The Maximum–Minimum Theorem
for the Heat Equation

Let $u(x,t)$ be a solution of the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

that is defined in a rectangle $R = \{(x,t) : a \le x \le b, 0 \le t \le T\}$. Let Γ be the union of the bottom of the rectangle, $\{(x,0) : a \le x \le b\}$, and its lateral sides $\{(a,t) : 0 \le t \le T\}$, $\{(b,t) : 0 \le t \le T\}$. Then

$$\max\{u(x,t) : (x,t) \in R\} = \max\{u(x,t) : (x,t) \in \Gamma\} \tag{2}$$

and

$$\min\{u(x,t) : (x,t) \in R\} = \min\{u(x,t) : (x,t) \in \Gamma\}. \tag{3}$$

The maximum–minimum theorem has a simple interpretation: if there is no heat sources inside a rod, then the temperature can not rise above the maximum of the initial temperatures (at the moment $t = 0$) and the temperatures at the ends of the rod ($x = a$ and $x = b$); similarly, the temperature inside a rod can not fall below the minimum of the initial temperatures and the temperatures at the ends of the rod.

Proof. First, we prove (2), and then the proof of (3) will be reduced to (2). The proof of the theorem involves a simple trick, which is not obvious.

Let us assume that (2) is not true, that is

$$M = \max\{u(x,t) : (x,t) \in R\} > \max\{u(x,t) : (x,t) \in \Gamma\} = m.$$

Notice that M can not be smaller than m: both M and m are maximal values of the same function, and M is the maximal value over a bigger set. So, if M and m are different, then $M > m$.

Introduce an auxiliary function (this is the trick!)

$$v(x,t) = u(x,t) + \epsilon(x-a)^2.$$

A positive number ϵ is chosen in such a way that

$$M_1 = \max\{v(x,t) : (x,t) \in R\} > \max\{v(x,t) : (x,t) \in \Gamma\} = m_1. \tag{4}$$

To show that (4) can be satisfied, we notice that $M_1 \ge M$ because $v(x,t) \ge u(x,t)$, and $m_1 \le m + \epsilon(b-a)^2$ because the maximum value of the function $\epsilon(x-a)^2$ on Γ equals exactly $\epsilon(b-a)^2$. Therefore, (4) is satisfied if $\epsilon(b-a)^2 < M - m$, or

$$\epsilon < \frac{M-m}{(b-a)^2}.$$

The function $v(x,t)$ assumes its maximal value at some point $(x_0,t_0) \in R$, $u(x_0,t_0) = M_1$. The point (x_0,t_0) does not lie on Γ (otherwise, we would have had $M_1 = m_1$!), so

$$a < x_0 < b, \quad t_0 > 0.$$

Now, we use the fact that (x_0,t_0) is a point of maximum of the function $v(x,t)$ in R:

$$v_x(x_0,t_0) = 0, \quad v_{xx}(x_0,t_0) \le 0, \quad v_t(x_0,t_0) \ge 0.$$

Notice that if $t_0 < T$ then $v_t(x_0,t_0) = 0$. In any case,

$$v_t(x_0,t_0) - v_{xx}(x_0,t_0) \ge 0.$$

On the other hand,

$$v_t(x_0,t_0) - v_{xx}(x_0,t_0) = u_t(x_0,t_0) - u_{xx}(x_0,t_0) - 2\epsilon = -2\epsilon < 0.$$

The contradiction shows that our assumption $M > m$ was wrong. That proves (2).

To derive (3), we notice that if a function $u(x,t)$ solves the heat equation then the function $-u(x,t)$ is also a solution of the same equation. Then, $\max(-u(x,t)) = -\min u(x,t)$. If one applies the maximum theorem to $-u(x,t)$, then one gets the minimum theorem for the function $u(x,t)$.
Q.E.D.

The maximum–minimum theorem has an important corollary. Consider solutions of the heat equation (1) that satisfy the boundary conditions

$$u(a,t) = \phi(t), \quad u(b,t) = \psi(t) \tag{5}$$

and the initial condition

$$u(x,0) = f(x). \tag{6}$$

This means that one prescribes the temperature distribution inside a rod at the initial moment of time, and one also prescribes the temperature at both ends of the rod.

Corollary (The Uniqueness Theorem). *The solution of the problem (1), (5), (6) is unique.*

Proof. Assume that two functions $u_1(x,t)$ and $u_2(x,t)$ satisfy the heat equation (1); they satisfy the same boundary conditions (5) and the same initial condition (6). Let $u(x,t) = u_1(x,t) - u_2(x,t)$. Then the function $u(x,t)$ solves the heat equation. In addition,

$$u(a,t) = 0, \quad u(b,t) = 0, \quad u(x,0) = 0.$$

Chose any positive number T, and let $R = \{(x,t) : a \le x \le b, 0 \le t \le T\}$. As in the maximum–minimum theorem, we denote by Γ the union of the bottom of the rectangle R and its lateral sides. Then $u(x,t) = 0$ when a point (x,t) belongs to Γ. So, the maximum–minimum theorem tells us that

$$\max\{u(x,t) : (x,t) \in R\} = \min\{u(x,t) : (x,t) \in R\} = 0.$$

This means that $u(x,t) = 0$ for $a \le x \le b$ and $0 \le t \le T$. The number T was chosen arbitrarily, so one can replace $0 \le t \le T$ by $t \ge 0$. The equality $u(x,t) = 0$ is equivalent to $u_1(x,t) = u_2(x,t)$.

To summarize, we have shown that any two solutions of the problem (1), (5), (6) are equal to each other. This means that the problem admits not more than one solution. Q.E.D.

Remark. The Uniqueness Theorem tells us that the problem (1), (5), (6) has *not more than one solution*. It does not address the question of whether or not a solution of this problem actually *exists*.

Problem. Show that the heat equation (1) has not more than one solution that satisfies the initial condition (6) and the boundary conditions

$$u_x(a,t) = \phi(t), \quad u_x(b,t) = \psi(t).$$

Here $\phi(t)$, $\psi(t)$, and $f(x)$ are given functions.

Hint. If a function $u(x,t)$ solves the heat equation, then $u_x(x,t)$ is also a solution.