Pairing of Zeros and Critical Points for Random Polynomials

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Model: $SU(2)$ Polynomials

$SU(2)$ polynomial:

$$p_N(z) \text{ def } = N \sum_{j=0}^{N} a_j \cdot \left( N \cdot j \right)^{1/2} z^j a_j \sim N(0, 1) C_{\text{indep}}.$$

Unique Gaussian ensemble for which the expected distribution of zeros is invariant under $SU(2)$.

$$M(z) \text{ def } = |p_N(z)|^2.$$

\{local minima of $M$\} $\leftrightarrow$ \{zeros of $p_N$\}.

\{saddle points of $M$\} $\leftrightarrow$ \{critical points of $p_N$\}.

Flow lines for $-\nabla M(z)$ accumulate at zeros of $p_N$. 

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$$p_N(z) \overset{\text{def}}{=} \sum_{j=0}^{N} a_j \cdot \binom{N}{j}^{1/2} z^j \quad a_j \sim N(0, 1)_\mathbb{C} \text{ indep.}$$
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Pairing of Zeros and Crits for $SU(2)$ Polynomials
“Proof” of Pairing

\[ \Delta G = \mu \Rightarrow dG(z) = e - \text{field from } \mu. \]

Fix \( p \mathcal{N} : \mathbb{S}^2 \rightarrow \mathbb{S}^2 \):

\[ \text{Def. } \text{Div} (p \mathcal{N}) = -\mathcal{N} \cdot \delta_{\infty} + \sum p \mathcal{N}(\xi) = 0 \delta_{\xi} \]

\[ \Delta \log |p \mathcal{N}(z)| = \text{Div} (p \mathcal{N}) \text{ (Poincaré-Lelong)} \]

Therefore,

\[ d \log |p \mathcal{N}(z)| = e - \text{field from Div} (p \mathcal{N}) \]

\[ \frac{d}{dz} p \mathcal{N}(z) = 0 \iff d \log |p \mathcal{N}(z)| = 0 \]
“Proof” of Pairing

- Electrostatics for measure $\mu$ on $S^2$: 

$\Delta G = \mu \implies dG(z) = e^{-\text{field from } \mu}$

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$S^2 \to S^2$:

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Pairing of Zeros and Crits for $SU(2)$ Polynomials

Figure: Zeros – black discs. Crits – blue squares.
Definition of Pairing of Zeros and Crits

Let \( p \) be a degree \( N \) polynomial and fix \( \epsilon > 0 \).
A zero at \( \xi \) and critical point at \( w \) are \( \epsilon \)-paired if
\[ N - 1 - \epsilon < |\xi - w| < N - 1 + \epsilon. \]
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![Diagram showing pairing of zeros and critical points](image-url)
Theorem (H - 2013)

Fix $\xi \in \mathbb{C} \setminus \{0\}$.

Let $p_N(z)$ be a degree $N \text{ SU}(2)$ polynomial conditioned to have $p_N(\xi) = 0$.

For each $\epsilon \in (0, \frac{1}{2})$, there exists $K = K(\epsilon)$ such that $P(\exists ! \epsilon - \text{paired with } \xi) \geq 1 - K \cdot N^{-3/2 + 3\epsilon}$. 

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Definition of Hermitian Gaussian Ensembles

\[ \text{Def.} \quad P_N = \{ \text{Poly. deg.} \leq N \} \]

Fix \( \phi \in C^\infty (\mathbb{C}, \mathbb{R}) \):

\[ \phi(z) \sim \log |z| \text{ as } |z| \to \infty \]

\( \phi(z) \) subharmonic

Each such \( \phi \) defines different \( HGE(\phi, N) \):

\[ L^2(\mathbb{C}, e^{-N \phi(z)} \Delta \phi(z) \, dz) \supseteq P_N \]

\[ \phi(z) = \log (1 + |z|^2) - SU(2) \text{ polynomials} \]

\[ \text{Def.} \quad \xi \in S^2 \text{ is distinguished in } HGE(\phi, N) \text{ if } d\phi(\xi) = 0. \]

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Definition of Hermitian Gaussian Ensembles

- Def. \( \mathcal{P}_N = \{ \text{Poly. deg.} \leq N \} \)

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  - $\phi(z) = \log \left(1 + |z|^2\right)$ — $SU(2)$ polynomials

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Definition of Hermitian Gaussian Ensembles

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- **Def.** $\xi \in S^2$ is distinguished in $HGE(\phi, N)$ if $d\phi(\xi) = 0$. 

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Pairing of Zeros and Critical Points
Proof Ingredients

Local scaling limit of $p_N$ is multiple of GAF:

$$p_N(\xi + u\sqrt{N}) \cdot e^{\sqrt{N}\phi(\xi + u/\sqrt{N})} \to \sum_{j \geq 0} a_j u^j \sqrt{j!}$$

in the sense of $C^\infty$ convergence of covariance kernels.

Poincaré Lelong $-\frac{1}{4}\pi \Delta \log |p_N| = Z_{p_N}$

Probabilistic analog for Gaussian fields:

$$E[Z_{p_N}(z)] = \frac{1}{4\pi} \Delta \log |K_N(z, z)|$$

Correlations

$$E[Z_{p_N} \otimes C_{p_N}]$$

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- Poincaré Lelong
  \[ - \frac{1}{4\pi} \Delta \log |p_N| = Z_{p_N} \]
Proof Ingredients

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$$\frac{1}{4\pi} \Delta \log |p_N| = Z_{p_N}$$

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$$\mathbb{E} [Z_{p_N}] (z) = \frac{1}{4\pi} \Delta \log |K_N(z, z)|$$
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- Probabilistic analog for Gaussian fields:

$$\mathbb{E} [Z_{p_N}](z) = \frac{1}{4\pi} \Delta \log |K_N(z, z)|$$

- Correlations

$$- \mathbb{E} [Z_{p_N} \otimes C_{p_N}] - \mathbb{E} [Z_{p_N}] \otimes \mathbb{E} [C_{p_N}]$$
Figure: Zeros and critical points near $\xi = 0$. For every critical point the sector predicted to contain its paired zero is shown.
Theorem (H - 2012)

Fix an $N^{-1/2}-$scaled normal coordinate around a distinguished point $\xi \in S^2$. 

Expected Nearest Neighbor Spacings near $\xi = 0$
**Theorem (H - 2012)**

*Fix an $N^{-1/2}$-scaled normal coordinate around a distinguished point $\xi \in S^2$. Consider $A \subseteq \mathbb{C}\setminus \{|w| \leq 1\}$.*
Expected Nearest Neighbor Spacings near $\xi = 0$

**Theorem (H - 2012)**

Fix an $N^{-1/2}$-scaled normal coordinate around a distinguished point $\xi \in S^2$. Consider $A \subseteq \mathbb{C} \setminus \{|w| \leq 1\}$. Define

$$\mathcal{X}_{A,N} := \# \text{ (zero,crit) pairs } (z, w)$$

so that

$$w \in A \text{ and } z = w + re^{it}$$
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$$r \in \left[|w|^{-1} \pm |w|^{-7/4}\right] \text{ and } t \in \left[\arg(w) \pm |w|^{-3/4}\right].$$
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We have, for each $\epsilon > 0$,

$$\mathbb{E} [\mathcal{X}_{A,N}] = \mathbb{E} [C_A] + O \left(\int_A |w|^{-1/4} \, dw \wedge d\overline{w}\right) + O(N^{-1/2+\epsilon}).$$