Problem 4.2.3(a)

The function is of the form

$$f(\mathbf{x}) = \mathbf{x}^T K\mathbf{x} - 2\langle \mathbf{f}, \mathbf{x}\rangle - c$$

where $\mathbf{x} = (x, y)^T$,

$$K = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}.$$

The matrix K is positive definite: its pivots are 1 and 3, so the function $f(\mathbf{x})$ takes the minimum value at $\mathbf{x}^* = K^{-1}\mathbf{f}$. One computes

$$K^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix},$$

so

$$\mathbf{x}^* = \begin{pmatrix} -2/3 \\ -1/6 \end{pmatrix}.$$

To find the minimum value, we compute

$$f(-2/3, -1/6) = -4/3.$$

Problem 4.3.5

First, let us find a basis in the space spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 \begin{pmatrix} 0 \\ 1 \\ -2 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix}.$$

To do that, we form the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & -2 & 3 \\ 0 & -1 & 2 \end{pmatrix}$$

and bring it to the row echelon form:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the rank of A equals 2, so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a space of dimension 2; vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis in this space.

Now, we form the Gram matrix

$$K = \begin{pmatrix} \langle \mathbf{v}_1, \mathbf{v}_1\rangle & \langle \mathbf{v}_1, \mathbf{v}_2\rangle \\ \langle \mathbf{v}_2, \mathbf{v}_1\rangle & \langle \mathbf{v}_2, \mathbf{v}_2\rangle \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$

and the vector

$$\mathbf{f} = \begin{pmatrix} \langle \mathbf{b}, \mathbf{v}_1 \rangle \\ \langle \mathbf{b}, \mathbf{v}_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We compute the vextor

$$\mathbf{x}^* = K^{-1}\mathbf{f} = \frac{1}{10} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}.$$

The vector in the space spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ that is the closest to $\mathbf{b}$ is

$$\mathbf{v} = 0.5\mathbf{v}_1 - 0.5\mathbf{v}_2 = (0, 5, 0.5, 0.5, 0.5)^T.$$

The distance between $\mathbf{b}$ and $\mathbf{v}$ equals

$$\sqrt{0.5^2 + 0.5^2 + 1.5^2 + 2.5^2} = \sqrt{9} = 3.$$

Problem 4.3.14(e)

In the matrix form, the problem is to find the least square solution of $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

One computes

$$K = A^T A = \begin{pmatrix} 3 & 1 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ -1 & 1 & -1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{f} = A^T\mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 1 \\ -2 \end{pmatrix}.$$

The solution is given by

$$\mathbf{x} = K^{-1}\mathbf{f} = \begin{pmatrix} 2/3 & -1/2 & -1/6 & 1/3 \\ -1/2 & 1 & 0 & -1/2 \\ -1/6 & 0 & 2/3 & 1/6 \\ 1/3 & -1/2 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2 \\ -1/3 \\ -4/3 \end{pmatrix}.$$

I have skipped the computation of K^{-1}.

Problem 4.4.8(a)

The exponential growth model says that $P = C\ln(at)$. taking the logarithm of both sides,

$$y = \ln P = \ln C + at = b + at \quad \text{where} \quad b = \ln C.$$

It is convenient to denote by t the time elapsed from 1900, in decades. we form a table

year	t	P = population in millions	$y = \ln P$
1900	0	76	4.33
1910	1	92	4.52
1920	2	106	4.66
1930	3	123	4.81
1940	4	131	4.88
1950	5	150	5.01

We construct

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 4.33 \\ 4.52 \\ 4.66 \\ 4.81 \\ 4.88 \\ 5.01 \end{pmatrix}.$$

Then

$$A^T A = \begin{pmatrix} 6 & 15 \\ 15 & 55 \end{pmatrix} \quad \text{and} \quad A^T \mathbf{y} = \begin{pmatrix} 28.21 \\ 72.84 \end{pmatrix}.$$

One has

$$\begin{pmatrix} b \\ a \end{pmatrix} = (A^T A)^{-1}(A^T \mathbf{y}) = \frac{1}{105} \begin{pmatrix} 55 & -15 \\ -15 & 6 \end{pmatrix} \begin{pmatrix} 28.21 \\ 72.84 \end{pmatrix} = \begin{pmatrix} 4.371 \\ 0.132 \end{pmatrix}.$$

Therefore, $a = 0.132$ and $C = e^b = e^{4.371} = 79.19$. The exponential growth model gives the prediction

$$P = 79.12 e^{0.132t}.$$

For the years 2000, 2010, and 2050, we take $t = 10$, $t = 11$, and $t = 15$, respectively, to get

year	projected population in millions
2000	296
2010	338
2050	573

Problem 4.4.12(c)

We find

$$L_1(t) = \frac{t(t-1)}{(-1) \times (-1-1)} = \frac{t(t-1)}{2}$$
$$L_2(t) = \frac{(t+1)(t-1)}{(0+1) \times (0-1)} = -(t+1)(t-1)$$
$$L_3(t) = \frac{t(t+1)}{1 \times (1+1)} = \frac{t(t+1)}{2},$$

and the Lagrange interpolation polynomial is

$$L(x) = L_1(t) + 2L_2(t) - L_3(t) = \frac{1}{2}t(t-1) - 2(t-1)(t+1) - \frac{1}{2}t(t+1).$$