Problem 3.5.7(b)

The matrix of the quadratic form is

$$\begin{pmatrix} 3 & -4 & 0.5 \\ -4 & -2 & 0 \\ 0.5 & 0 & 1 \end{pmatrix}.$$

It is not positive definite because one of its diagonal entries is negative.

Problem 3.5.10

a) All pivots of a positive definite matrix are positive; the determinant is equal to their product, so it is positive.
b) All diagonal entries of a positive definite matrix are positive; the trace is equal to their sum, so it is also positive.
c) Let

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

be a symmetric matrix. Suppose that $a + d > 0$ (the trace) and $ad - b^2 > 0$ (the determinant.) Notice that the numbers a and d must be of the same sign: otherwise, $ad \le 0$ and $ad - b^2 \le 0$. They are positive because $a + d > 0$. In particular, $a > 0$. Conclusion: $a > 0$ and $ad - b^2 > 0$; therefore, the matrix is positive definite.
d) Let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 100 \end{pmatrix}.$$

The trace of A equals 98; its determinant is equal to 100, and A is not positive definite.

Problem 3.6.29

a) To find whether vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, we form a matrix

$$\begin{pmatrix} 1 & 0 & -1+i \\ i & 1+i & 1+i \\ 0 & 2 & -1 \end{pmatrix}.$$

Subtract i times the first row from the second row to get

$$\begin{pmatrix} 1 & 0 & -1+i \\ 0 & 1+i & 2+2i \\ 0 & 2 & -1 \end{pmatrix}.$$

Subtract $2/(2+i)$ times the second row from the third row:

$$\begin{pmatrix} 1 & 0 & -1+i \\ 0 & 1+i & 2+2i \\ 0 & 0 & -5 \end{pmatrix}.$$

All pivots in the last matrix are non-zero, so the vectors are linearly independent.
b) Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis in $\mathbb{C}^3$: they are linearly independent and

there are three of them.
c)

$$||\mathbf{v}_1|| = \sqrt{1+1} = \sqrt{2}, \quad ||\mathbf{v}_2|| = \sqrt{2=4} = \sqrt{6}, \quad ||\mathbf{v}_3|| = \sqrt{2+2+1} = \sqrt{5}.$$

d)

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = i(1-i) = 1+i, \quad \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = -1 - i + i(1-i) = 0,$$

$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = (1+i)(1-i) - 2 = 0.$$

The vector $\mathbf{v}_3$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$.
e) The vectors do not form an orthogonal basis in $\mathbb{C}^3$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal.

Problem 3.6.31

False: the vector $(1,1)^T$ belongs to the set, but the vector $i(1,1)^T = (i,i)^T$ does not belong to it.

Problem 4.1.3(e)

The vector $\mathbf{b} = (-1,2)^T$ lies on the line $2x + y = 0$, so the closest point to $\mathbf{b}$ on the line is $(-1,2)$.