Problem 5.5.7

Let $\mathbf{v}_1 = (1, -1, 2, 5)^T$ and $\mathbf{v}_2 = (2, 1, 0, -1)^T$. Then

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 4 \times 1 \times 2 + 3 \times (-1) \times 1 + 2 \times 2 \times 0 + 5 \times (-1) = 0,$$

so the vestors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Therefore, the projection of $\mathbf{v}$ onto the space spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\begin{aligned}
\frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{v}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 &= \frac{4-6-4+10}{4+3+8+25}\mathbf{v}_1 + \frac{8+6-2}{16+3+1}\mathbf{v}_2 \\
&= 0.1\mathbf{v}_1 + 0.6\mathbf{v}_2 = \begin{pmatrix} 1.3 \\ 0.5 \\ 0.2 \\ -0.1 \end{pmatrix}.
\end{aligned}$$

Problem 5.6.3(b)

Form the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ -2 & 0 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{pmatrix},$$

the rows of which are $\mathbf{v}_1^T$, $\mathbf{v}_2^T$, and $\mathbf{v}_3^T$ where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

A vector $\mathbf{w}$ is orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ if $A\mathbf{w} = \mathbf{0}$. The problem boild down to finding a basis in the kernel of A. We bring the matrix A to the row echelon form, first adding twice of the first row to the second row and adding the first row to the third row; then subtracting the second row from the third row:

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 4 & -1 & 4 \\ 0 & 4 & -1 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 4 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank equals 2, so the kernel of A has dimension 2. A basis consists of two vectors; w_3 and w_4 can be taken as free variables.
Answer: $(1/2, 1/4, 1, 0)^T$ and $(-1, -1, 0, 1)^T$.

Problem 5.6.20(b)

The matrix of the system is

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & -2 \\ 2 & -3 & 5 \end{pmatrix}.$$

The compatibility condition is that the vector $(a, b, c)^T$ is orthogonal to the kernel of A^T. To find this kernel, we bring the matrix

$$A^T = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 5 & -3 \\ 3 & -2 & 5 \end{pmatrix}$$

to the row echelon form:

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 7 & -7 \\ 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank of A^T equals 2, so its kernel is spanned by one vector. To find it, one sets $x_3 = 1$, and by back substitution one gets $x_2 = 1$ and $x_1 = -1$. Therefore, the compatibility condition is that the vector $(a, b, c)^T$ is orthogonal to $(-1, 1, 1)^T$, or

$$-a + b + c = 0$$

.

Problem 8.2.20

If λ is an eigenvalue for A then $A\mathbf{v} = \lambda\mathbf{v}$ for some vector $\mathbf{v} \neq \mathbf{0}$. Then

$$A^2\mathbf{v} = a(A\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v},$$

so λ^2 is an eigenvalue for A^2.

Problem 8.3.2(f)

One computes

$$\det(A-\lambda I) = -(\lambda+6)(2-\lambda)(6-\lambda)+32(2-\lambda) = (2-\lambda)(\lambda^2-4) = -(\lambda-2)^2(\lambda+2).$$

The matrix has two eigenvalues: $\lambda_1 = \lambda_2 = 2$, of multiplicity 2, and $\lambda_3 = -2$, of multiplicity 1. First, find the eigenspace V_2. One has

$$A - 2I = \begin{pmatrix} -8 & 0 & -8 \\ -4 & 0 & -4 \\ 4 & 0 & 4 \end{pmatrix};$$

its low echelon form is

$$\begin{pmatrix} -8 & 0 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and one can take

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

as a basis in V_2. The dimension on V_2 equals 2, which is the multiplicity of λ_1, so the eigenvalue 2 is complete. Now, we turn to the eigenspace V_{-2}. One has

$$A + 2I = \begin{pmatrix} -4 & 0 & -8 \\ -4 & 4 & -4 \\ 4 & 0 & 8 \end{pmatrix};$$

its row echelon form equals

$$\begin{pmatrix} -4 & 0 & -8 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

The space V_{-2} is one-dimensional, and it is spanned by the vector

$$\mathbf{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}.$$