in principle be observed from the exterior frame of reference.

Equation (26) within the body has the solution

$$\rho = r + \frac{m}{2R_0^3}R^3 - \frac{3m}{2R_0}R, \tag{63}$$

this being fitted to the exterior solution (47) at the boundary.

We have considered a model for which the pressure is zero, and even in this case we have no catastrophic strong compression of the matter. In real objects, the process of gravitational contraction will be even weaker. Since, as was shown above, the growth in the density of the collapsing body in the comoving frame is halted, the relativistic theory of gravitation contains a new phenomenon – gravitational restraint. It is by virtue of this phenomenon, with allowance for the mechanism of formation of neutron stars, that it is in principle impossible for objects with density greater than 10^{16} g/cm^3 to be formed. If objects with greater density exist at all, they can only have a primordial origin. Thus, in principle, a singularity does not arise during gravitational contraction in the relativistic theory of gravitation.

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MAXWELL–BLOCH EQUATION AND THE INVERSE SCATTERING METHOD

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The inverse scattering method is used to construct general solutions of the Maxwell–Bloch system, these solutions being determined by specification of the polarization as $t \to -\infty$. The solutions are classified. An approximate solution is obtained for the mixed boundary-value problem for the Maxwell–Bloch system describing the phenomenon of superfluorescence (generation of a pulse from initial fluctuations of the polarization in a mirrorless laser).

Introduction

Among the numerous nonlinear equations of mathematical physics now known to which the inverse scattering method can be applied, an important part is played by the system of Maxwell–Bloch (MB) equations. We shall write this system in the form

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\mathscr{E} = \langle\rho\rangle, \tag{I.1}$$

$$\frac{\partial \rho}{\partial t} + 2i\lambda\rho = N\mathscr{E}, \tag{I.2}$$

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$$\frac{\partial N}{\partial t}=-\frac{1}{2}(\mathscr{E}^*\rho+\mathscr{E}\rho^*). \tag{I.3}$$

Here, $\mathscr{E}=\mathscr{E}(t,x)$ is a complex function of the coordinate x and the time t, and $\rho=\rho(t,x,\lambda)$ and $N=N(t,x,\lambda)$ are complex and real functions of x, t, and the additional parameter λ. The brackets $\langle\,\rangle$ denote averaging over λ with given weight function $g(\lambda)>0$,

$$\int_{-\infty}^{\infty} g(\lambda)\,d\lambda=1, \quad \langle\rho\rangle=\int_{-\infty}^{\infty} g(\lambda)\,\rho(t,x,\lambda)\,d\lambda. \tag{I.4}$$

Equations (I.1)-(I.3) arise in many problems of physics. The most important is the application of this system to the problem of the propagation of an electromagnetic wave in a medium with distributed two-level atoms, in particular, to the problem of self-induced transparency, and also to laser type problems – of quantum amplifiers and superfluorescence. In all these cases, $\mathscr{E}(t,x)$ is the complex envelope of an electromagnetic wave of fixed polarization, and N and ρ are elements of the density matrix of the atomic subsystem,

$$\hat{\rho}=\begin{bmatrix} N & \rho \\ \rho^* & -N \end{bmatrix}. \tag{I.5}$$

The parameter λ is the deviation of the transition frequency of the atom from its mean value, and the function g describes the shaped of the spectral line. Equations (I.2)-(I.3) can be written in the matrix form

$$\frac{\partial\hat{\rho}}{\partial t}=i[-\mathrm{I}\lambda+H,\hat{\rho}], \tag{I.6}$$

$$\mathrm{I}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H=\frac{i}{2}\begin{bmatrix} 0 & \mathscr{E} \\ -\mathscr{E}^* & 0 \end{bmatrix}. \tag{I.7}$$

The MB system (I.1)-(I.3) became well known after the papers of Lamb [1, 2]. In the following years, the MB equations were the subject of a certain number of studies, among which we should mention [3] of Ablowits, Kaup, and Newell. In this paper, in particular, it was established that the inverse scattering method can be applied to the MB equations; in fact, the MB equations are one of the first examples of the successful use of the method; the solutions of Eqs. (I.1)-(I.3) describing the physical phenomenon of self-induced transparency have also been fairly fully studied. In [3], as in the majority of other studies, a very particular class of solutions of the MB system was considered; for it $\rho\to 0$ as $t\to-\infty$. These solutions are sufficient to describe the propagation of an electromagnetic wave in a stable absorbing medium (the case of self-induced transparency), but, in general, are not sufficient for describing waves in an unstable medium, which is characteristic of laser problems. This is explained by the fact that until recently the inverse scattering method has not been applied systematically to laser problems (an exception is [4], in which the asymptotic behavior of pulse propagation in a long quantum amplifier in the absence of fluctuations of the polarization is considered). The foundation of the systematic application of the inverse scattering method to laser problems was laid in [5], which introduced the concepts of "spontaneous" and "causal" solutions of the system (I.1)-(I.3). In the present paper, we present general solutions of the MB system, these being determined in particular by the specification of $\rho(t,x,\lambda)$ as $t\to-\infty$, and we classify these solutions. Our solutions describe in principle the important phenomenon of superfluorescence – the generation in a mirrorless laser of a pulse from initial fluctuations of the polarization; however, the solution of this problem leads to a mixed boundary-value problem for the system (I.1)-(I.3). As a rule, mixed problems are not amenable to solution by the inverse scattering method. We have succeeded, apparently for the first time, in obtaining an effective approximate solution of the mixed problem describing superfluorescence for lasers of not too great a length. We note that for such lasers in the case of an infinitely narrow line an approximate solution to the superfluorescence problem can be obtained in an elementary manner; further, the construction developed in the present paper makes it possible to justify more rigorously the method proposed in [6, 7]. We note also that the development of the technique of the inverse scattering method for the system (I.1)-(I.3) includes not only the solution of the mixed problem but also has a number of features that are nonstandard from the point of view of the inverse scattering method and are of independent interest.

1. General Structure of the Solution

We consider the following overdetermined system of linear equations for the matrix function $\Psi(t,x,\lambda)$:

$$\frac{\partial \Psi}{\partial t}=i(-I\lambda+H)\Psi, \tag{1.1}$$

$$\frac{\partial \Psi}{\partial x}+i\left(-I\lambda+H+\frac{1}{4}\int_{-\infty}^{\infty}\frac{\hat{\rho}(t,x,\eta)g(\eta)}{\eta-\lambda}d\eta\right)\Psi=0, \tag{1.2}$$

and require Eqs. (1.1) and (1.2) to have a common fundamental solution. Equations (1.1) and (1.2) can be written in the form

$$\frac{\partial \Psi}{\partial t}=U\Psi, \quad \frac{\partial \Psi}{\partial x}=V\Psi, \tag{1.3}$$

where

$$U=i(-I\lambda+H), \quad V=i\left(I\lambda-H-\frac{1}{4}\int_{-\infty}^{\infty}\frac{\hat{\rho}(t,x,\eta)g(\eta)}{\eta-\lambda}d\eta\right). \tag{1.4}$$

The conditions of compatibility of the system (1.3) have the form

$$\frac{\partial U}{\partial x}-\frac{\partial V}{\partial t}+[U,V]=0. \tag{1.5}$$

By direct calculation we verify

PROPOSITION 1. Equation (1.5) with U and V specified by (1.4) is equivalent to the system of Maxwell–Bloch equations (I.1)–(I.3).

Proposition 1 ensures that the inverse scattering method can be applied to the system (I.1)–(I.3). In the usual form of the method [8], it is assumed that U and V are rational functions of the spectral parameter λ. In our case, a rational dependence holds only if $g(\lambda)=\sum g_n\delta(\lambda-\lambda_n)$, i.e., the line shape consists of a discrete set of infinitely narrow lines. However, following [3], it is expedient to consider the general case of a continuous function $g(\lambda)$, when the function V has a cut along the real λ axis.

We consider Eq. (1.1) in the class of coefficients $\mathscr{E}(t,x)$ that decrease rapidly with respect to t and satisfy the condition

$$\int_{-\infty}^{\infty}|\mathscr{E}(t,x)|\,dt<\infty \tag{1.6}$$

(physically, this corresponds to considering pulse processes). We pose for (1.1) the scattering problem, introducing sets of Jost functions – solutions $\chi^{\pm}$ of Eq. (1.1) that are determined by the asymptotic behaviors

$$\chi^{\pm}\to\exp(-iI\lambda t), \quad t\to\pm\infty, \tag{1.7}$$

and determine the scattering matrix S in accordance with

$$\chi^{-}=\chi^{+}S, \quad S=S(x,\lambda). \tag{1.8}$$

As is well known (see [10]), the S matrix has the form

$$S=\begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}, \tag{1.9}$$

where $|a|^2+|b|^2=1$. The function $a(x,\lambda)$ is analytic in the upper half-plane Im $\lambda > 0$ and, the condition (1.6) being satisfied, has there a finite number J of zeros λ_j, these being eigenvalues of the spectral problem (1.1). The corresponding eigenfunctions can be determined by the asymptotic behaviors

$$\Psi_j\to\begin{pmatrix} 0 \\ \exp(i\lambda_j t) \end{pmatrix}, \quad t\to\infty,$$

and then

$$\Psi_j\to\begin{pmatrix} c_j\exp(-i\lambda_j t) \\ 0 \end{pmatrix}, \quad t\to-\infty.$$

The function $c(x,\lambda)=b^*(x,\lambda)/a(x,\lambda)$ and the set of constants λ_j, c_j form the "scattering data" for the spectral problem (1.1). If the function $\mathscr{E}(t,x)$ satisfies the condition

$$|\mathscr{E}(t,x)| < \alpha \exp(2\gamma t) \tag{1.10}$$

as $t \to -\infty$, and $\gamma > \max \operatorname{Im} \lambda_j$, then the matrix S can be analytically continued into the strip $\operatorname{Im} \lambda < \gamma$ of the upper half-plane. In this case, $c_j = b^*|_{\lambda=\lambda_j}$. We denote $M_j = (c/a)|_{\lambda=\lambda_j}$ and construct the function

$$F(t,x) = \sum_{j=1}^{J} M_j \exp(-i\lambda_j t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} c(x,\lambda)\exp(-i\lambda t)\,d\lambda. \tag{1.11}$$

If the condition (1.10) is satisfied, the function $c(x, \lambda)$ is meromorphic in the strip $\gamma > \operatorname{Im}\lambda > 0$, and (1.11) can be rewritten in the form

$$F(t,x) = \frac{1}{2\pi}\int_{-\infty+i\gamma}^{\infty+i\gamma} c(x,\lambda)\exp(-i\lambda t)\,d\lambda. \tag{1.12}$$

The construction from $\mathscr{E}(t, x)$ of the set of scattering data $c(x, \lambda)$, λ_j, c_j (the solution of the direct scattering problem) defines a mapping $\mathscr{E} \to F$. This mapping is one to one – to invert it, it is necessary to solve for all t the system of integral equations (Marchenko equations)

$$K_1(t,\tau,x) = F(t+\tau,x) + \int_{-\infty}^{t} K_2(t,\xi,x)F(\tau+\xi,x)\,d\xi, \tag{1.13}$$

$$K_2(t,\tau,x) = -\int_{-\infty}^{t} K_1(t,\xi,x)F^*(\tau+\xi,x)\,d\xi. \tag{1.14}$$

Then

$$\mathscr{E}(t,x) = 4K_1(t,t,x), \tag{1.15}$$

$$\int_{-\infty}^{t} |\mathscr{E}(t,x)|^2\,dt = -4K_2(t,t,x). \tag{1.16}$$

The mapping $\mathscr{E} \rightleftharpoons F$ is remarkable in that the function F, as shown below, satisfies a linear integrodifferential equation.

It follows from Eqs. (I.2) and (I.3) that

$$\frac{\partial}{\partial t}(|\rho|^2 + N^2) = 0.$$

In what follows, we shall set

$$|\rho|^2 + N^2 = 1. \tag{1.17}$$

Going to the limits $t \to \pm\infty$, we obtain

$$\hat{\rho} \to \exp(-iI\lambda t)\hat{\rho}^{\pm}\exp(iI\lambda t). \tag{1.18}$$

Here, $\rho^{\pm}(x, \lambda)$ are t-independent matrices

$$\hat{\rho}^{\pm} = \begin{bmatrix} \nu^{\pm}(x,\lambda) & r^{\pm}(x,\lambda) \\ (r^{\pm}(x,\lambda))^* & -\nu^{\pm}(x,\lambda) \end{bmatrix}. \tag{1.19}$$

From (I.2),

$$\rho(t,x,\lambda) = \exp(-2i\lambda t)\int_{-\infty}^{t} N(\tau,x,\lambda)\mathscr{E}(\tau,x)\exp(2i\lambda\tau)\,d\tau + r^-(x,\lambda)\exp(-2i\lambda t). \tag{1.20}$$

In the limit $t \to -\infty$, we can in (1.20) make the substitution $N \to \nu^-(x, \lambda)$. The field $\mathscr{E}$ now satisfies the linearized equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\mathscr{E}(t,x) = \int_{-\infty}^{t} d\tau \int_{-\infty}^{\infty} \nu^-(x,\lambda)g(\lambda)\exp[2i\lambda(\tau-t)]\mathscr{E}(\tau,x)\,d\lambda + \int_{-\infty}^{\infty} r^-(x,\lambda)\exp(-2i\lambda t)g(\lambda)\,d\lambda. \tag{1.21}$$

PROPOSITION 2. The function $\hat{F}(t, x) = \frac{1}{4}F(2t, x)$, where F is the kernel of the Marchenko equations, satisfies the linear inhomogeneous equation (1.21).

To prove this, we consider the common solution Ψ_0 of Eqs. (1.1) and (1.2) (note that the Jost function $\chi^{\pm}$ does not satisfy Eq. (1.2)) and decompose it with respect to the functions $\chi^{\pm}$:

$$\Psi_0=\chi^+\Phi^+=\chi^-\Phi^-. \tag{1.22}$$

The functions $\Phi^{\pm}(x, \lambda)$ for given $\hat{\rho}^{\pm}(x, \lambda)$ satisfy linear equations that can be obtained by going in (1.2) to the limit $t \to \pm\infty$. We denote

$$\hat{R}^{\pm}(x,\lambda)=\lim_{t\to\pm\infty}\int_{-\infty}^{\infty}\frac{\exp[iI(\lambda-\eta)t]\hat{\rho}^{\pm}(x,\eta)\exp[-iI(\lambda-\eta)t]}{\eta-\lambda}g(\eta)\,d\eta. \tag{1.23}$$

Using the well-known formula

$$\lim_{t\to\pm\infty}\int_{-\infty}^{\infty}\frac{f(\eta)}{\eta-\lambda}\exp(i\eta t)\,d\eta=\pm\pi i f(\lambda),$$

we obtain

$$\hat{R}^{\pm}(x,\lambda)=\begin{bmatrix} N^{\pm}(x,\lambda) & \mp\pi i r^{\pm}(x,\lambda)g(\lambda) \\ \pm\pi i(r^{\pm}(x,\lambda))^*g(\lambda) & -N^{\pm}(x,\lambda)\end{bmatrix}\quad N^{\pm}(x,\lambda)=\int_{-\infty}^{\infty}\frac{\nu^{\pm}(x,\eta)}{\eta-\lambda}g(\eta)\,d\eta. \tag{1.24}$$

The function $\Phi^{\pm}(x, \lambda)$ satisfies the equation

$$\frac{\partial\Phi^{\pm}}{\partial x}-i\lambda I\Phi^{\pm}+\frac{i}{4}\hat{R}^{\pm}\Phi^{\pm}=0. \tag{1.25}$$

Comparing (1.22) and (1.8), we find

$$S=\Phi^+(\Phi^-)^{-1}. \tag{1.26}$$

From (1.25) and (1.26),

$$\frac{\partial S}{\partial x}-i\lambda[IS]+\frac{i}{4}(R^+S-SR^-)=0. \tag{1.27}$$

Substituting in (1.27) the S matrix in the form (1.9), and making simple calculations, we obtain for a and b

$$\frac{\partial a}{\partial x}=\frac{ia}{2}\int_{-\infty}^{\infty}\frac{\nu^+(x,\eta)-\nu^-(x,\eta)}{\eta-\lambda+i0}g(\eta)\,d\eta, \tag{1.28}$$

$$\frac{\partial b}{\partial x}=-2i\lambda b^*+\frac{ib^*}{2}\int_{-\infty}^{\infty}\frac{\nu^+(x,\eta)-\nu^-(x,\eta)}{\eta-\lambda+i0}g(\eta)\,d\eta+\frac{\pi}{2}g(\lambda)ar^-. \tag{1.29}$$

From (1.27) we have for $c=b^*/a$

$$\frac{\partial c}{\partial x}-i\varphi(x,\lambda)c=\frac{\pi}{2}g(\lambda)r^-(x,\lambda), \tag{1.30}$$

where

$$\varphi(x,\lambda)=2\lambda-\frac{1}{2}\int_{-\infty}^{\infty}\frac{\nu^-(x,\eta)g(\eta)}{\eta-\lambda+i0}\,d\eta \tag{1.31}$$

is a function that is analytic in the upper half-plane of λ.

The general solution of Eq. (1.30) is

$$c(x,\lambda)=c_1(x,\lambda)+c_2(x,\lambda), \tag{1.32}$$

$$c_1(x,\lambda)=c_0(\lambda)\exp[i\zeta(x,\lambda)], \tag{1.33}$$

$$c_2(x,\lambda)=\frac{\pi}{2}g(\lambda)\int_0^x\exp[i(\zeta(x,\lambda)-\zeta(y,\lambda))]r^-(y,\lambda)\,dy. \tag{1.34}$$

Here, $c_0(\lambda)=c(0,\lambda)\zeta(x,\lambda)=\int_0^x\varphi(y,\lambda)\,dy$ is a function analytic in the upper half-plane Im $\lambda > 0$. Equation (1.30) proves Proposition 2 in the case when there is no discrete spectrum in the problem (1.1); for making a

Fourier transformation in (1.30), we see after simple calculations that $\hat{F}(t, x) = F(2t, x)$ satisfies Eq. (1.21).

If there is a discrete spectrum, we use the following device for the proof: We construct sets of functions $c_j(x, \lambda)$, $f_j(x, \lambda)$ that are analytic in the strip $0 < \mathrm{Im}\,\gamma < \gamma$ and converge on the real axis:

$$c_j^0(\lambda) \to c_0(\lambda), \quad f_j(x, \lambda) \to g(\lambda) r^-(x, \lambda), \quad j \to \infty, \quad \mathrm{Im}\,\lambda = 0.$$

It follows from Eqs. (1.32)-(1.34) that for all j the solutions of Eq. (1.30) are analytic in the strip $0 < \mathrm{Im}\,\lambda < \gamma$. Applying to them Eq. (1.12), we see that all $\hat{F}_j(t, x)$ satisfy Eq. (1.21). Going to the limit $j \to \infty$, we see that this is true for $\hat{F}(t, x)$, and this completes the proof of Proposition 2.

From Eqs. (1.28)-(1.29), we can obtain the law of variation of the scattering data with respect to x. It follows directly from (1.28) that the zeros do not depend on x:

$$\partial \lambda_j / \partial x = 0. \tag{1.35}$$

For $M_j = c_j / a_j$ we have

$$M_j(x) = M_j^0 \exp[i\zeta(x, \lambda_j)]. \tag{1.36}$$

Equations (1.28) and (1.29) are formally linear, but in reality they are nonlinear because the functions $\nu^{\pm}(x, \lambda)$ are not independent.

We represent the matrix $\hat{\rho}(t, x, \lambda)$ in the form

$$\hat{\rho} = \Psi_0 \hat{\rho}_0 \Psi_0^{-1}. \tag{1.37}$$

Substituting (1.37) in (I.6) and taking into account (1.1), we see that the matrix $\hat{\rho}_0$ does not depend on the time. Going to the limit $t \to \pm\infty$ and using Eqs. (1.22), (1.7), and (1.18), we obtain

$$\hat{\rho}^{\pm} = \Phi^{\pm} \hat{\rho}_0 (\Phi^{\pm})^{-1}.$$

Hence and using (1.26) we find

$$\hat{\rho}^{+} = S \rho^{-} S^{-1}. \tag{1.38}$$

From (1.38),

$$\nu^{+} = \nu^{-}(|a|^2 - |b|^2) - a^* b^* (r^-)^* - a b r^-, \tag{1.39}$$

$$r^{+} = 2 b^* a \nu^{-} - (b^2 r^-)^* + a^2 r^-. \tag{1.40}$$

Substituting (1.39) and (1.40) in (1.28) and (1.29), we obtain the necessary equations for $a(x, \lambda)$, $b^*(x, \lambda)$, which need not be written out here. In the special case $r^- = 0$, $\nu^- = -1$ they are identical to the equations constructed in [3]. Note that to construct the general solution of the MB equation it is necessary to solve Eq. (1.30) for the function $c(x, \lambda)$, which, as in [3], is a linear but now inhomogeneous equation.

After transition to the function $F(t, x)$ and solution of the integral equations (1.13)-(1.14), the expressions (1.32)-(1.34) give the general solution of the MB equations (I.1)-(I.3).

2. Causal and Spontaneous Solutions

We now turn to the interpretation of our general solution. It is determined by the scattering data at $x=0$: $c_0(\lambda)$, λ_j, M_j^0, and also by the specification of the function $r^-(x, \lambda)$; in addition, it is necessary to specify the sign of

$$\nu^-(x, \lambda) = \pm\sqrt{1 - |r^-(x, \lambda)|^2}. \tag{2.1}$$

The minus sign in (2.1) describes the propagation of waves in a stable medium in which the lower level of the atomic subsystem is more populated than the upper. In contrast, the plus sign in (2.1) means that waves propagate in an unstable medium with population inversion. We shall assume that this medium occupies the positive half-axis $0 \le x < \infty$, and that in the limit $t \to -\infty$ there was "prepared" in this medium a certain state of the atomic subsystem characterized by the fluctuation polarization $r^-(x, \lambda)$ and population $\nu^-(x, \lambda)$. Subsequently, this polarization, evolving, is the source of generation of the electromagnetic field $\mathscr{E}(t, x)$. In addition, there is incident on the medium from without at the point $x = 0$ an electromagnetic field pulse $\mathscr{E}_0(t)$, which determines after the solution of the direct scattering problem the function $c_0(\lambda)$ and the parameters λ_j, M_j^0.

Two classes of particular solutions of the MB equations are naturally distinguished. Suppose $r^-(x, \lambda) = 0$; then the solution is completely determined by the incident pulse $\mathscr{E}_0(t)$. Such solutions, to which the term $c_1(x, \lambda)$ corresponds in (1.32), we shall call causal. This can be motivated as follows. Suppose the pulse entering the medium satisfies the condition $\mathscr{E}_0(t)=0$ for $t < t_0$; then $c_0(\lambda)$ is a function analytic in the upper half-plane and has poles at the points $\lambda = \lambda_j$ with residues M_j. It can be verified (we shall not dwell on the proof of this fact, which can be deduced from the integral representations for the Jost functions) that the function c_0 admits as $\lambda \to \infty$, $\mathrm{Im}\,\lambda > 0$ the estimate

$$|c_0(\lambda)| < c_0'' \exp(-2\,\mathrm{Im}\,\lambda t_0)/|\lambda|.$$

It can be seen from (1.31) that in the limit $\lambda \to \infty$

$$\varphi(x, \lambda) \to 2\lambda, \quad \zeta(x, \lambda) \simeq 2\lambda x.$$

Thus, the integrand in (1.12) has for $\mathrm{Im}\,\lambda > 0$ the asymptotic behavior $\exp[i\lambda(2t_0 - t + 2x)]$, so that $F(t, x) = 0$ for $t < 2(t_0 + x)$. It now follows from the Marchenko equations (1.13)-(1.14) that $\mathscr{E}(t, x)=0$ for $t < t_0 + x$. This means that the causal solution for a potential of finite range has a front which propagates into the medium with the velocity of light, in complete agreement with the notion of causality.

Now suppose there is no incident pulse: $\mathscr{E}_0(t)=0$. Then the solution is entirely determined by the specification of the polarization fluctuations $r^-(x, \lambda)$, and we shall call it spontaneous. In this case, the function $c(x, \lambda)$ is determined by (1.34). For spontaneous solutions $a(0, \lambda)=1$,

$$a(x, \lambda) = \exp\left[\frac{i}{4}\int_0^x dy \int_{-\infty}^{\infty} \frac{v^+(y, \eta) - v^-(y, \eta)}{\eta - \lambda + i0} g(\eta)\, d\eta\right] \tag{2.2}$$

and $a(x, \lambda) \neq 0$ for $\mathrm{Im}\,\lambda > 0$. Thus, for spontaneous solutions the spectral problem (1.1) does not have a discrete spectrum. The general solution is a linear superposition of a causal and a spontaneous solution.

We consider one further special class of soliton solutions, for which $r^-(x, \lambda) = c_0(\lambda) = 0$; they are entirely determined by the discrete spectrum of the problem (1.1). Formally, soliton solutions may be of the causal type, but for $\nu^- > 0$ they have a strongly degenerate nature, are unstable, and therefore do not have physical meaning.

We rewrite the expression (1.31) for $\varphi(x, \lambda)$ in the form

$$\varphi(x, \lambda) = 2\lambda - \frac{1}{2}\int_{-\infty}^{\infty} d\eta \frac{v^-(x, \eta)}{\eta - \lambda} g(\eta) - \frac{\pi}{2} i g(\lambda) v^-(x, \lambda). \tag{2.3}$$

In a stable medium, $\nu^- < 0$ and the exponential in the causal solution (1.33) is a decreasing function as $x \to \infty$, so that $c_1 \to 0$ as $x \to \infty$. At sufficiently great distances, the causal solution becomes a purely soliton solution; in the language of nonlinear optics, it goes over into a set of interacting 2π pulses. This is the essence of the phenomenon of self-induced transparency.

In a spontaneous solution in a stable medium, the integral in (1.34) is determined in the limit $x \to \infty$ by the neighborhood of the point x. We have

$$c_2(x, \lambda) \to \frac{\pi i g(\lambda)}{2\varphi(x, \lambda)} r^-(x, \lambda). \tag{2.4}$$

For a value of the polarization $r^-(x, \lambda)$ which is uniformly small with respect to x, a spontaneous solution remains uniformly small. In a medium with population inversion, when $\nu^- > 0$, the asymptotic behavior of the general solution as $x \to \infty$ has the form

$$c(x, \lambda) = \exp[i\zeta(x, \lambda)]\left[c_0(\lambda) + \frac{\pi}{2} g(\lambda) \int_0^{\infty} r^-(y, \lambda) \exp[-i\zeta(y, \lambda)]\, dy\right]. \tag{2.5}$$

In the general case, the expression in the square brackets in (2.5) is not equal to zero. At the same time, $c(x, \lambda)$ increases exponentially as $x \to \infty$. Since $c=b^*/a$, and the coefficients a and b are related by $|a|^2+|b|^2=1$, it follows that as $c \to \infty$, $a \to 0$, $|b| \to 1$, so that the S matrix takes the form

$$S = \begin{bmatrix} 0 & \exp[i\alpha(x, \lambda)] \\ \exp[-i\alpha(x, \lambda)] & 0 \end{bmatrix}, \quad \mathrm{Im}\,\alpha(x, \lambda) = 0. \tag{2.6}$$

Substituting (2.6) in (1.39), we see that now as $x \to \infty$ $v^+ \to -v^-$. This means that over sufficiently great lengths there is a reversal of the population of the medium, the population inversion becoming normal population.

If the initial pulse is chosen specially in such a way that

$$c_0(\lambda)=-\frac{\pi}{2}g(\lambda)\int_0^\infty r^-(y,\lambda)\exp[-i\zeta(y,\lambda)]dy, \tag{2.7}$$

then $c(x, \lambda)$ remains bounded for all x and complete reversal of the population of the medium does not occur. In particular, this is true for purely soliton solutions, when $r^- = 0$, $c_0 = 0$. However, all these situations are unstable – a small deviation from fulfillment of the condition (2.7) is sufficient for there to be complete reversal of the polarization. A purely soliton solution is unstable. This instability explains the well-known paradox of superluminal propagation of pulses in a medium with population inversion. It is easy to calculate the one-soliton solution (see, for example, [9]), for which

$$\mathscr{E}(t,x)=2\eta\operatorname{sech}\left\{\eta\left[t-t_0-x\left(1-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g(\nu)d\nu}{\eta^2+\nu^2}\right)\right]\right\}. \tag{2.8}$$

Here, η and t_0 are independent parameters. Although the soliton (2.8) has superluminal velocity, no information can be transmitted by means of it, since the shape of the soliton on the complete axis $-\infty < t < \infty$ can be uniquely recovered from its behavior in an arbitrarily small neighborhood of any instant of time. If there is the slightest deviation from the form (2.8), there arises a breaking front of the medium population, this propagating with the velocity of light. It is for this reason that all purely soliton solutions are unstable.

3. The Case of an Infinitely Narrow Line

From the point of view of physics, a very interesting case is that in which the spectral line is infinitely narrow, i.e.,

$$g(\lambda)=\delta(\lambda). \tag{3.1}$$

This case can also be realized for a finite line width if the electromagnetic field pulse $\mathscr{E}(t, x)$ is sufficiently narrow (as we shall see below, this is the case for any form of the initial pulse at the exit of a long laser). In the case (3.1), Eqs. (I.1)-(I.3) take the form

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\mathscr{E}=\rho,\quad \frac{\partial\rho}{\partial t}=N\mathscr{E},\quad \frac{\partial N}{\partial t}=-\frac{1}{2}(\mathscr{E}^*\rho+\mathscr{E}\rho^*). \tag{3.2}$$

In the limit $\mathscr{E}\to 0$, Eqs. (3.2) have only two solutions $\rho = 0$, $N = \pm 1$, these corresponding to unstable and stable media in the complete absence of fluctuations. To construct the general solution of the system (3.2), it is convenient to consider the limit from the situation with finite line width.

To simplify the treatment, we assume that the line profile $g(\lambda)$ is described by the Lorentz function

$$g(\lambda)=\frac{\varepsilon}{\pi(\lambda^2+\varepsilon^2)}, \tag{3.3}$$

and that the initial fluctuations $r^-(x, \lambda)$ are small, so that we can set $\nu^-(x, \lambda) = \pm 1$. Then

$$\varphi(\lambda)=2\lambda+\frac{\nu^-}{2(\lambda+i\varepsilon)}, \tag{3.4}$$

and as $\varepsilon \to 0$ the exponential

$$\exp[i\zeta(x,\lambda)]=\exp\left[i\left(2\lambda+\frac{\nu^-}{2(\lambda+i\varepsilon)}\right)x\right]$$

acquires an essential singularity at the point $\lambda = 0$. We consider first the case of a medium with normal population, $\nu^- = -1$. Since $g(\varepsilon) \to \infty$ as $\varepsilon \to 0$, the asymptotic expression (2.4) is satisfied in this limit for all $x > 0$. Substituting (3.4) in (2.4), we obtain

$$c_1(x,\lambda)=\frac{i\varepsilon r^-(x,\lambda)}{(\lambda-i\varepsilon)[4\lambda(\lambda+i\varepsilon)-1]}. \tag{3.5}$$

After substitution in (1.11) we find that the function

$$F_{sp}(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} c_2(x,\lambda)\exp(-i\lambda t)\,d\lambda$$

tends for any finite t to zero as $\varepsilon \to 0$. Therefore, spontaneous solutions do not exist in a medium with normal population in the case of an infinitely narrow line.

For causal solutions, we have

$$F_{ca}(t,x)=\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} c_0(\lambda)\exp\left[i\left(2\lambda x-\frac{x}{2(\lambda+i\varepsilon)}\right)-i\lambda t\right]d\lambda. \tag{3.6}$$

Thus, the integration with respect to λ is taken around the essential singularity in the upper half-plane.

It can be seen from (1.33) that in the limit $\varepsilon \to 0$ the function $c(x, \lambda)$ acquires a removable singularity $c(x, \lambda) = 0$. With allowance for $c=b^*/a$, $|a|^2+|b|^2=1$ it follows from this that $b(x, 0)=0$, $|a(x, 0)| =1$. From (1.39), we now have $\nu^+ = \nu^-$. Further, from Eqs. (1.28) and (1.29) we find

$$\frac{\partial a}{\partial x}=0,\quad \frac{\partial b}{\partial x}=i\left(2\lambda-\frac{1}{2\lambda}\right),\quad b=\exp\left[ix\left(2\lambda-\frac{1}{2\lambda}\right)\right].$$

Qualitatively, the propagation of a causal pulse in a medium with normal population and infinitely narrow line differs from the case of a line of finite width by the absence of damping of the nonsoliton part of the solution (therefore, this case can be called undamped). Nevertheless, in this case too there is a separation of the soliton part of the solution due to the dispersion spreading of its nonsoliton part.

Going over to a medium with population inversion, $\nu^- = 1$, we find for the causal contribution to F an expression analogous to (3.6):

$$F_{ca}(t,x)=\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} c_0(\lambda)\exp\left[ix\left(2\lambda+\frac{1}{2(\lambda+i\varepsilon)}\right)-i\lambda t\right]d\lambda. \tag{3.7}$$

Qualitatively, a causal solution in the case with population inversion and narrow line does not differ from the case of a line of finite width; in both cases there is transition from population inversion to normal population. In a medium with population inversion and narrow line spontaneous solutions exist.

For $g(\lambda) = \delta(\lambda)$, the main contribution to the integral (1.34) as $\lambda \to 0$ is made by the neighborhood of the point $x = 0$. Therefore, we can make the approximate substitution $r^-(x, \lambda) = r^-(0, \lambda) = r(\lambda)$. With allowance for this, for the spontaneous contribution to the kernel F_{sp} of the Marchenko equation we have

$$F_{sp}(t,x)=-\frac{i\varepsilon}{2\pi}\int_{-\infty}^{\infty}\frac{r(\lambda)}{\lambda-i\varepsilon}\,\frac{1-\exp[i\varphi(\lambda)x]}{1+4\lambda(\lambda+i\varepsilon)}\exp(-i\lambda t)\,d\lambda. \tag{3.8}$$

We assume that r is analytic in a certain disk with center at the origin and radius ε_0, and we deform the contour in such a way that it passes below the origin around a circle of radius ε_0. In the limit $\varepsilon \to 0$, the integral around the new contour – it does not depend on ε – tends to zero. However, in making the deformation of the contour we passed through the singular point $\lambda = -i\varepsilon$, whose residue must be taken into account. In the limit $\varepsilon \to 0$, we have

$$F_{sp}(t,x)=\frac{1}{4\pi}\oint r(\lambda)\exp\left[i\left(2\lambda+\frac{\nu^-}{2\lambda}\right)\gamma-i\lambda t\right]d\lambda. \tag{3.9}$$

The integration in (3.9) is around a circle of small radius with center at the origin. The expression (3.9) describes the spontaneous solutions as $g(\lambda) \to \delta(\lambda)$; it is obvious that $F(t, x) = 0$ at $x = 0$.

In (3.9), $r(\lambda)$ must be understood as the germ of a function analytic in the neighborhood of $\lambda = 0$. As can be seen from Eqs. (3.2), $\rho \to 0$ as $t \to -\infty$. This holds for the total polarization $\langle\rho\rangle$ in Eqs. (I.1)–(I.3). For the simplest of the solutions of the type (3.9), $r(\lambda) = r_0 = \text{const}$. In this case, $F_{sp}(t, x)$ has a self-similar nature:

$$F_{sp}(t,x)=\frac{r_0 x}{2\sqrt{2x(t-2x)}}\,I_1\left(\sqrt{2x(t-2x)}\right). \tag{3.10}$$

We note also that in the limit $\varepsilon \to 0$ the length of the region in which there is reversal of the population inversion tends to zero. Therefore, for all x we obtain from (1.28) $\nu^+ = -\nu^- = -1$. From (1.28) and (1.29) we now have

$$a(x,\lambda)=\exp(-ix/2\lambda),\quad b(x,\lambda)=\exp(2i\lambda x). \tag{3.11}$$

Thus, $a(x,\lambda)$ has for spontaneous solutions an essential singularity at the point $\lambda = 0$. It is well known [10] that for $\int_{-\infty}^{\infty} |\mathscr{E}(t,x)|dt<\infty$ the coefficient $a(x,\lambda)$ is continuous up to the real axis. This means that in the case $g(\lambda) = \delta(\lambda)$ all spontaneous solutions are weakly damped,

$$\int_{-\infty}^{\infty} |\mathscr{E}(t,x)|dt=\infty.$$

The expressions (3.9) and (3.10) were given without proof in [5]. The latter admits a curious interpretation. Since $|a|=1$ on the real axis and has a zero of infinite order at the point $\lambda = +i0$, the spontaneous solution can be interpreted as a limiting case of a purely soliton solution – the fusion of an infinite number of solitons of infinitesimally small amplitude. With the spectral problem (1.1) there is associated an infinite set of trace formulas (see [10]), these having for purely soliton solutions the form

$$P_k=\sigma_k\sum_{j=1}^{J}(\lambda_j^k)^*-\lambda_j^k,\quad \sigma_k=\mathrm{const}(k),$$

where

$$P_1=\int_{-\infty}^{\infty}|\mathscr{E}(t,x)|^2dt,\quad P_2=\frac{i}{2}\int_{-\infty}^{\infty}(\mathscr{E}^*\mathscr{E}_t-\mathscr{E}\mathscr{E}_t^*)(t,x)\,dt,\dots.$$

Since for J identical solitons $(\lambda_j=i\lambda_0)$ $P_k\sim J\lambda_0^k$, it follows that in such a limit only the invariant P_1 (if we set $\lambda_0 \sim 1/J$) can be nonzero.

Indeed, it follows from Eqs. (3.1)-(3.3) that

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty}|\mathscr{E}(t,x)|^2dt=\int_{-\infty}^{\infty}g(\lambda)\,[\nu^-(x,\lambda)-\nu^+(x,\lambda)]\,d\lambda.$$

Hence, for spontaneous solutions $\mathscr{E}(t,0)=0$, $g(\lambda)=\delta(\lambda)$, $\nu^+=-\nu^-=-1$ we have

$$P_1=\int_{-\infty}^{\infty}|\mathscr{E}(t,x)|^2dt=2x,\quad P_k=0,\quad k>1.$$

4. Mixed Problem

For the system of MB equations (1.1)-(1.3) it is natural on physical grounds to pose the mixed problem determined by the initial and boundary conditions

$$\mathscr{E}(t,0)=\mathscr{E}_1(t), \tag{4.1}$$

$$\mathscr{E}(0,x)=\mathscr{E}_2(x), \tag{4.2}$$

$$\rho(0,x,\lambda)=\rho_0(x,\lambda). \tag{4.3}$$

For a finite sample measuring L, this problem is posed in the half-strip $0 \le x \le L$, $t \ge 0$, it being assumed that $\mathscr{E}_1(t)\to 0$, $t\to\infty$.

The inverse scattering method is not adapted to the solution of the problem (4.1)-(4.3). We can solve only the asymptotic mixed problem described above in the half-plane $x \ge 0$, $-\infty < t < \infty$ with boundary condition $\mathscr{E}_0(t)$ and asymptotic condition $\rho(t,x,\lambda) \to r^-(x,\lambda)\exp(-2i\lambda t)$ as $t \to -\infty$.

It is however possible to attempt to reduce the mixed problem (4.1)-(4.3) to that asymptotic problem by specifying the condition $\mathscr{E}_3(t)=\mathscr{E}(t,0)$, $t<0$ and $r^-(x,\lambda)$ in such a way that the conditions (4.2) and (4.3) are reconstructed at $t = 0$. For this, it is necessary to solve the mixed problem with the conditions (4.2) and (4.3) "backward in time," imposing the subsidiary condition $\mathscr{E}(t,x)\to 0$ as $t \to -\infty$.

When the sign of the time is reversed, so is the direction of the characteristics; in solving this problem, it is therefore incorrect to specify the value of $\mathscr{E}_1(t)$ at x = 0. The boundary condition must be imposed at x = L:

$$\mathscr{E}_4(t)=\mathscr{E}(t, L), \tag{4.4}$$

and chosen such that $\mathscr{E}(t, x)\to 0$ as $t \to -\infty$. From the solution of this problem, it is possible to determine $\mathscr{E}_3(t)$ and $r^-(x, \lambda)$, thereby reducing the mixed problem (4.2)-(4.3) to the asymptotic problem.

In the general case, the solving of this problem is in no way simpler than that of the original (4.2)-(4.3). However, if the initial conditions (4.2), (4.3) are sufficiently small, the problem (4.2)-(4.4) can be solved "backward in time," by using a linear approximation. We give the results of this solution, assuming, as before, $g(\lambda)=\varepsilon/\pi(\lambda^2+\varepsilon^2)$ and specifying

$$\mathscr{E}_2(x)=0, \quad \rho_0(x, \lambda)=\rho_\xi \exp(i\xi x), \quad |\rho_\xi|\ll 1. \tag{4.5}$$

From the solution of the linear equations, we find

$$\mathscr{E}_3(t)=\frac{\rho_\xi}{p_1-p_2}[\exp(-p_2t)+\theta(-t-L)\exp(-p_1t)], \tag{4.6}$$

$$\mathscr{E}_4(t)=\frac{\rho_\xi}{p_1-p_2}\left\{\exp(i\xi L)[\exp(-p_2t)-\exp(-p_1t)(1-\theta(-t))]+\theta(t)\sqrt{\frac{L}{-t}}\frac{J_1(2\sqrt{-Lt})}{2\varepsilon+p_1}\exp(p_1L+2\varepsilon t)\right\},$$

$$p_{1,2}=\frac{i\xi-2\varepsilon\pm\sqrt{(i\xi+2\varepsilon)^2+4}}{2}, \qquad \theta(t)=0\ (t\leqslant 0), \quad \theta(t)=1\ (t>0). \tag{4.7}$$

We note the behavior of the auxiliary pulse $\mathscr{E}_3(t)$ incident on the medium. For $-t < L$, this pulse grows exponentially, the argument of the exponential being equal to the growth rate $\gamma_\xi = \operatorname{Re} p_1$ of the instability with wave vector ξ (for $-t < L$, the field at the point x = 0 "does not know" of the existence of the pulse $\mathscr{E}_4(t)$). But for $-t > L$, the pulse $\mathscr{E}_4(t)$ begins to have an effect, and the field in the complete volume of the sample begins to decrease. Thus, the field at the point x = 0 reaches the maximal value $\mathscr{E}_{max}=\mathscr{E}_3(-L+0)$. The condition of applicability of the linear approximation, $\mathscr{E}_{max}\ll 1$, gives in order of magnitude a criterion for the applicability of the linear treatment:

$$|\rho_\xi|\exp(L\gamma_{max})\ll 1, \quad \gamma_{max}=\max_\xi \gamma_\xi. \tag{4.8}$$

In [7], lasers satisfying the condition (4.8) were said to be "moderately long." We do not give the expression for $r^-(x, \lambda)$, which is cumbersome.

Knowing $\mathscr{E}_3(t)$, we can, solving the direct scattering problem with the potential $\mathscr{E}_0(t)=\mathscr{E}_1(t)$, $t>0$, and $\mathscr{E}_0(t)=\mathscr{E}_3(t)$, $t<0$, find $c_0(\lambda)$, and also the elements λ_j, M_j^0 of the discrete spectrum. Further, knowing $r^-(x, \lambda)$, we can recover $c_2(x, \lambda)$ from the expressions (1.31) and (1.34), and with it $c(x, \lambda)$. Further, from (1.11) we recover $F(t, x)$, and the value of the field $\mathscr{E}(t, x)$ can be obtained after solution of the Marchenko equations (1.13)-(1.15).

The procedure we have described makes it possible to solve approximately the mixed problem (4.1)-(4.3) for moderately long lasers satisfying the condition (4.8). It is clear that this solution is strongly non-unique – we could impose the boundary condition for the pulse incident on the medium at any point x > L instead of the point x = L. However, this would mean that the conditions of applicability of the theory were less well satisfied.

5. Application to the Problem of Superfluorescence

One of the physically important problems posed for the MB equation is that of superfluorescence. It is described by the mixed problem (4.1)-(4.3) under the additional simplifying assumptions

$$\mathscr{E}_1(t)=0, \quad \mathscr{E}_2(x)=0. \tag{5.1}$$

The function $\rho_0(x, \lambda)$ is determined by the quantum fluctuations in the laser. As is shown in [11, 12], this function can be assumed to be a random one with Gaussian distribution, and

$$\overline{\rho_0(x,\lambda)\rho_0^*(y,\mu)}=\frac{4\delta(x-y)\delta(\lambda-\mu)}{N_0 g(\lambda)}, \tag{5.2}$$

where N_0 is the total number of active atoms in the system, a large parameter (in a typical situation, $N_0 > 10^8$).

It follows from (5.2) that the initial condition $\rho_0 \sim 1/\sqrt{N_0}$ is a small parameter, so that if

$$L\gamma_{max} \ll \frac{1}{2}\ln N_0, \quad \gamma_{max} = \max_i \gamma_i \tag{5.3}$$

the scheme described in the previous section can be used. In the first step of the scheme, we must solve the auxiliary mixed problem in the linear approximation. The function $\mathcal{E}_0(t)$ obtained as a result of this calculation will be small on the complete axis $-\infty < t < \infty$. Solving to this accuracy (in the first Born approximation) the direct scattering problem, we can readily calculate $c(x, \lambda)$, a discrete spectrum being absent, and then calculate $F(t, x)$.

In the superfluorescence problem, one can take into account the specific boundary conditions (5.1) and use a simpler method to calculate the kernel $F(t, x)$. We use the identity of the equations for $\hat{F}(t, x)$ and $\mathcal{E}(t, x)$ and show that

$$F(2t, 0) = {}^1/_4\mathcal{E}_0(t). \tag{5.4}$$

Indeed, $\mathcal{E}_0(t)$ being small, we calculate the S matrix in the first Born approximation. For this, we use the definition (1.8) of the S matrix and Eq. (1.1). It is readily verified that the function $\Psi^- = \Psi(t, 0, \lambda)$ satisfies the integral equation

$$\Psi^-(t,\lambda) = \Psi_0^-(t,\lambda) + i\Psi_0^-(t,\lambda)\int_{-\infty}^{t} (\Psi_0^-(\tau,\lambda))^{-1} H\Psi^-(\tau,\lambda)\,d\tau, \tag{5.5}$$

where we choose the function $\Psi_0^- = \exp(-iI\lambda t)$ as the zeroth approximation for $\Psi^-(t, \lambda)$:

$$\Psi^-(t,\lambda) = \Psi_0^-(t,\lambda) + \Psi_1^-(t,\lambda) + \dots . \tag{5.6}$$

Substituting (5.6) in (5.5), for the first Born approximation we obtain the expression

$$\Psi_1^-(t,\lambda) = i\Psi_0^-(t,\lambda)\int_{-\infty}^{t} (\Psi_0^-(\tau,\lambda))^{-1} H\Psi_0^-(\tau,\lambda)\,d\tau. \tag{5.7}$$

Using (1.8), and also taking into account (5.7) and the actual form of the matrix H, we obtain for the S matrix the expression

$$S = \begin{bmatrix} 1 & \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{E}_0(\tau)\exp(2i\lambda\tau)\,d\tau \\ \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{E}_0^*(\tau)\exp(-2i\lambda\tau)\,d\tau & 1 \end{bmatrix}. \tag{5.8}$$

Hence

$$c_0(\lambda) = \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{E}_0(\tau)\exp(2i\lambda\tau)\,d\tau. \tag{5.9}$$

From the definition of the kernel of the Marchenko equation (1.11) and the expression (5.9) we obtain an expression for $F(2t, 0)$:

$$F(2t, 0) = {}^1/_4\mathcal{E}_0(t). \tag{5.10}$$

It follows from this that in the region $0 \leqslant x \leqslant L$, $-\infty < t < \infty$

$$F(2t, x) = {}^1/_4 E(t, x), \tag{5.11}$$

where $E(t, x)$ is the solution of the linearized MB equations with boundary condition $\mathcal{E}_0(t)$.

We take $g(\lambda)$ to be the Lorentz function (3.3); in the physics literature, the notation is $T_2^* = 1/\varepsilon$ [13], the so-called inhomogeneous broadening time. In this case, the expression for the kernel $F(t, x)$ when $t > x$ has the form

$$F(2t, x) = \frac{\pi}{4T_2^*}\int_0^x dx' \int_{-\infty}^{\infty} d\lambda \frac{G(t, x-x', \lambda)\rho_0(x', \lambda)}{\lambda^2 + (1/T_2^*)^2},$$
$$G(t, x, \lambda) = \theta(t-x)\left[I_0(2\sqrt{x(t-x)}) + (i\lambda + 1/T_2^*)\int_0^{t-x} dt' I_0(2\sqrt{xt'})\exp[(i\lambda + 1/T_2^*)(t-t'-x)]\right]\exp(-t/T_2^*). \tag{5.12}$$

We consider the limit $\varepsilon \to 0$, the interesting one from the point of view of applications. Then the kernel

F(t, x) becomes

$$F(2t, x)=\frac{1}{4}\int_0^x \rho_0(y)\,I_0(2\sqrt{(x-y)(t-x+y)})\,dy. \tag{5.13}$$

The kernel (5.13) increases exponentially when $t \gg x$. In this region, the dependence of F on x and t becomes self-similar:

$$F(2t, x)=x\mathcal{F}(2\sqrt{x(t-x)}). \tag{5.14}$$

where

$$\mathcal{F}(y)=\frac{\rho_0(0)\,I_1(y)}{2y}. \tag{5.15}$$

For $t \lessapprox x$, the kernel F is small with respect to the parameter $1/\sqrt{N_0}$, and therefore we can ignore it in studying the region of the solution of the MB equations containing the bulk of the energy.

The condition (5.3) guarantees the applicability and uniformity of the self-similar approximation (5.14) in the complete region $0\leqslant t<\infty$, $0\leqslant x\leqslant L$. Therefore, the solution of the MB equations tends to a self-similar one, and this gives further justification for the results of [6, 7] and agrees with experiment (see, for example, the review [14]).

6. Analysis of the Self-Similar Solution

The MB equations (I.1)-(I.3) in the case of an infinitely narrow line $g(\lambda) = \delta(\lambda)$ admit the self-similar ansatz

$$\mathcal{E}(t, x)=x\mathcal{E}(z), \quad N(t, x)=n(z), \quad \rho(t, x)=\rho(z), \tag{6.1}$$

where $z=2\sqrt{x(t-x)}$ is the similarity variable. The functions $\mathcal{E}(z)$, $n(z)$, $\rho(z)$ satisfy the system of equations

$$z\mathcal{E}'+2\mathcal{E}=2\rho, \tag{6.2}$$

$$2\rho'=zn\mathcal{E}, \tag{6.3}$$

$$2n'=-z\rho\mathcal{E}. \tag{6.4}$$

The similarity variable can take both real, $t > x$, as well as imaginary, $t < x$, values. To imaginary z there corresponds the noncausal region in the x, t coordinates.

The system (6.2)-(6.4) has a one-parameter family of solutions nonsingular at the origin and completely determined by $\mathcal{E}_0=\mathcal{E}(0)$ and the sign of $n_0 = n(0)$. For it follows from (6.3) and (6.4) that

$$n^2+\rho^2=1. \tag{6.5}$$

From given $\mathcal{E}_0$ we determine $\rho_0=\mathcal{E}_0$ from (6.2), and from (6.5) we find $n_0 = \pm\sqrt{1-\rho_0}$. Note that the system (6.2)-(6.4) is invariant with respect to the substitution

$$\hat{z}=iz, \quad \hat{n}(\hat{z})=-n(z), \quad \hat{\mathcal{E}}(\hat{z})=\mathcal{E}(z), \hat{\rho}(\hat{z})=\rho(z), \tag{6.6}$$

and it is therefore sufficient to calculate the solutions for positive n_0. Qualitatively, the behavior of the solutions of the system (6.4)-(6.2) depends on $\mathcal{E}_0$. In particular, there is a solution which is asymmetric with respect to the substitution (6.6). To it there corresponds an initial condition invariant with respect to the transformation (6.6): $\mathcal{E}_0=1$, $\rho_0=1$, $n_0=0$. For $\mathcal{E}_0=1$, the solution becomes asymmetric and has the asymptotic behaviors

$$\mathcal{E}(z)\simeq\frac{\mathcal{E}^\pm}{|z|^{3/2}}\sin\left(|z|\sqrt{2}-\frac{3\sqrt{2}}{128}(\mathcal{E}^\pm)^2\ln|z|+\beta^\pm\right), \quad 1-|n(z)|\simeq\frac{\mathcal{E}^\pm}{16|z|}\cos^2\left(|z|\sqrt{2}-\frac{3\sqrt{2}}{128}(\mathcal{E}^\pm)^2\ln|z|+\beta^\pm\right),$$

where $\mathcal{E}^\pm$, $\beta^\pm$ are constants whose values depend on $\mathcal{E}_0$, the sign + corresponds to $z \to \infty$ and the sign − to $z \to i\infty$. To asymptotically small $\mathcal{E}_0$ ($\ln\mathcal{E}_0\ll-1$) there correspond strongly asymmetric solutions. In this case, the point z_0, the start of the region of large variations of $\mathcal{E}$ (the first zero $n(z_0) = 0$), is logarithmically far from the point $z = 0$, and $\mathcal{E}^-=2^{3/4}\mathcal{E}_0/\sqrt{\pi}$.

To values $\mathcal{E}_0>1$ there corresponds a two-parameter family of singular solutions of the system (6.2)-(6.4), which we shall not consider.

Note that the system (6.2)-(6.4) can be reduced to a single second-order equation.

Setting

$$n=\cos\Phi,\quad \rho=\sin\Phi,\quad \mathscr{E}=\frac{2}{z}\Phi',\quad \Phi=\Phi(z), \tag{6.7}$$

we obtain

$$\Phi''+\frac{1}{z}\Phi'=\sin\Phi. \tag{6.8}$$

This equation is satisfied by self-similar solutions of the sine–Gordon equation determined by the initial conditions $\Phi_0=\Phi(0)$, $\Phi'(0)=0$; these solutions can be expressed in terms of the classical Painlevé transcendents [15].

In the laboratory coordinates, each value of the similarity variable z for fixed t corresponds to two values of the coordinate:

$$x_{1,2}=\frac{t\pm\sqrt{t^2-z^2}}{2}. \tag{6.9}$$

Suppose that at some point z_0 the function $n(z_0)$ has changed sign for the first time. This means that $N(t, x)$ changes sign for the first time at the time $t_0 = 2z_0$ at the point $x_0 = z_0$. In accordance with (6.9), two waves travel away with increasing t from the point x_0 in opposite directions, the condition $N = 0$ holding on their fronts. The position of the point depends logarithmically on the similarity parameter Φ_0. The smaller Φ_0, the larger x_0. In the limit $t \to \infty$, the front of the wave traveling to the left moves in accordance with the law

$$x=z_0^2/t+O(z_0^4/t^3). \tag{6.10}$$

The front of the second wave moves in accordance with

$$x=t-O(z_0^2/t). \tag{6.11}$$

The first situation is realized in the superfluorescence effect; the second, in the case of a quantum amplifier.

As was established in Sec. 3 (Eq. (3.9)), the spontaneous solutions are completely characterized by the function $r(\lambda)$. We consider the simplest case $r(\lambda) = r_0 = \text{const}$. Then the kernel $F(t, x)$ of the Marchenko equations takes the form (5.9). The spontaneous solutions corresponding to such a kernel are self-similar. To see this, we substitute the expression (5.9) in Eqs. (1.13)-(1.14) and make the change of variables

$$i\xi=\sqrt{2x(t-x)}. \tag{6.12}$$

We obtain

$$\mathscr{K}_1(\xi,\eta,x)=s_0\mathscr{F}(\sqrt{\xi^2+\eta^2})+s_0\int_\xi^\infty \mathscr{K}_2(\xi,\eta,x)\mathscr{F}(\sqrt{\zeta^2+\eta^2})\zeta\,d\zeta,\quad \mathscr{K}_2(\xi,\eta,x)=-s_0^*\int_\xi^\infty \mathscr{K}_1(\xi,\zeta,x)\mathscr{F}(\sqrt{\zeta^2+\eta^2})\zeta\,d\zeta, \tag{6.13}$$

$\mathscr{F}(y)=J_1(y)/y$, $s_0=r_0/2$, $K_i=x\mathscr{K}_i$. The free term and kernel $\mathscr{F}$ of the inhomogeneous system of equations (6.12)-(6.13) does not depend on x, and therefore the solution of this system, which exists and is unique, is also independent of x.

The electromagnetic field $\mathscr{E}$ can be expressed in terms of the solutions of the system of equations (6.12)-(6.13) as follows:

$$\mathscr{E}(t,x)=4x\mathscr{K}_1(\xi,\xi), \tag{6.14}$$

$$\int_{-\infty}^{t}|\mathscr{E}(t,x)|^2dt=-4\mathscr{K}_2(\xi,\xi),\quad \xi=\xi(t,x). \tag{6.15}$$

The solution (6.14) is a self-similar solution for the Maxwell–Bloch system, and the variable $-i\xi\sqrt{2}$ is a similarity variable.

To establish the connection between the kernel and the solution of the system (6.2)-(6.4), it is sufficient to find the dependence of $\mathscr{E}_0$ on s_0. To this end, we transform the system of integral equations (6.12)-(6.13) to algebraic form. First, we use the Gegenbauer composition formula for Bessel functions [16]:

$$\frac{J_k(\sqrt{\xi^2+\eta^2})}{(\xi^2+\eta^2)^{k/2}}=2^k(k-1)!\sum_{l=0}^{\infty}(k+l)\frac{J_{k+l}(r)}{\eta^l}\frac{J_{k+l}(\xi)}{\xi^l}C_l^k(0), \tag{6.16}$$

where $C_l^k(0)$ is the value of the Gegenbauer polynomial at the origin. In the self-similar case, the kernel $\mathcal{F}$ of Eqs. (6.12)-(6.13) can be represented in the form

$$\mathcal{F}(\sqrt{\xi^2+\eta^2})=\frac{2}{\xi\eta}\sum_{k=1}^{\infty}(-1)^{k-1}J_{2k-1}(\xi)J_{2k-1}(\eta). \tag{6.17}$$

Then, assuming regularity of the solutions $\mathcal{K}_i$ with respect to ξ, η, we expand them in von Neumann series [17]:

$$\mathcal{K}_1(\xi,\eta)=\sum_{k=1}^{\infty}\psi_k(\xi)J_{2k-1}(\eta)\frac{2}{\xi\eta}, \tag{6.18}$$

$$\mathcal{K}_2(\xi,\eta)=\sum_{k=1}^{\infty}\varphi_k(\xi)J_{2k-1}(\eta)\frac{2}{\xi\eta}. \tag{6.19}$$

Note that in the expansions (6.18)-(6.19) we have retained only the Bessel functions with odd index. It can be shown that the coefficients corresponding to the Bessel functions with even index vanish identically by virtue of the integral equations (6.12)-(6.13). We substitute (6.17)-(6.19) in (6.12)-(6.13). The system of integral equations can now be reduced to an infinite system of linear algebraic equations by using the biorthogonality property of the Bessel and Neumann functions:

$$\int_{\Gamma} J_m(z)O_k(z)\,dz=a_k\delta_{km}, \tag{6.20}$$

where $a_0=2\pi i$, $a_k=\pi i$ for $k > 0$, and the contour Γ passes once round the origin in the complex z plane. Multiplying the integral equation by $O_k(\eta)$ and integrating with respect to η along the contour Γ, we obtain

$$\psi_k(\xi)=s_0(2k-1)(-1)^{k-1}\left[J_{2k-1}(\xi)+2\sum_{l=0}^{\infty}Q_{kl}(\xi)\varphi_l(\xi)\right],\quad \varphi_k(\xi)=-s_0^*(2k-1)(-1)^{k-1}2\sum_{l=0}^{\infty}Q_{kl}(\xi)\psi_l(\xi), \tag{6.21}$$

where

$$Q_{kl}(\xi)=\int_{\xi}^{\infty}J_{2k-1}(\eta)J_{2l-1}(\eta)\frac{d\eta}{\eta}. \tag{6.22}$$

This integral can be calculated and is equal to

$$Q_{km}(\xi)=\frac{1-\delta_{km}}{4(m+k-1)}\left[\xi\frac{J_{2m-2}(\xi)J_{2k-1}(\xi)-J_{2m-1}(\xi)J_{2k-2}(\xi)}{k-m}+\right.$$
$$\left.2J_{2m-1}(\xi)J_{2k-1}(\xi)\right]+\frac{\delta_{km}}{2(2m-1)}\left[J_0^2(\xi)+J_{2m-1}^2(\xi)+2\sum_{j=1}^{2m-2}J_j^2(\xi)\right] \tag{6.23}$$

It is possible to write down a formal solution of Eqs. (6.21) that is the analog of an infinite soliton solution:

$$\mathcal{K}_2(\xi,\xi)=\frac{1}{2\xi}\frac{\partial}{\partial\xi}\ln[\det(E+|s_0|^2B^2)], \tag{6.24}$$

where

$$B_{km}(\xi)=(2k-1)(-1)^{k-1}Q_{km}(\xi),\quad E_{km}=\delta_{km}. \tag{6.25}$$

Thus,

$$\int_{\xi}^{\infty}\eta|\mathcal{E}(\eta)|^2d\eta=-\frac{16}{\xi}\frac{\partial}{\partial\xi}\ln[\det(E+|s_0|^2B^2)]. \tag{6.26}$$

Moreover, the solution (6.24) makes it possible to determine $\mathcal{E}(\xi)$ up to the sign:

$$\mathcal{E}(\xi)=\frac{4s_0}{|s_0|}\sqrt{\frac{1}{\xi}\frac{\partial}{\partial\xi}\frac{1}{\xi}\frac{\partial}{\partial\xi}\ln[\det(E+|s_0|^2B^2)]}.$$

We calculate $\mathcal{E}(0)$, seeking for this purpose a solution of the system (6.21) in the form of a series in powers of ξ:

$$\varphi_k(\xi)=\sum_{m=1}^{\infty}\xi^m\varphi_{km}, \quad \psi_k(\xi)=\sum_{m=1}^{\infty}\xi^m\psi_{km}, \quad Q_{mk}(\xi)=\frac{\delta_{mk}}{2(2m-1)}+\sum_{j=1}^{\infty}q_{mkj}\xi^j. \tag{6.27}$$

In the first order in ξ,

$$\psi_{m1}=s_0(2m-1)(-1)^{m-1}\left(\frac{1}{2^m}\frac{\delta_{m1}}{m!}+\frac{\delta_{mk}}{2m-1}\varphi_{k1}\right), \quad \varphi_{m1}=s_0{}^*(-1)^{m-1}\delta_{mk}\psi_{k1},$$

whence

$$\psi_{m1}=\delta_{m1}\frac{s_0}{2(1+|s_0|^2)}, \quad \varphi_{m1}=-\delta_{m1}\frac{|s_0|}{2(1+|s_0|^2)}. \tag{6.28}$$

Substituting (6.28) in (6.27) and then in (6.18)-(6.19), we obtain

$$\mathscr{E}(0)=\frac{2s_0}{1+|s_0|^2}, \qquad \int_0^{\infty}\xi|\mathscr{E}(\xi)|^2d\xi=\frac{8|s_0|^2}{1+|s_0|^2}.$$

Therefore,

$$n(0)=\frac{|s_0|^2-1}{|s_0|^2+1}.$$

The dependence of $\mathscr{E}_0$ on s_0 is not single valued. To each $\mathscr{E}_0$ there correspond two values, s_0 and $1/s_0^*$. This is due to the ambiguity in the choice of the sign of n_0. Thus, s_0 completely parametrizes the solution, and the transformation $s_0 = 1/s_0^*$ is equivalent to the transformation (6.6) for the initial data.

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