Problem 2.2.4

Form the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ -1 & 1 & 3 \end{pmatrix}.$$

To show that the vectors $(1, 2, -1)^T$, $(2, 0, 1)^T$, and $(0, -1, 3)^T$ span $\mathbb{R}^3$ is the same as to show that the rank of A equals 3. To find the rank of A, we bring it to the row echelon form by performing row operations:

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 0 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & -4 & -1 \\ 0 & 0 & 2.25 \end{pmatrix}.$$

There are three non-zero pivots in the row echelon matrix, so the rank equals 3 indeed.

Problem 2.2.18

Let $u(x)$ be a solution to

$$u'' = x + u. \tag{1}$$

Let $v(x) = 2u(x)$. Then $v'' - v = 2(u'' - u) = 2x$, and $v'' - v - x = x \neq 0$. We conclude that $v(x)$ is not a solution to the equation (1). Therefore, the set of solutions to (1) does not form a vector space (otherwise, a multiple of every solution would be a solution.)

Problem 2.3.3(a)

Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The question of whether the vector $(1, -2, -3)^T$ lies in the span of $(1, 1, 0)^T$ and $(0, 1, 1)^T$ is equivalent to the question of whether the system

$$A\mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$$

has a solution. To answer this question, we form the augmented matrix

$$\begin{pmatrix} 1 & 0 & | & 1 \\ 1 & 1 & | & -2 \\ 0 & 1 & | & -3 \end{pmatrix},$$

and, by performing row operations, bring A to the row echelon form:

$$\begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & -3 \\ 0 & 1 & | & -3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & -3 \\ 0 & 0 & | & 0 \end{pmatrix}.$$

The system is solvable, so *the vector* $(1, -2, -3)^T$ *lies in the span of* $(1, 1, 0)^T$ *and* $(0, 1, 1)^T$.

Problem 2.3.33(a)

One notices that $(2-x^2)+(x^2+x-2)=x$, so

$$3(2-x^2)-(3x)+3(x^2+x-2)=0.$$

Answer: the functions $2-x^2$, $3x$, and x^2+x-2 are linearly dependent.

Problem 2.4.8(a)

We bring the matrix A to the row echelon form:

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -6 & 5 & -4 \end{pmatrix}.$$

There are two free variables, x_3, and x_4, so the dimension of the space of solutions equals 2. We solve the system by back substitution:

$$x_2=\frac{5}{6}x_3-\frac{2}{3}x_4,$$

$$x_1=-2x_2+x_3-x_4=-\frac{2}{3}x_3+\frac{1}{3}x_4.$$

One can set $x_3=6$, $x_4=0$ for the first basis vector and $x_3=0$, $x_4=3$ for the second basis vector (the reason why I took the numbers 6 and 3 is not to have fractions in the answer; one can take any non-zero numbers for 6 and 3.) Then, the basis is

$$\begin{pmatrix} -4 \\ 5 \\ 6 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}.$$