Chapter 1

Complex integration

1.1 Complex number quiz

1. Simplify $\frac{1}{3+4i}$.

2. Simplify $|\frac{1}{3+4i}|$.

3. Find the cube roots of 1.

4. Here are some identities for complex conjugate. Which ones need correction? $\overline{z+w} = \bar{z}+\bar{w}$, $\overline{z-w} = \bar{z}-\bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, $\overline{z/w} = \bar{z}/\bar{w}$. Make suitable corrections, perhaps changing an equality to an inequality.

5. Here are some identities for absolute value. Which ones need correction? $|z+w| = |z|+|w|$, $|z-w| = |z|-|w|$, $|zw| = |z||w|$, $|z/w| = |z|/|w|$. Make suitable corrections, perhaps changing an equality to an inequality.

6. Define $\log(z)$ so that $-\pi < \Im \log(z) \leq \pi$. Discuss the identities $e^{\log(z)} = z$ and $\log(e^w) = w$.

7. Define $z^w = e^{w \log z}$. Find i^i.

8. What is the power series of $\log(1+z)$ about $z = 0$? What is the radius of convergence of this power series?

9. What is the power series of $\cos(z)$ about $z = 0$? What is its radius of convergence?

10. Fix w. How many solutions are there of $\cos(z) = w$ with $-\pi < \Re z \leq \pi$.

1.2 Complex functions

1.2.1 Closed and exact forms

In the following a region will refer to an open subset of the plane. A differential form $p\,dx + q\,dy$ is said to be closed in a region R if throughout the region

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}. \tag{1.1}$$

It is said to be exact in a region R if there is a function h defined on the region with

$$dh = p\,dx + q\,dy. \tag{1.2}$$

Theorem. An exact form is closed.

The converse is not true. Consider, for instance, the plane minus the origin. The form $(-y\,dx + x\,dy)/(x^2 + y^2)$ is not exact in this region. It is, however, exact in the plane minus the negative axis. In this region

$$\frac{-y\,dx + x\,dy}{x^2 + y^2} = d\theta, \tag{1.3}$$

where $-\pi/2 < \theta < \pi/2$.

Green's theorem. If S is a bounded region with oriented boundary ∂S, then

$$\int_{\partial S} p\,dx + q\,dy = \int\int_S (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})\,dx\,dy. \tag{1.4}$$

Consider a region R and an oriented curve C in R. Then $C \sim 0$ (C is homologous to 0) in R means that there is a bounded region S such that S and its oriented boundary ∂S are contained in R such that $\partial S = C$.

Corollary. If $p\,dx + q\,dy$ is closed in some region R, and if $C \sim 0$ in R, then

$$\int_C p\,dx + q\,dy = 0. \tag{1.5}$$

If C is an oriented curve, then $-C$ is the oriented curve with the opposite orientation. The sum $C_1 + C_2$ of two oriented curves is obtained by following one curve and then the other. The difference $C_1 - C_2$ is defined by following one curve and then the other in the reverse direction.

Consider a region R and two oriented curves C_1 and C_2 in R. Then $C_1 \sim C_2$ (C_1 is homologous to C_2) in R means that $C_1 - C_2 \sim 0$ in R.

Corollary. If $p\,dx + q\,dy$ is closed in some region R, and if $C_1 \sim C_2$ in R, then

$$\int_{C_1} p\,dx + q\,dy = \int_{C_2} p\,dx + q\,dy. \tag{1.6}$$

1.2.2 Cauchy-Riemann equations

Write $z = x + iy$. Define partial differential operators

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y} \tag{1.7}$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y} \tag{1.8}$$

The justification for this definition is the following. Every polynomial in x, y may be written as a polynomial in $z, \bar{z}$, and conversely. Then for each term in such a polynomial

$$\frac{\partial}{\partial z} z^m \bar{z}^n = m z^{m-1} \bar{z}^n \tag{1.9}$$

and

$$\frac{\partial}{\partial \bar{z}} z^m \bar{z}^n = z^m \, n \bar{z}^{n-1}. \tag{1.10}$$

Let $w = u + iv$ be a function $f(z)$ of $z = x + iy$. Suppose that this satisfies the system of partial differential equations

$$\frac{\partial w}{\partial \bar{z}} = 0. \tag{1.11}$$

In this case we say that $f(z)$ is an analytic function of z in this region. Explicitly

$$\frac{\partial(u + iv)}{\partial x} - \frac{\partial(u + iv)}{\partial iy} = 0. \tag{1.12}$$

This gives the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1.13}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1.14}$$

1.2.3 The Cauchy integral theorem

Consider an analytic function $w = f(z)$ and the differential form

$$w\,dz = f(z)\,dz = (u + iv)\,(dx + idy) = (u\,dx - v\,dy) + i(v\,dx + u\,dy). \tag{1.15}$$

According to the Cauchy-Riemann equations, this is a closed form.

Theorem (Cauchy integral theorem) If $f(z)$ is analytic in a region R, and if $C \sim 0$ in R, then

$$\int_C f(z)\,dz = 0. \tag{1.16}$$

Example: Consider the differential form $z^m\,dz$ for integer $m \neq 1$. When $m \geq 0$ this is defined in the entire complex plane; when $m < 0$ it is defined in the punctured plane (the plane with 0 removed). It is exact, since

$$z^m\,dz = \frac{1}{m+1} dz^{m+1}. \tag{1.17}$$

On the other hand, the differential form dz/z is closed but not exact in the punctured plane.

1.2.4 Polar representation

The exponential function is defined by

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1.18}$$

It is easy to check that

$$e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)). \tag{1.19}$$

Sometimes it is useful to represent a complex number in the polar representation

$$z = x + iy = r(\cos(\theta) + i\sin(\theta)). \tag{1.20}$$

This can also be written

$$z = re^{i\theta}. \tag{1.21}$$

From this we derive

$$dz = dx + i\,dy = dr\,e^{i\theta} + rie^{i\theta}\,d\theta. \tag{1.22}$$

This may also be written

$$\frac{dz}{z} = \frac{dr}{r} + i\,d\theta. \tag{1.23}$$

Notice that this does not say that dz/z is exact in the punctured plane. The reason is that the angle θ is not defined in this region. However dz/z is exact in a cut plane, that is, a plane that excludes some line running from the origin to infinity.

Let $C(0)$ be a circle of radius r centered at 0. We conclude that

$$\int_{C(0)} f(z)\,dz = \int_0^{2\pi} f(z)z\,i\,d\theta. \tag{1.24}$$

In particular,

$$\int_{C(0)} \frac{1}{z}\,dz = \int_0^{2\pi} i\,d\theta = 2\pi i. \tag{1.25}$$

By a change of variable, we conclude that for a circle $C(z)$ of radius r centered at z we have

$$\int_{C(z)} \frac{1}{\xi - z}\,d\xi = 2\pi i. \tag{1.26}$$

1.2.5 Branch cuts

Remember that

$$\frac{dz}{z} = \frac{dr}{r} + i\, d\theta \tag{1.27}$$

is exact in a cut plane. Therefore

$$\frac{dz}{z} = d\log(z) \tag{1.28}$$

in a cut plane,

$$\log(z) = \log(r) + i\theta \tag{1.29}$$

Two convenient choices are $0 < \theta < 2\pi$ (cut along the positive axis and $-\pi < \theta < \pi$ (cut along the negative axis).

In the same way one can define such functions as

$$\sqrt{z} = \exp(\frac{1}{2}\log(z)). \tag{1.30}$$

Again one must make a convention about the cut.

1.3 Complex integration and residue calculus

1.3.1 The Cauchy integral formula

Theorem. (Cauchy integral formula) Let $f(\xi)$ be analytic in a region R. Let $C \sim 0$ in R, so that $C = \partial S$, where S is a bounded region contained in R. Let z be a point in S. Then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\, d\xi. \tag{1.31}$$

Proof: Let $C_\delta(z)$ be a small circle about z. Let R' be the region R with the point z removed. Then $C \sim C_\delta(z)$ in R'. It follows that

$$\frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i}\int_{C_\delta(z)} \frac{f(\xi)}{\xi - z}\, d\xi. \tag{1.32}$$

It follows that

$$\frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\, d\xi - f(z) = \frac{1}{2\pi i}\int_{C_\delta(z)} \frac{f(\xi) - f(z)}{\xi - z}\, d\xi. \tag{1.33}$$

Consider an arbitrary $\epsilon > 0$. The function $f(\xi)$ is continuous at $\xi = z$. Therefore there is a δ so small that for ξ on $C_\delta(z)$ the absolute value $|f(\xi) - f(z)| \leq \epsilon$. Then the integral on the right hand side has integral with absolute value bounded by

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\epsilon}{\delta}\, \delta d\theta = \epsilon. \tag{1.34}$$

Therefore the left hand side has absolute value bounded by ϵ. Since ϵ is arbitrary, the left hand side is zero.

1.3.2 The residue calculus

Say that $f(z)$ has an isolated singularity at z_0. Let $C_\delta(z_0)$ be a circle about z_0 that contains no other singularity. Then the residue of $f(z)$ at z_0 is the integral

$$\mathrm{res}(z_0) = \frac{1}{2\pi i}\int_{C_\delta(z_0)} f(z)\,dz. \tag{1.35}$$

Theorem. (Residue Theorem) Say that $C \sim 0$ in R, so that $C = \partial S$ with the bounded region S contained in R. Suppose that $f(z)$ is analytic in R except for isolated singularities $z_1, \ldots, z_k$ in S. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \mathrm{res}(z_j). \tag{1.36}$$

Proof: Let R' be R with the singularities omitted. Consider small circles $C_1, \ldots, C_k$ around these singularities such that $C \sim C_1 + \cdots + C_k$ in R'. Apply the Cauchy integral theorem to $C - C_1 - \ldots - C_k$.

If $f(z) = g(z)/(z - z_0)$ with $g(z)$ analytic near z_0 and $g(z_0) \neq 0$, then $f(z)$ is said to have a pole of order 1 at z_0.

Theorem. If $f(z) = g(z)/(z - z_0)$ has a pole of order 1, then its residue at that pole is

$$\mathrm{res}(z_0) = g(z_0) = \lim_{z \to z_0} (z - z_0) f(z). \tag{1.37}$$

Proof. By the Cauchy integral formula for each sufficiently small circle C about z_0 the function $g(z)$ satisfies

$$g(z_0) = \frac{1}{2\pi i}\int_C \frac{g(z)}{z - z_0}\,d\xi = \frac{1}{2\pi i}\int_C f(z)\,dz. \tag{1.38}$$

This is the residue.

1.3.3 Estimates

Recall that for complex numbers we have $|zw| = |z||w|$ and $|z/w| = |z|/|w|$. Furthermore, we have

$$||z| - |w|| \leq |z \pm w| \leq |z| + |w|. \tag{1.39}$$

When $|z| > |w|$ this allows us to estimate

$$\frac{1}{|z \pm w|} \leq \frac{1}{|z| - |w|}. \tag{1.40}$$

Finally, we have

$$|e^z| = e^{\Re z}. \tag{1.41}$$

1.3.4 A residue calculation

Consider the task of computing the integral

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x^2+1}\, dx \tag{1.42}$$

where k is real. This is the Fourier transform of a function that is in L^2 and also in L^1. The idea is to use the analytic function

$$f(z) = e^{-ikz} \frac{1}{z^2+1}. \tag{1.43}$$

The first thing is to analyze the singularities of this function. There are poles at $z = \pm i$. Furthermore, there is an essential singularity at $z = \infty$.

First look at the case when $k \leq 0$. The essential singularity of the exponential function has the remarkable feature that for $\Im z \geq 0$ the absolute value of e^{-ikz} is bounded by one. This suggests looking at a closed oriented curve C_a in the upper half plane. Take C_a to run along the x axis from $-a$ to a and then along the semicircle $z = ae^{i\theta}$ from $\theta = 0$ to $\theta = \pi$. If $a > 1$ there is a singularity at $z = i$ inside the curve. So the residue is the value of $g(z) = e^{-ikz}/(z+i)$ at $z = i$, that is, $g(i) = e^k/(2i)$. By the residue theorem

$$\int_{C_a} e^{-ikz} \frac{1}{z^2+1}\, dz = \pi e^k \tag{1.44}$$

for each $a > 1$.

Now let $a \to \infty$. The contribution from the semicircle is bounded by

$$\int_0^{\pi} \frac{1}{a^2-1}\, a\, d\theta = \pi \frac{a}{a^2-1}. \tag{1.45}$$

We conclude that for $k \leq 0$

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x^2+1}\, dx = \pi e^k. \tag{1.46}$$

Next look at the case when $k \geq 0$. In this case we could look at an oriented curve in the lower half plane. The integral runs from a to $-a$ and then around a semicircle in the lower half plane. The residue at $z = -i$ is $e^{-k}/(-2i)$. By the residue theorem, the integral is $-\pi e^{-k}$. We conclude that for all real k

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x^2+1}\, dx = \pi e^{-|k|}. \tag{1.47}$$

1.4 Problems

1. Evaluate
$$\int_0^{\infty} \frac{1}{\sqrt{x}(4+x^2)}\, dx$$
by contour integration. Show all steps, including estimation of integrals that vanish in the limit of large contours.

2. In the following problems $f(z)$ is analytic in some region. We say that $f(z)$ has a root of multiplicity m at z_0 if $f(z) = (z - z_0)^m h(z)$, where $h(z)$ is analytic with $h(z_0) \neq 0$. Find the residue of $f'(z)/f(z)$ at such a z_0.

3. Say that $f(z)$ has several roots inside the contour C. Evaluate

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz.$$

4. Say that

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a polynomial. Furthermore, suppose that C is a contour surrounding the origin on which

$$|a_k z^k| > |f(z) - a_k z^k|.$$

Show that on this contour

$$f(z) = a_k z^k g(z)$$

where

$$|g(z) - 1| < 1$$

on the contour. Use the result of the previous problem to show that the number of roots (counting multiplicity) inside C is k.

5. Find the number of roots (counting multiplicity) of $z^6 + 3z^5 + 1$ inside the unit circle.

1.5 More residue calculus

1.5.1 Jordan's lemma

Jordan's lemma says that for $b > 0$ we have

$$\frac{1}{\pi} \int_0^{\pi} e^{-b\sin(\theta)}\, d\theta \leq \frac{1}{b}. \tag{1.48}$$

To prove it, it is sufficient to estimate twice the integral over the interval from 0 to $\pi/2$. On this interval use the inequality $(2/\pi)\theta \leq \sin(\theta)$. This gives

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-2b\theta/\pi}\, d\theta = \frac{1}{b}(1 - e^{-b}) \leq \frac{1}{b}. \tag{1.49}$$

1.5.2 A more delicate residue calculation

Consider the task of computing the integral

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x-i}\, dx \tag{1.50}$$

where k is real. This is the Fourier transform of a function that is in L^2 but not in L^1. The idea is to use the analytic function

$$f(z) = e^{-ikz} \frac{1}{z-i}. \tag{1.51}$$

The first thing is to analyze the singularities of this function. There is a pole at $z = i$. Furthermore, there is an essential singularity at $z = \infty$.

First look at the case when $k < 0$. Take C_a to run along the x axis from $-a$ to a and then along the semicircle $z = ae^{i\theta}$ in the upper half plane from $\theta = 0$ to $\theta = \pi$. If $a > 1$ there is a singularity at $z = i$ inside the curve. So the residue is the value of $g(z) = e^{-ikz}$ at $z = i$, that is, $g(i) = e^k$. By the residue theorem

$$\int_{C_a} e^{-ikz} \frac{1}{z-i}\, dz = 2\pi i e^k \tag{1.52}$$

for each $a > 1$.

Now let $a \to \infty$. The contribution from the semicircle is bounded using Jordan's lemma:

$$\int_0^{\pi} e^{ka\sin(\theta)} \frac{1}{a-1}\, a\, d\theta \leq \pi \frac{1}{-ka} \frac{a}{a-1} \tag{1.53}$$

We conclude that for $k < 0$

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x-i}\, dx = 2\pi i e^k. \tag{1.54}$$

Next look at the case when $k < 0$. In this case we could look at an oriented curve in the lower half plane. The integral runs from a to $-a$ and then around a semicircle in the lower half plane. The residue is zero. We conclude using Jordan's lemma that for $k < 0$

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{x-i}\, dx = 0. \tag{1.55}$$

1.5.3 Cauchy formula for derivatives

Theorem. (Cauchy formula for derivatives) Let $f(\xi)$ be analytic in a region R including a point z. Let C be an oriented curve in R such that for each sufficiently small circle $C(z)$ about z, $C \sim C(z)$ in R. Then the mth derivative satisfies

$$\frac{1}{m!} f^{(m)}(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{m+1}}\, d\xi. \tag{1.56}$$

Proof: Differentiate the Cauchy integral formula with respect to z a total of m times.

1.5.4 Poles of higher order

If $f(z) = g(z)/(z - z_0)^m$ with $g(z)$ analytic near z_0 and $g(z_0) \neq 0$ and $m \geq 1$, then $f(z)$ is said to have a pole of order m at z_0.

Theorem. If $f(z) = g(z)/(z - z_0)^m$ has a pole of order m, then its residue at that pole is

$$\text{res}(z_0) = \frac{1}{(m-1)!} g^{(m-1)}(z_0). \tag{1.57}$$

Proof. By the Cauchy formula for derivatives for each sufficiently small circle C about z the $m - 1$th derivative satisfies

$$\frac{1}{(m-1)!} g^{(m-1)}(z_0) = \frac{1}{2\pi i} \int_C \frac{g(z)}{(z - z_0)^m} \, dz = \frac{1}{2\pi i} \int_C f(z) \, dz. \tag{1.58}$$

The expression given in the theorem is evaluated in practice by using the fact that $g(z) = (z - z_0)^m f(z)$ for z near z_0, performing the differentiation, and then setting $z = z_0$. This is routine, but it can be tedious.

1.5.5 A residue calculation with a double pole

Confider the task of computing the integral

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{(x^2+1)^2} \, dx \tag{1.59}$$

where k is real. This is the Fourier transform of a function that is in L^2 and also in L^1. The idea is to use the analytic function

$$f(z) = e^{-ikz} \frac{1}{(z^2+1)^2}. \tag{1.60}$$

The first thing is to analyze the singularities of this function. There are poles at $z = \pm i$. Furthermore, there is an essential singularity at $z = \infty$.

First look at the case when $k \leq 0$. Consider a closed oriented curve C_a in the upper half plane. Take C_a to run along the x axis from $-a$ to a and then along the semicircle $z = ae^{i\theta}$ from $\theta = 0$ to $\theta = \pi$. If $a > 1$ there is a singularity at $z = i$ inside the curve. The pole there is of order 2. So the residue is calculated by letting $g(z) = e^{-ikz}/(z+i)^2$, taking the derivative, and evaluating at $z = i$, that is, $g'(i) = (1-k)e^k/(4i)$. By the residue theorem

$$\int_{C_a} e^{-ikz} \frac{1}{z^2+1} \, dz = \frac{1}{2}\pi(1-k)e^k \tag{1.61}$$

for each $a > 1$.

Now let $a \to \infty$. The contribution from the semicircle vanishes in that limit. We conclude that for $k \leq 0$

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{(x^2+1)^2} \, dx = \frac{1}{2}\pi(1-k)e^k. \tag{1.62}$$

Next look at the case when $k \geq 0$. In this case we could look at an oriented curve in the lower half plane. The integral runs from a to $-a$ and then around a semicircle in the lower half plane. The residue at $z = -i$ is $-(1+k)e^{-k}/(4i)$. By the residue theorem, the integral is $-\pi(1+k)e^{-k}/2$. We conclude that for all real k

$$\int_{-\infty}^{\infty} e^{-ikx} \frac{1}{(x^2+1)^2}\, dx = \frac{1}{2}\pi(1+|k|)e^{-|k|} \tag{1.63}$$

1.6 The Taylor expansion

1.6.1 Radius of convergence

Theorem. Let $f(z)$ be analytic in a region R including a point z_0. Let $C(z_0)$ be a circle centered at z_0 such that $C(z_0)$ and its interior are in R. Then for z in the interior of $C(z_0)$

$$f(z) = \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(z_0)(z-z_0)^m. \tag{1.64}$$

Proof. For each fixed ξ the function of z given by

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 + z_0 - z} = \frac{1}{(\xi - z_0)} \frac{1}{1 - \frac{z-z_0}{\xi-z_0}} = \frac{1}{(\xi - z_0)} \sum_{m=0}^{\infty} \left(\frac{z-z_0}{\xi-z_0}\right)^m. \tag{1.65}$$

has a geometric series expansion. Multiply by $(1/2\pi i) f(\xi)\, d\xi$ and integrate around $C(z_0)$. On the left hand side apply the Cauchy integral formula to get $f(z)$.

In each term in the expansion on the right hand side apply the Cauchy formula for derivatives in the form

$$\frac{1}{2\pi i} \int_{C(z_0)} \frac{f(\xi)}{(\xi - z_0)^{m+1}}\, d\xi = \frac{1}{m!} f^{(m)}(z_0). \tag{1.66}$$

This theorem is remarkable because it shows that the condition of analyticity implies that the Taylor series always converges. Furthermore, take the radius of the circle $C(z_0)$ as large as possible. The only constraint is that there must be a function that is analytic inside the circle and that extends $f(z)$. Thus one must avoid the singularity of this extension of $f(z)$ that is closest to z_0. This explains why the radius of convergence of the series is the distance from z_0 to this nearest singularity.

The reason that we talk of an analytic extension is that artificialities in the definition of the function, such as branch cuts, should not matter. On the other hand, singularities such as poles, essential singularities, and branch points are intrinsic. The radius of convergence is the distance to the nearest such intrinsic singularity.

If one knows an analytic function near some point, then one knows it all the way out to the radius of convergence of the Taylor series about that point.

But then for each point in this larger region there is another Taylor series. The function is then defined out to the radius of convergence associated with that point. The process may be carried out over a larger and larger region, until blocked by some instrinsic singularity. It is known as analytic continuation.

If the analytic continuation process begins at some point and winds around a branch point, then it may lead to a new definition of the analytic function at the original point. This appears to lead to the necessity of introducing an artifical branch cut in the definition of the function. However this may be avoided by introducing the concept of Riemann surface.

1.6.2 Riemann surfaces

When an analytic function has a branch cut, it is an indicator of the fact that the function should not be thought of not as a function on a region of the complex plane, but instead as a function on a Riemann surface. A Riemann surface corresponds to several copies of a region in the complex plane. These copies are called sheets. Where one makes the transition from one sheet to another depends on the choice of branch cut. But the Riemann surface itself is independent of the notion of sheet.

As an example, take the function $w = \sqrt{z}$. The most natural definition of this function is as a function on the curve $w^2 = z$. This is a kind of two-dimensional parabola in a four dimensional space of two complex variables. The value of the square root function on the point with coordinates z, w on the parabola $w^2 = z$ is w. Notice that if z is given, then there are usually two corresponding values of w. Thus if we want to think of $\sqrt{z}$ as a function of a complex variable, then it is ambiguous. But if we think of it as a function on the Riemann surface, then it is perfectly well defined. The Riemann surface has two sheets. If we wish, we may think of sheet I as the complex plane cut along the negative axis. Sheet II is another copy of the complex plane cut along the negative axis. As z crosses the cut, it changes from one sheet to the other. The value of w also varies continuously, and it keeps track of what sheet the z is on.

As another example, take the function $w = \sqrt{z(z-1)}$. The most natural definition of this function is as a function on the curve $w^2 = z(z-1)$. This is a kind of two-dimensional circle in a four dimensional space of two complex variables. The value of the function on the point with coordinates z, w on the circle $w^2 = z(z-1)$ is w. Notice that if z is given, then there are usually two corresponding values of w. Thus if we want to think of $\sqrt{z(z-1)}$ as a function of a complex variable, then it is ambiguous. But if we think of it as a function on the Riemann surface, then it is perfectly well defined. The Riemann surface has two sheets. If we wish, we may think of sheet I as the complex plane cut between 0 and 1. Sheet II is another copy of the complex plane, also cut between 0 and 1. As z crosses the cut, it changes from one sheet to the other. The value of w also varies continuously, and it keeps track of what sheet the z is on.

As a final example, take the function $w = \log z$. The most natural definition of this function is as a function on the curve $\exp(w) = z$. This is a two-dimensional curve in a four dimensional space of two complex variables. The value of the logarithm function on the point with coordinates z, w on the curve $\exp(w) = z$ is w. Notice that if z is given, then there are infinitely many corresponding values of w. Thus if we want to think of $\log z$ as a function of a complex variable, then it is ambiguous. But if we think of it as a function on the Riemann surface, then it is perfectly well defined. The Riemann surface has infinitely many sheets. If we wish, we may think of each sheet as the complex plane cut along the negative axis. As z crosses the cut, it changes from one sheet to the other. The value of w also varies continuously, and it keeps track of what sheet the z is on. The infinitely many sheets form a kind of spiral that winds around the origin infinitely many times.