Multinomial coefficients

notes from Math 447–547 lectures

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1 Multi-sets and multinomial coefficients

A *multinomial coefficient* is associated with each (finite) multiset taken from the set of natural numbers. Such a multi-set is given by a list $k_1, \ldots, k_n$, where numbers may be repeated, and where order does not matter.

If the elements of the multiset are $k_1, k_2, \ldots, k_n$ and $k_1+k_2+k_3+\cdots+k_n = k$, then the multinomial coefficient is

$$\binom{k}{k_1\ k_2\ k_3\ \cdots k_n} = \frac{k!}{\prod_{i=1}^{k} k_i!}. \tag{1}$$

A multiset of natural numbers determines a *type vector* $p_1, p_2, p_3, \ldots$. Thus p_j is the number of times j occurs in the list $k_1, \ldots, k_n$. Then $p_1 1 + p_2 2 + p_3 3 + \cdots = k$. The multinomial coefficient is also given by

$$\binom{k}{k_1\ k_2\ k_3\ \cdots k_n} = \frac{k!}{\prod_j (j!)^{p_j}}. \tag{2}$$

There is a problem when the multiset has 0s in it, since the type vector does not determine the number of zeros. However, suppose that each $k_i \geq 1$. Thus the multiset consists only of strictly positive natural numbers. In that case the type vector determines the multiset. Furthermore, it follows that $p_1 + p_2 + p_3 + \cdots = n$.

A multiset taken from the set of strictly positive natural numbers with sum k is called a *integer partition* of k. Each number k_i in the sum is called a *part*. We write $p(k)$ for the number of integer partitions of k and $p(k, n)$ for the number of integer partitions of k into n parts.

2 Functions and surjective functions

Let A have k points and B have n points. Let f be a function from A to B. Then f defines a multi-set taken from B. This may be given by listing some elements of B (repetitions allowed, order irrelevant). Or it may be given by a multi-index N, a function from B to the natural numbers. The value of N on a point y in B is the number of points x with $f(x) = y$. The total number of such multi-indices is $\left(\binom{n}{k}\right)$.

The multi-index N determines a multi-set $k_1, \ldots, k_n$ taken from the natural numbers. This multi-set consists of the values of the function N. It has less information in it than the multi-index N, since it has lost the information about which elements of B index the numbers.

The number of functions from A to B with given multi-index N is determined by the values $k_1, \ldots, k_n$ of the multi-index N. It is given by the multinomial coefficient

$$\binom{k}{N} = \binom{k}{k_1\ k_2\ k_3\ \cdots k_n} = \frac{k!}{\prod_{i=1}^{k} k_i!}. \tag{3}$$

To prove this, think of B as being ordered. Then for each function $f : A \to B$ with the given multi-indices the inverse images form an ordered family of n subsets of A with cardinalities k_i. Give in addition an ordering of each subset. The pairs consisting of a suitable function together with orderings of the subsets correspond to orderings of A. This shows that the number of functions times $k_1! \cdots k_n!$ is equal to $k!$.

One obvious consequence of this is that

$$n^k = \sum_N \binom{k}{N}. \tag{4}$$

The total number of functions is obtained by summing the number of functions corresponding to each multi-index.

Each *set partition* of A partitions A into disjoint non-empty sets. Each set in the set partition is called a *block*. Say that there are n blocks. Write $B(n)$ for the number of set partitions, and $S(k,n)$ for the number of set partitions into n blocks.

Now consider the case of surjective functions from A to B. This determines a multi-index N on B with each value at least one. The number of such multi-indices is $\left(\binom{n}{k-n}\right)$. In particular the values $k_i \geq 1$, so we have an integer partition of k into n parts. Furthermore, the surjective function determines a set partition of A into n blocks. The number of set partitions is $S(k,n)$. The number of surjective functions is thus $n!S(k,n)$, since to determine the function all one has to do is to give the image of each of the n blocks.

From this we see that

$$n!S(k,n) = \sum_{N \geq 1} \binom{k}{N}. \tag{5}$$

The total number of surjective functions is obtained by summing the number of functions corresponding to each multi-index that only strictly positive values.

3 Set partitions and integer partitions

For each set partition of A we have a corresponding integer partition which consists of the sizes of the blocks of the set partition. The corresponding type vector p_j is just the number of blocks of size j.

The claim is that the number of set partitions of A with given integer partition is given by the formula

$$\binom{k}{k_1\ k_2\ k_3\ \cdots k_n}\frac{1}{\prod_j p_j!} = \frac{k!}{\prod_j (j!)^{p_j}}\frac{1}{\prod_j p_j!}. \tag{6}$$

The proof is the following. Consider an n element set with multi-index N having values given by the integer partition. Consider a partition of A into blocks with sizes given by the integer partition. Each block of size k_i must map into a point in B with multi-index value k_i. The surjective function from A to B with multi-index N is determined by two kinds of data: the partition and the mappings from the p_j blocks of given size j to the p_j points in B with multi-index value j. Therefore the number of partitions times $\prod_j p_j!$ is the number of functions.

As a consequence we get that the number of set partitions of a k element set A into n parts is

$$S(k,n) = \sum_P \frac{k!}{\prod_j (j!)^{p_j}}\frac{1}{\prod_j p_j!}, \tag{7}$$

where the sum is over all type vectors $p_1, p_2, p_3, \ldots$ with $p_1 + p_2 + p_3 + \cdots = n$ and $p_1 1 + p_2 3 + p_3 3 + \cdots = k$, in other words, over all integer partitions of k into n parts.

4 Two kinds of multinomial coefficient

There is another, more illuminating way, to get this kind of result. We look at the number of surjective functions $f : A \to B$ that define a given integer partition $p_1, p_2, p_3, \ldots$. This will be the number of surjective functions per multi-index on B times the number of multi-indices on B per integer partition. We know that the number of surjective functions with multi-index N is given by a multinomial coefficient. However a type vector is itself a special kind of multi-index, one defined on the strictly positive natural numbers. So the number of multi-indices on B giving a particular type vector is also given by a multinomial coefficient

$$\binom{n}{P} = \frac{n!}{\prod_j p_j!}. \tag{8}$$

The result is that the number of surjective functions with given integer partition is the product of two multinomial coefficients

$$\binom{k}{N}\binom{n}{P} = \frac{k!}{\prod_j (j!)^{p_j}}\frac{n!}{\prod_j p_j!}. \tag{9}$$

In particular, we recover a variant on the previous result:

$$n!S(k,n) = \sum_P \frac{k!}{\prod_j (j!)^{p_j}}\frac{n!}{\prod_j p_j!}, \tag{10}$$

where the sum is over all type vectors $p_1, p_2, p_3, \ldots$ with $p_1 + p_2 + p_3 + \cdots = n$ and $p_1 1 + p_2 2 + p_3 3 + \cdots = k$, in other words, over all integer partitions of k into n parts. This is a remarkable formula: It writes the number of surjective functions as a sum over integer partitions of terms each of which is a product of two multinomial coefficients.

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