Problem 2.5.1(d)

To solve the equation $Ax = b$, we form the augmented matrix

$$\begin{pmatrix} 1 & -1 & 0 & 1 & | & b_1 \\ -1 & 0 & 1 & -1 & | & b_2 \\ 1 & -2 & 1 & 1 & | & b_3 \\ 1 & 2 & -3 & 1 & | & b_4 \end{pmatrix}.$$

By doing elementary row operations, we bring the matrix to the row echelon form:

$$\begin{pmatrix} 1 & -1 & 0 & 1 & | & b_1 \\ 0 & -1 & 1 & 0 & | & b_2 + b_1 \\ 0 & -1 & 1 & 0 & | & b_3 - b_1 \\ 0 & 3 & -3 & 0 & | & b_4 - b_1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 & 1 & | & b_1 \\ 0 & -1 & 1 & 0 & | & b_2 + b_1 \\ 0 & 0 & 0 & 0 & | & b_3 - b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & | & b_4 + 3b_2 + 2b_1 \end{pmatrix}.$$

The range is the set of vectors b such that $b_3 = 2b_1 + b_2$ and $b_4 = -2b_1 - 3b_2$. The range of A is spanned by the vectors $(1, 0, 2, -2)^T$ and $(0, 1, 1, -3)^T$. The kernel of A is the set of vectors x such that $x_2 = x_3$ and $x_1 = x_2 - x_4 = x_3 - x_4$. It is spanned by the vectors $(1, 1, 1, 0)^T$ and $(-1, 0, 0, 1)^T$.

Problem 2.5.5(a)

For a particular solution of the equation

$$x - y + 3z = 1,$$

one can take $\mathbf{x}^* = (1, 0, 0)^T$. The kernel of the matrix

$$\begin{pmatrix} 1 & -1 & 3 \end{pmatrix}$$

is two-dimensional, and it is spanned by the vectors $(1, 1, 0)^T$ and $(-3, 0, 1)^T$. Then the general solution of the equation can be written as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

Problem 2.5.17

Let

$$\mathbf{x} = c_1 \mathbf{x}_1^* + c_2 \mathbf{x}_2^*$$

be a linear combination of $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$. Then

$$A\mathbf{x} = c_1 A\mathbf{x}_1^* + c_2 A\mathbf{x}_2^* = c_1 \mathbf{b} + c_2 \mathbf{b} = (c_1 + c_2)\mathbf{b}.$$

If $\mathbf{b} \neq \mathbf{0}$ then $\mathbf{x}$ solves the equation $A\mathbf{x} = \mathbf{b}$ when $c_1 + c_2 = 1$. If $\mathbf{b} = \mathbf{0}$ then $\mathbf{x}$ solves the equation for all values of c_1 and c_2.

Problem 2.5.26

The set of vectors from the problem is the range of the matrix

$$A = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

As the range of a 4×4-matrix, it is a linear subspace in $\mathbb{R}^4$. Its dimension equals the rank of the matrix A. To find the rank, we bring it to the row echelon form:

$$\begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 3 & 2 & 4 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 7/3 & -7/3 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 7/3 & -7/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rank of the matrix equals 3; therefore, the dimension of the space in question equals 3.

Problem 3.1.21(b)

The formula

$$\int_{-1}^{1} f(x)g(x)x\,dx$$

does not define an inner product on $C^0[-1,1]$ because

$$\int_{-1}^{1} [f(x)]^2 x\,dx$$

is not necessarily positive.