Problem 5.1.6

The formula $\langle \mathbf{v}, \mathbf{w}\rangle = av_1w_1 + bv_2w_2$ defines an inner product when both a and b are positive. The vectors $\mathbf{v} = (1, 2)^T$ and $\mathbf{w} = (-1, 1)^T$ form a basis in $\mathbb{R}^2$. Then,

$$\langle \mathbf{v}, \mathbf{w}\rangle = -a + 2b,$$

so $\mathbf{v}$ and $\mathbf{w}$ are orthogonal when $a = 2b$.
Answer: $a = 2b > 0$.

Problem 5.1.27

(a) One computes

$$\begin{aligned}
\langle P_0(t), P_1(t)\rangle &= \int_0^1 (t - (2/3))tdt = \frac{t^3}{3} - \frac{t^2}{3}\bigg|_0^1 = 0,\\
\langle P_0(t), P_2(t)\rangle &= \int_0^1 (t^2 - (6/5)t + (3/10))tdt = \frac{t^4}{4} - \frac{2t^3}{5} + \frac{3t}{20}\bigg|_0^1 = 0,\\
\langle P_1(t), P_2(t)\rangle &= \int_0^1 (t - (2/3))(t^2 - (6/5)t + (3/10))tdt\\
&= \int_0^1 (t^4 - (28/15)t^3 + (11/10)t^2 - (1/5)t)dt\\
&= \frac{t^5}{5} - \frac{7t^4}{15} + \frac{11t^3}{30} - \frac{t^2}{10}\bigg|_0^1 = 0.
\end{aligned}$$

(b) To find an orthonormal basis, we compute

$$\begin{aligned}
||P_0||^2 &= \int_0^1 tdt = \frac{1}{2},\\
||P_1||^2 &= \int_0^1 (t - (2/2))^2 tdt = \frac{1}{36},\\
||P_2||^2 &= \int_0^1 (t^2 - (6/5)t + (3/10))^2 tdt = \frac{1}{600},
\end{aligned}$$

so

$$\begin{aligned}
&Q_0(t) = \sqrt{2},\\
&Q_1(t) = 6P_1(t) = 6t - 4,\\
&Q_2(t) = \sqrt{600}P_2(t) = \sqrt{600}(t^2 - (6/5)t + (3/10))
\end{aligned}$$

is the corresponding orthonormal basis.
(c) To find the representation of t^2 as a linear combination of P_0, P_1, and P_2 one computes

$$\begin{aligned}
\langle P_0(t), t^2\rangle &= \int_0^1 t^3 dt = \frac{1}{4},\\
\langle P_1(t), t^2\rangle &= \int_0^1 (t - (2/3))t^3 dt = \frac{1}{30},\\
\langle P_2(t), t^2\rangle &= \int_0^1 (t^2 - (6/5)t + (3/10))t^3 dt = \frac{1}{600}.
\end{aligned}$$

Therefore,

$$t^2 = \frac{\langle P_0(t), t^2\rangle}{||P_0||^2} P_0 + \frac{\langle P_1(t), t^2\rangle}{||P_1||^2} P_1 + \frac{\langle P_2(t), t^2\rangle}{||P_2||^2} P_2 = 0.5P_0(t) + 1.2P_1(t) + P_2(t).$$

Problem 5.2.5

We set

$$\mathbf{v}_1 = \mathbf{w}_1 = (1, -1, -1, 1, 1)^T.$$

Then

$$\mathbf{v}_2 = \mathbf{w}_2 - \frac{\langle \mathbf{w}_2, \mathbf{v}_1\rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \frac{-5}{5}\begin{pmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}$$

and

$$\mathbf{v}_3 = \mathbf{w}_3 - \frac{\langle \mathbf{w}_3, \mathbf{v}_1\rangle}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\langle \mathbf{w}_3, \mathbf{v}_2\rangle}{||\mathbf{v}_2||^2}\mathbf{v}_2 = \mathbf{w}_3 - 4\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 2 \\ -2 \end{pmatrix}.$$

Problem 5.3.1(d)

Let

$$\mathbf{u}_1 = \begin{pmatrix} -1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix}$$

be the column vectors of the matrix. One checks that

$$\langle \mathbf{u}_1, \mathbf{u}_2\rangle = \langle \mathbf{u}_1, \mathbf{u}_3\rangle = \langle \mathbf{u}_2, \mathbf{u}_3\rangle = 0$$

and

$$||\mathbf{u}_1||^2 = ||\mathbf{u}_2||^2 = ||\mathbf{u}_3||^2 = (-1/3)^2 + (2/3)^2 + (2/3)^2 = 1.$$

Therefore, the matrix is orthogonal. To find out, whether it is proper orthogonal, one should compute the determinant of the matrix. For doing that, we add twice of the first row to both the second and the third rows to get

$$\begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Then we subtract twice of the second row from the third row; the resulting upper triangular matrix is

$$\begin{pmatrix} -1/3 & 2/3 & 2/3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{pmatrix}.$$

The determinant equals $(-1/3) \times 1 \times (-3) = 1$, so the matrix is proper orthogonal.

Problem 5.3.27(d)

Let

$$A = (\,\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3\,) = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

We apply the Gram–Schmidt orthogonalization process to the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$.

Step 1. $\mathbf{w}_1 = r_{11}\mathbf{u}_1$ where $r_{11} = ||\mathbf{w}_1||$. We get

$$r_{11} = \sqrt{2} \quad \text{and} \quad \mathbf{u}_1 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

Step 2. $\mathbf{w}_2 = r_{12}\mathbf{u}_1 + r_{22}\mathbf{u}_2$ where

$$r_{12} = \langle \mathbf{w}_2, \mathbf{u}_1 \rangle = -2/\sqrt{2} = -\sqrt{2}.$$

Then

$$r_{22}\mathbf{u}_2 = \mathbf{w}_2 - r_{12}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$r_{22} = 1 \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Step 3. $\mathbf{w}_3 = r_{13}\mathbf{u}_1 + r_{23}\mathbf{u}_2 + r_{33}\mathbf{u}_3$ where

$$r_{13} = \langle \mathbf{w}_3, \mathbf{u}_1 \rangle = -4/\sqrt{2} = -2\sqrt{2} \quad \text{and} \quad r_{23} = \langle \mathbf{w}_3, \mathbf{u}_2 \rangle = 2.$$

Then

$$r_{33}\mathbf{u}_3 = \mathbf{w}_3 - r_{13}\mathbf{u}_1 - r_{23}\mathbf{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore

$$r_{33} = \sqrt{2} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

Finally, $A = QR$ where

$$Q = (\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3\,) = \begin{pmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

and

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & -2\sqrt{2} \\ 0 & 1 & 2 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$