Problem 1.8.5

Subtract b times the first equation from the second equation and 3 times the first equation from the third equation to get

$$\begin{cases} x_1 & +x_2 & +bx_3 & = 1 \\ & (3-b)x_2 & -(1+b^2)x_3 & = -2-b \\ & x_2 & +(1-3b)x_3 & = c-3 \end{cases}$$

Interchange the second and the third equations:

$$\begin{cases} x_1 & +x_2 & +bx_3 & = 1 \\ & x_2 & +(1-3b)x_3 & = c-3 \\ & (3-b)x_2 & -(1+b^2)x_3 & = -2-b \end{cases}$$

Subtract $(3-b)$ times the second equation from the third equation:

$$\begin{cases} x_1 & +x_2 & +bx_3 & = 1 \\ & x_2 & +(1-3b)x_3 & = c-3 \\ & & -(4b^2-10b+4)x_3 & = -2-b-(3-b)(c-3) \end{cases}$$

If $4b^2-10b+4 \neq 0$ then the system has a unique solution. The equation $4b^2-10b+4=0$ has solutions $b=2$ and $b=1/2$ (use the quadratic formula.) If $b=2$ or $b=1/2$, and $-2-b-(3-b)(c-3)=0$ then the system has infinitely many solutions. Otherwise, it has no solutions. The equation $-2-b-(3-b)(c-3)=0$ has the solution

$$c = \frac{-2-b}{3-b} + 3,$$

so $c=-1$ when $b=2$ and $c=2$ when $b=1/2$.

Answer: The system has a unique solution if $b \neq 2$ and $b \neq 1/2$; the system has infinitely many solutions if $b=2$ and $c=-1$ or if $b=1/2$ and $c=2$; the system has no solutions if $b=2$ and $c \neq -1$ or if $b=1/2$ and $c \neq 2$.

Problem 1.8.7(c)

Subtract the first row from the second row and add the first row to the third row:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, subtract the second row from the third row to get

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is in row echelon form; it has 2 non-zero pivots, so the rank of the matrix equals 2.

Problem 1.9.1(g)

Denote by A the matrix from the problem. First, we perform operations Row2-Row1, Row3-2Row1, Row4-5Row1, and Row5-2Row1:

$$\det A = \det \begin{pmatrix} 1 & -2 & 1 & 4 & -5 \\ 0 & 3 & -3 & -1 & 2 \\ 0 & 3 & -3 & -6 & 12 \\ 0 & 9 & -5 & -15 & 30 \\ 0 & 6 & -2 & -4 & 9 \end{pmatrix}.$$

Next, we perform operations Row3-Row2, Row4-3Row2, and Row5-2Row2:

$$\det A = \det \begin{pmatrix} 1 & -2 & 1 & 4 & -5 \\ 0 & 3 & -3 & -1 & 2 \\ 0 & 0 & 0 & -5 & 10 \\ 0 & 0 & 4 & -12 & 24 \\ 0 & 0 & 4 & -2 & 5 \end{pmatrix}.$$

Now, we interchange the third and the fourth rows:

$$\det A = -\det \begin{pmatrix} 1 & -2 & 1 & 4 & -5 \\ 0 & 3 & -3 & -1 & 2 \\ 0 & 0 & 4 & -12 & 24 \\ 0 & 0 & 0 & -5 & 10 \\ 0 & 0 & 4 & -2 & 5 \end{pmatrix}.$$

Subtract the third row from the fifth row:

$$\det A = -\det \begin{pmatrix} 1 & -2 & 1 & 4 & -5 \\ 0 & 3 & -3 & -1 & 2 \\ 0 & 0 & 4 & -12 & 24 \\ 0 & 0 & 0 & -5 & 10 \\ 0 & 0 & 0 & -10 & -19 \end{pmatrix}.$$

Finally, add twice of the fourth row to the fifth row:

$$\det A = -\det \begin{pmatrix} 1 & -2 & 1 & 4 & -5 \\ 0 & 3 & -3 & -1 & 2 \\ 0 & 0 & 4 & -12 & 24 \\ 0 & 0 & 0 & -5 & 10 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = -3 \times 4 \times (-5) = 60.$$

Problem 7.1.5(g)

$(1, -1, 2)^T$ is a normal vector to the plane $x - y + 2z = 0$, so the orthogonal projection of a vector $(x, y, z)^T$ on this plane is of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + c \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} x + c \\ y - c \\ z + 2c \end{pmatrix}. \tag{1}$$

One has

$$(x+c)-(y-c)+2(z+2c)=0;$$

$x-y+2z+6c=0$, and

$$c=-\frac{1}{6}x+\frac{1}{6}y-\frac{1}{3}z.$$

Substituting this value of c into (1), we get that the projection of a vector $(x,y,z)^T$ on the plane $x-y+2z=0$ is the vector

$$\begin{pmatrix} \frac{5}{6}x+\frac{1}{6}y-\frac{1}{3}z \\ \frac{1}{6}x+\frac{5}{6}y+\frac{1}{3}z \\ -\frac{1}{3}x+\frac{1}{3}y+\frac{1}{3}z \end{pmatrix}.$$

This linear function is given by the matrix

$$\begin{pmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Problem 7.1.7

The linear function L is given by a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The equation

$$L\begin{pmatrix} 1 \\ 2 \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

means

$$\begin{cases} a+2b=2 \\ c+2d=-1 \end{cases}$$

The equation

$$L\begin{pmatrix} 2 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

means

$$\begin{cases} 2a+b=0 \\ 2c+d=-1 \end{cases}$$

We solve these equations for a, b, c, and d: $a=-2/3$, $b=4/3$, $c=d=-1/3$.
Answer:

$$L\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -2/3 & 4/3 \\ -1/3 & -1/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$