by $\prod_{p \in S} 1_{A_p}$. This gives

$$\prod_{p \in S} 1_{A_p} \prod_{p \in P \setminus S} (1 - 1_{A_p}) = \sum_{S \subseteq J \subseteq P} (-1)^{|J \setminus S|} \prod_{p \in J} 1_{A_p}. \tag{14}$$

Theorem 5 (Generalized Inclusion-Exclusion for counting) *Let $N_=(S)$ be the number of points in $A_=(S)$, and let $N_\supseteq(J)$ be the number of points in $A_\supseteq(J)$. Then*

$$N_=(S) = \sum_{S \subseteq J \subseteq P} (-1)^{|J|-|S|} N_\supseteq(J). \tag{15}$$