## Problem 9.1.11

The matrix of the system is

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}.$$

One finds its eigenvalues: $\lambda_1 = 5$ and $\lambda_2 = -5$; the corresponding eigenvectors are $\mathbf{u}_1 = (2,1)^T$ and $\mathbf{u}_2 = (1,-2)^T$. Notice that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to each other as they should be: the matrix $A$ is symmetric. The general solution to the system $\dot{\mathbf{u}} = A\mathbf{u}$ is

$$c_1\mathbf{u}_1 e^{5t} + c_2\mathbf{u}_2 e^{-5t}.$$

The constants $c_1$ and $c_2$ are found from the initial condition:

$$\mathbf{w} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2.$$

One uses orthogonality to get

$$c_1 = \frac{\langle \mathbf{w}, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} = \frac{4}{5}, \quad c_2 = \frac{\langle \mathbf{w}, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} = \frac{7}{5}.$$

Finally,

$$\mathbf{u}(t) = \begin{pmatrix} 1.6 \\ 0.8 \end{pmatrix} e^{5t} + \begin{pmatrix} 1.4 \\ -2.8 \end{pmatrix} e^{-5t},$$

or

$$u(t) = 1.6e^{5t} + 1.4e^{-5t}, \quad v(t) = 0.8e^{5t} - 2.8e^{-5t}.$$

## Problem 9.1.16

The system can be written as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where

$$A = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of $A$ are 1 and $\pm 2i$; the corresponding eigenvectors are $(0,0,1)^T$ and $(\pm i, 1, 0)^T$. The eigenvector $(i,1,0)^T$ gives rise to a solution

$$\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} e^{2it}$$

of the system. Both the real and the imaginary part of this solution are solutions of the system. They are equal to

$$\begin{pmatrix} -\sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(2t) \\ \sin(2t) \\ 0 \end{pmatrix}.$$

The general real solution of the system is

$$c_1 \begin{pmatrix} -\sin(2t) \\ \cos(2t) \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \cos(2t) \\ \sin(2t) \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix},$$

or

$$x = -c_1 \sin(2t) + c_2 \cos(2t), \quad y = c_1 \cos(2t) + c_2 \sin(2t), \quad z = c_3 e^t.$$

In the formulas for $x$ and $y$ one can factor out

$$r = \sqrt{c_1^2 + c_2^2}.$$

If one denotes by $\phi$ such an angle that

$$\sin\phi = \frac{c_1}{r} \quad \text{and} \quad \cos\phi = \frac{c_2}{r}$$

then one can use the sine/cosine of the sum formulas to write the answer in the form

$$x = r\cos(2t+\phi), \quad y = r\sin(2t+\phi), \quad z = c_3 e^t.$$

One can interpret these formulas in the following way. The motion of fluid particles takes place on the cylinders $x^2 + y^2 = r^2$. Particles in the $xy$-plane ($c_3 = 0$) are moving along circles centered at the origin, in the counterclockwise direction. Above the $xy$-plane ($c_3 > 0$,) particles are moving along helix-like trajectories in the upward direction. Below the $xy$-plane ($c_3 < 0$,) particles are moving along helix-like trajectories in the downward direction.

## Problem 9.1.26(f)

The matrix of the system is upper triangular, so one can solve it starting from the last variable. One has $\dot{u}_3 = -u_3$, so

$$u_3 = c_3 e^{-t}.$$

The equation for $u_2$ is

$$\frac{du_2}{dt} = -u_2 + c_3 e^{-t}.$$

We move $u_2$ to the left hand side of the last equation and multiply the resulting equation by the integrating factor $e^t$ to get

$$\frac{d}{dt}\left(e^t u_2\right) = c_3.$$

Therefore, $e^t u_2 = c_3 t + c_2$ or

$$u_2 = c_2 e^{-t} + c_3 t e^{-t}.$$

The equation for $u_1$ is

$$\frac{du_1}{dt} + u_1 = (c_2 + c_3)e^{-t} + c_3 t e^{-t}.$$

One multiplies the last equation by $e^t$ and integrates the resulting equation:

$$e^t u_1 = c_1 + (c_2 + c_3)t + \frac{c_3}{2}t^2.$$

Finally,

$$u_1 = c_1 e^{-t} + (c_2 + c_3)te^{-t} + \frac{c_3}{2}t^2 e^{-t}.$$

## Problem 9.2.1(e)

The matrix of the system is

$$A = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & 1 \\ -3 & -3 & 2 \end{pmatrix}.$$

We compute

$$\begin{aligned} \det(A - \lambda I) &= (2+\lambda)^2(2-\lambda) + 3 + 3 - 3(2+\lambda) - (2-\lambda) - 3(2+\lambda) \\ &= -\lambda_3 - 2\lambda^2 - \lambda = -\lambda(\lambda+1)^2. \end{aligned}$$

The eigenvalues are 0 and $-1$. The eigenvalues are non-positive, and the eigenvalue 0 is complete because it is simple. We conclude that the origin is a stable equilibrium point of the system but it is not asymptotically stable.

## Problem 9.3.3(b)

The matrix of the system is

$$\begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix};$$

its eigenvalues are equal to $-3 \pm \sqrt{2}$. They are different, and both are negative. Therefore, $\mathbf{0}$ is a stable node.