Problem 8.3.15(d)

The eigenvalues of the matrix are $\lambda_1 = -2$, $\lambda_2 = 1$, and $\lambda_3 = 3$ (eigenvalues of an upper triangular matrix are its diagonal entries.) One finds the corresponding eigenvectors (I skip this step:)

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} -0.1 \\ -0.5 \\ 1 \end{pmatrix}.$$

We form the matrix with $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ being its column vectors:

$$S = \begin{pmatrix} 1 & 1 & -0.1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{pmatrix}.$$

The inverse to S can be found by performing elementary row operations:

$$\begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -0.5 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 0 & | & 1 & 0 & 0.1 \\ 0 & 1 & 0 & | & 0 & 1 & 0.5 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 1 & -1 & -0.4 \\ 0 & 1 & 0 & | & 0 & 1 & 0.5 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{pmatrix},$$

so

$$S^{-1} = \begin{pmatrix} 1 & -1 & -0.4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 1 & -0.1 \\ 0 & 1 & -0.5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & -0.4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 8.3.21(d)

If $\lambda_1 = -1 + 2i$ is an eigenvalue, and $\mathbf{v}_1 = (1 + i, 3i)^T$ is an eigenvector then $\lambda_2 = \bar{\lambda}_1 = -1 - 2i$ is also an eigenvalue, and $\mathbf{v}_2 = \bar{\mathbf{v}}_1 = (1 - i, -3i)^T$ is the corresponding eigenvector We form the matrix with $\mathbf{v}_1$ and $\mathbf{v}_2$ being its column vectors:

$$S = \begin{pmatrix} 1+i & 1-i \\ 3i & -3i \end{pmatrix}.$$

The inverse to S equals

$$S^{-1} = -\frac{1}{6i} \begin{pmatrix} -3i & -1+i \\ -3i & 1+i \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & -1-i \\ 3 & -1+i \end{pmatrix},$$

and

$$A = \frac{1}{6} \begin{pmatrix} 1+i & 1-i \\ 3i & -3i \end{pmatrix} \begin{pmatrix} -1+2i & 0 \\ 0 & -1-2i \end{pmatrix} \begin{pmatrix} 3 & -1-i \\ 3 & -1+i \end{pmatrix} = \begin{pmatrix} -3 & 4/3 \\ -6 & 1 \end{pmatrix}.$$

Problem 8.4.14(d)

First, we look for the eigenvalues:

$$\begin{aligned}\det(A-\lambda I) &= (3-\lambda)(2-\lambda)^2-(2-\lambda)-(2-\lambda)=(2-\lambda)(\lambda^2-5\lambda+4)\\ &= (2-\lambda)(\lambda-1)(\lambda-4).\end{aligned}$$

The eigenvalues are $\lambda_1=1$, $\lambda_2=2$, and $\lambda_3=4$. Then we find eigenvectors:

$$\mathbf{v}_1=\begin{pmatrix}1\\1\\1\end{pmatrix},\quad \mathbf{v}_2=\begin{pmatrix}0\\-1\\1\end{pmatrix},\quad \text{and}\quad \mathbf{v}_3=\begin{pmatrix}-2\\1\\1\end{pmatrix}.$$

To form the orthogonal matrix Q one has to normalize the eigenvectors:

$$\mathbf{u}_1=\frac{\mathbf{v}_1}{||\mathbf{v}_1||}=\begin{pmatrix}\sqrt{3}/3\\\sqrt{3}/3\\\sqrt{3}/3\end{pmatrix},\quad \mathbf{u}_2=\frac{\mathbf{v}_2}{||\mathbf{v}_2||}=\begin{pmatrix}0\\-\sqrt{2}/2\\\sqrt{2}/2\end{pmatrix},\quad\text{and}$$

$$\mathbf{u}_3=\frac{\mathbf{v}_3}{||\mathbf{v}_3||}=\begin{pmatrix}-\sqrt{6}/3\\\sqrt{6}/6\\\sqrt{6}/6\end{pmatrix}.$$

Finally,

$$Q=\begin{pmatrix}\sqrt{3}/3 & 0 & -\sqrt{6}/3\\ \sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{6}/6\\ \sqrt{3}/3 & \sqrt{2}/2 & \sqrt{6}/6\end{pmatrix},$$

and

$$A=Q\begin{pmatrix}1&0&0\\0&2&0\\0&0&4\end{pmatrix}Q^T.$$

Problem 8.4.35

The matrix of the quadratic form is

$$A=\begin{pmatrix}2 & 1/2 & 1\\ 1/2 & 2 & 0\\ 1 & 0 & 2\end{pmatrix}.$$

The problem of finding the maximum and the minimum of the quadratic form, subject to the constraint $x^2+y^2+z^2=1$, is equivalent to the problem of finding the maximal and the minimal eigenvalues of A. We compute

$$\det(A-\lambda I)=(2-\lambda)^3-(2-\lambda)-\frac{1}{4}(2-\lambda)=(2-\lambda)(\lambda^2-4\lambda+11/4),$$

so $\lambda_1=2$, and two other eigenvalues are determined from the equation

$$\lambda^2-4\lambda+\frac{11}{4}=0.$$

Solutions of the last equation are

$$\lambda_{2,3}=\frac{4\pm\sqrt{16-11}}{2}=2\pm\frac{\sqrt{5}}{2}.$$

Out of all three eigenvalues, $2+\sqrt{5}/2$ is the largest and $2-\sqrt{5}/2$ is the smallest.
Answer: The largest value of the quadratic form is $2+\sqrt{5}/2$, the smallest value is $2-\sqrt{5}/2$.

Problem 8.6.1(f)

One computes

$$\det(A-\lambda I) = -\lambda^2(\lambda+1) - 2 + (\lambda+1) - 2\lambda = -(\lambda+1)(\lambda^2+1),$$

so the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_{2,3} = \pm i$. One finds that $\mathbf{v}_1 = (1,0,1)^T$ is an eigenvector that corresponds to $\lambda_1 = -1$. The normalized eigenvector $\mathbf{u}_1 = \mathbf{v}_1/||\mathbf{v}_1||$ equals $(\sqrt{2}/2, 0, \sqrt{2}/2)$. One can take

$$U_1 = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}$$

as a unitary matrix with $\mathbf{u}_1$ being its first column. Then

$$\begin{aligned} U_1^\dagger A U_1 &= \begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 0 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & \sqrt{2} & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}. \end{aligned}$$

One forms a 2×2 matrix

$$A_1 = \begin{pmatrix} -1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$$

by removing the first row and the first column from the matrix $U_1^\dagger A U_1$. The eigenvalues of A_1 are $\pm i$; $\mathbf{u}_2 = ((1-i)/2, \sqrt{2}/2)^T$ is a normalized eigenvector that corresponds to the eigenvalue i. Then

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-i)/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & (-1-i)/2 \end{pmatrix}$$

is a unitary matrix, the first column of which equals $(1,0,0)^T$, and the second column of which equals $(0, \mathbf{u}_2^T)^T$. The matrix U_2 is symmetric, so

$$U_2^\dagger = \bar{U}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+i)/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & (-1+i)/2 \end{pmatrix}.$$

Let

$$\begin{aligned} U = U_1 U_2 &= \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-i)/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & (-1-i)/2 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{2}/2 & -1/2 & \sqrt{2}(1+i)/4 \\ 0 & (1-i)/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1/2 & -\sqrt{2}(1+i)/4 \end{pmatrix}. \end{aligned}$$

Then

$$U^\dagger AU = U_2^\dagger (U_1^\dagger AU_1)U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+i)/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & (-1+i)/2 \end{pmatrix} \begin{pmatrix} -1 & \sqrt{2} & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-i)/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & (-1-i)/2 \end{pmatrix} = \begin{pmatrix} -1 & \sqrt{2}(1-i)/2 & 1 \\ 0 & i & -\sqrt{2}(1+i) \\ 0 & 0 & -i \end{pmatrix}.$$

Finally,

$$A = U \begin{pmatrix} -1 & \sqrt{2}(1-i)/2 & 1 \\ 0 & i & -\sqrt{2}(1+i) \\ 0 & 0 & -i \end{pmatrix} U^\dagger$$

is a Schur decomposition of the matrix A.