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Interchannel interaction of optical solitons

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We study interaction between two solitons from different frequency channels propagating in an optical fiber. The interaction may be viewed as an inelastic collision, in which energy is lost to continuous radiation due to small but finite third order dispersion. We develop a perturbation theory with two small parameters: the third order dispersion coefficient d_3, and the reciprocal of the interchannel frequency difference, $1/\beta$. We find that amplitude of the leading contribution to radiation emitted during the collision is proportional to d_3/β^2. The source term for this radiation is of the form that would be generated by a variation in the second order dispersion coefficient. In addition, the only other effects up to the combined third order of the perturbation theory are phase changes and position shifts of the solitons. Solitons propagating in a given frequency channel interact via radiation emitted due to collisions with many solitons from other frequency channels. We show that this intrachannel interaction effect, induced by many interchannel collisions, is identical to the radiation mediated intrachannel interaction effect observed for solitons propagating under the influence of disorder in the second order dispersion coefficient.

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I. INTRODUCTION

Modern high speed optical fiber communication systems extensively use multifrequency channel technology (wavelength division multiplexing, WDM, see, e.g., Ref. [1]). One of the major limitations on the performance of WDM systems is caused by the nonlinear interchannel interaction of data signals from different channels. We investigate this phenomenon using the conventional optical soliton as an example. In an ideal case soliton bit patterns from different channels would not experience any distortion due to the elastic character of soliton-soliton interaction. However, there exist other phenomena that are able to break the elastic nature of the interchannel interaction. The leading effect of this kind is associated with third order dispersion, which is the linear dependence of the chromatic dispersion on the wavelength of the carrier frequency. In this inelastic case, collisions between solitons from different frequency channels (interchannel collisions) lead to emission of radiation, corruption of the soliton shape, shift in soliton position (soliton walk off from the assigned time slot), and other undesirable effects. Moreover, the radiation emitted due to interchannel collisions might, in its turn, lead to interaction between solitons from the same frequency channel (intrachannel interaction). Therefore, it is important to have a realistic estimation for the intensity of the radiation emitted, as well as for the change in the soliton parameters due to the interchannel interaction.

The interaction between ideal solitons can be modeled using the ideal nonlinear Schrödinger equation (NLSE) [2,3]. This ideal interaction has been studied in detail [4–6], and is by now well understood. In contrast, accurate analysis of interchannel interactions between nonideal solitons is a long-standing problem, which was never addressed in the past. The main problem in this case is to develop a perturbation theory around the multisoliton solutions of the ideal nonlinear Schrödinger equation. In spite of existence of exact expressions for the multisoliton solutions of the ideal NLSE, direct perturbative analysis around these solutions, to the best of our knowledge, has not yet been successful.

In this paper we solve this long-standing problem and analyze the effects of inelastic collisions between solitons from different frequency channels in the presence of third order dispersion. (Figure 1 shows a cartoon of the collision process.) We calculate the spectrum and intensity of the radiation emitted in the frequency channels of the colliding solitons as a result of the collision. We also calculate the collision induced change in the soliton parameters. To

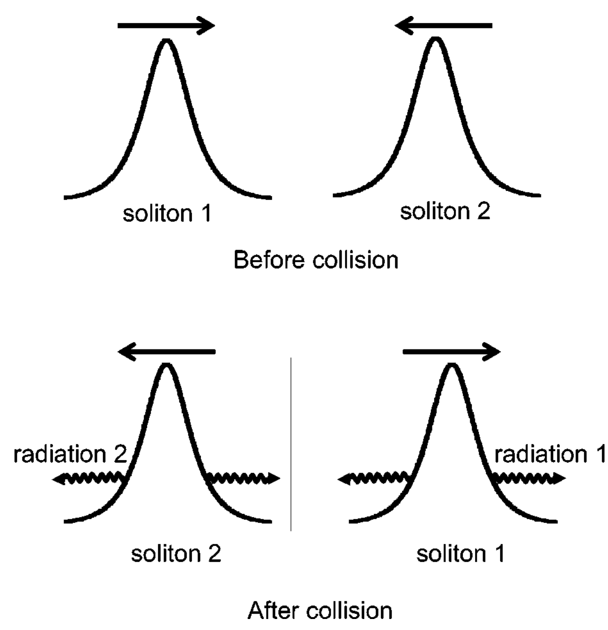

FIG. 1. Schematic description of the collision between two solitons from different frequency channels. The straight arrows denote the group velocity of the solitons, and the curly arrows denote radiation emitted by each of the two solitons in its own frequency channel.

achieve this goal we, first, find by means of the singular perturbation technique proposed by Kaup [7] a single-soliton solution, which is stationary (independent of position along the fiber), taking into account third order dispersion. It is assumed that even if an initial pulse pumped into the optical fiber is not exactly of this stationary form, it evolves into this stationary solution after a transient [8]. The form of this stationary solution is compared with results obtained in earlier studies of single-soliton propagation in the presence of third order dispersion [9,10]. We then use two such stationary pulses, propagating with different group velocities, as the initial condition for the collision problem. To find the effects of the collision we develop a perturbation theory with *two* small parameters: the dimensionless third order dispersion coefficient d_3 and the reciprocal of the dimensionless interchannel frequency difference $1/\beta$. (Nonanalytical terms are neglected in this perturbation theory, which is the standard case. However, it is possible to show that even if such terms do exist their contribution to any collision induced effects can be neglected if d_3 is sufficiently small.)

We find that the amplitude of the emitted radiation is proportional to d_3/β^2. The source term for this radiation has the same form that would be produced by a variation with respect to distance along the fiber of the second order dispersion coefficient. The soliton amplitude and the phase velocity do not acquire any change in perturbative corrections up to third order. *This explains why accurate analysis up to the relatively high, third, order is necessary.* It is also found that soliton propagation in a given channel experiencing many collisions with solitons from other channels can be extracted from the same equation that describes propagation of solitons in a fiber with weak disorder in the second order dispersion coefficient. (The latter problem was addressed in Ref. [11].) This observation means that results obtained in Ref. [11] can be readily applied to describe the intrachannel interaction between solitons caused by many interchannel collisions.

The material in the manuscript is organized as follows. The stationary single-soliton solution of the nonlinear Schrödinger equation in the presence of nonzero third-order dispersion is perturbatively constructed in Sec. II. This stationary solution is then used as the initial condition for the two-soliton interchannel collision problem. Section III is devoted to description of the two-soliton collision. The general formulation of the perturbation method is presented in Sec. III A, while Secs. III B, III C, and III D describe the first, second, and third orders of the perturbation theory, respectively. Section IV is reserved for conclusions. Some auxiliary calculations are detailed in three Appendixes.

II. THE EFFECT OF THIRD ORDER DISPERSION ON A SINGLE SOLITON

A. Introduction

Propagation of an electric field wave packet $\Psi(t,z)$ through an optical fiber under the influence of third order dispersion is described by the following modification of the nonlinear Schrödinger equation (see Ref. [2], p. 44):

$$i\partial_z\Psi+\partial_t^2\Psi+2|\Psi|^2\Psi=id_3\partial_t^3\Psi\ . \tag{2.1}$$

Here, z is the dimensionless position along the fiber, $z=x(\kappa P_0/2)$, x is the actual position along the fiber, P_0 is the peak soliton power, and κ is the Kerr nonlinearity coefficient. The dimensionless retarded time is $t=\tau/\tau_0$, where τ is the retarded time associated with the reference channel and τ_0 is the soliton width. The spectral width ν_0 is given by $\nu_0=1/(\pi^2\tau_0)$, and the channel spacing by $\Delta\nu=\beta\nu_0$. The pulse envelope is $\Psi=E/\sqrt{P_0}$, where E is the actual electric field. The dimensionless second and third order dispersion coefficients are given by $d=-1=\beta_2/(\kappa P_0\tau_0^2)$ and $d_3=\beta_3/(3\kappa P_0\tau_0^3)$, where β_2 and β_3 are the second and third order chromatic dispersion coefficients, respectively. The typical setup for a short pulse optical fiber experiment is $\tau_0=0.5$ ps, $\beta_2=-1$ ps^2/km, $\beta_3=0.1$ ps^3/km, $\kappa=10$ W^{-1} km^{-1}, $P_0=0.4W$, $\Delta\nu=2.03\times10^{12}$ Hz, and the total energy of the soliton is 4×10^{-13} J. These values correspond to $d_3=0.066\,667$, and $\beta=10$.

The term $id_3\partial_t^3\Psi$ (where d_3 is a small constant) on the right-hand side of Eq. (2.1) accounts for the effect of third order dispersion. Terms of higher order, such as terms with higher temporal derivatives, and terms accounting for types of nonlinearity other than the Kerr nonlinearity, can be neglected in the majority of practical cases. Equation (2.1) is global in the sense that it explains simultaneous propagation through many frequency channels. Notice that for $d_3\neq0$, Eq. (2.1) is not integrable. However, in many practical cases $d_3\ll1$, allowing a perturbative calculation about the integrable $d_3=0$ limit.

Fiber losses in Eq. (2.1) are neglected. In practice, this can be achieved by compensating for losses in a fiber span by means of distributed optical amplification, e.g., Raman amplification. Compensation for fiber losses γ can also be achieved without distributed amplification [12] by a proper choice of the parameters $\Delta\nu$, β_2, β_3, and τ_0. Indeed, if $\tau_0\ll\sqrt{\beta_2/\gamma}$ and $\Delta\nu\gg(\gamma\tau_0^2)/\beta_3$, the term in Eq. (2.1) responsible for losses can be neglected. Note that Eq. (2.1) can also be used for description of optical pulse dynamics in two types of fiber links with in-line amplifiers: (i) optical links lumped optical amplifiers and fiber spans with exponentially decreasing spatial dispersion profile [13]; (ii) optical links with distributed in-line amplification.

B. Stationary single-soliton solution of Eq. (2.1)

Let us assume that $d_3\ll1$, and derive perturbatively a z-independent (stationary) single-soliton solution of Eq. (2.1). Combination of two such well separated in t stationary solutions will later be used as the initial condition for the two-soliton collision problem.

When $d_3=0$, the single-soliton solution of Eq. (2.1) in a given frequency channel (i.e., characterized by the frequency shift β relative to the reference channel) is described by

$$\psi_\beta(t,z)=\eta_\beta\frac{\exp[i\alpha_\beta+i\beta(t-y_\beta)+i(\eta_\beta^2-\beta^2)z]}{\cosh[\eta(t-y_\beta-2\beta z)]}, \tag{2.2}$$

where α_β, η_β, and y_β are the soliton phase, amplitude, and position, respectively. We shall call this solution the ideal, or bare soliton, solution, and refer to Eq. (2.1) with $d_3=0$ as the ideal NLSE.

To find a stationary single-soliton solution of Eq. (2.1) with $d_3\neq 0$ one first makes the following substitution:

$$\Psi_\beta(t,z)=e^{i\chi_\beta}\tilde{\Psi}_\beta(x), \tag{2.3}$$

where $x=\tilde{\eta}_\beta\tau_\beta$,

$$\tau_\beta=t-y_\beta-2\beta(1+3d_3\beta/2)z, \tag{2.4}$$

$$\tilde{\eta}_\beta=(1+3d_3\beta)^{-1/2}\eta_\beta \tag{2.5}$$

and

$$\chi_\beta=\alpha_\beta+\beta(t-y_\beta)+[\eta_\beta^2-\beta^2(1+d_3\beta)]z. \tag{2.6}$$

This substitution accounts for effects caused by the $d_3\beta$ shift in the second order dispersion coefficient in channel β. Thus, our perturbation theory does not require smallness of the shift in the second order dispersion coefficient. From Eqs. (2.4) and (2.5) one can see that the two main effects of the $d_3\beta$ shift in the second order dispersion coefficient are a $3d_3\beta/2$ increase in the group velocity (for $\beta>0$) and an increase of the pulse width by a factor of $(1+3d_3\beta)^{1/2}$ (for $\beta>0$). Using this substitution one obtains the following equation for the function $\tilde{\Psi}_\beta$:

$$\eta_\beta^2\tilde{\Psi}''_\beta-\eta_\beta^2\tilde{\Psi}_\beta+2|\tilde{\Psi}_\beta|^2\tilde{\Psi}_\beta=\frac{id_3\eta_\beta^3}{(1+3d_3\beta)^{3/2}}\tilde{\Psi}'''_\beta. \tag{2.7}$$

Let us now assume that $d_3\ll 1$ and look for perturbative solution of Eq. (2.7),

$$\tilde{\Psi}_\beta(x)=\tilde{\Psi}_{\beta 0}(x)+\tilde{\Psi}_{\beta 1}(x)+\cdots, \tag{2.8}$$

where

$$\tilde{\Psi}_{\beta 0}(x)=\frac{\eta_\beta}{\cosh(x)} \tag{2.9}$$

is the zero order solution, and $\tilde{\Psi}_{\beta 1}$ is the first order (in d_3) correction. Substituting $\tilde{\Psi}_\beta(x)$ into Eq. (2.7) one arrives at

$$\hat{L}_{\eta_\beta}\begin{pmatrix}\tilde{\Psi}_{\beta 1}(x)\\ \tilde{\Psi}^*_{\beta 1}(x)\end{pmatrix}=\frac{id_3\eta_\beta^4}{(1+3d_3\beta)^{3/2}}\tilde{\Psi}'''_{\beta 0}(x)\begin{pmatrix}1\\1\end{pmatrix}, \tag{2.10}$$

where the operator $\hat{L}_{\eta_\beta}$ is

$$\hat{L}_{\eta_\beta}\equiv\eta_\beta^2\left[(\partial_x^2-1)\hat{\sigma}_3+\frac{2}{\cosh^2(x)}(2\hat{\sigma}_3+i\hat{\sigma}_2)\right]. \tag{2.11}$$

This operator describes evolution of a linear perturbation around the single soliton (2.2) of the ideal NLSE. The complete set of the eigenfunctions of $\hat{L}_{\eta_\beta}$ includes the infinite (continuous) set of unlocalized modes φ_k and $\overline{\varphi}_k$, obeying

$$\hat{L}_{\eta_\beta}\varphi_k=(k^2+\eta_\beta^2)\varphi_k,\quad \hat{L}_{\eta_\beta}\overline{\varphi_k}=-(k^2+\eta_\beta^2)\overline{\varphi_k}. \tag{2.12}$$

There are also four discrete (localized) modes in the spectrum, f_0, f_1, f_2, and f_3, defined as

$$\hat{L}_{\eta_\beta}f_0=0,\quad \hat{L}_{\eta_\beta}f_1=0,$$

$$\hat{L}_{\eta_\beta}f_2=-2\eta_\beta^2 f_1,\quad \hat{L}_{\eta_\beta}f_3=-2\eta_\beta^2 f_0. \tag{2.13}$$

The complete system of eigenfunctions of the ideal NLSE, found by Kaup [7], is described in Appendix A. The unlocalized eigen-functions can be written as $\varphi_k(x)=f_{k/\eta_\beta}(x)$ and $\overline{\varphi_k}(x)=\overline{f_{k/\eta_\beta}}(x)$, where f_k and $\overline{f_k}$ are the eigenfunctions of $\hat{L}_{\eta_\beta}$ at $\eta_\beta=1$ introduced in Appendix A.

It is natural to expand $\tilde{\Psi}_{\beta 1}(x)$ in series in the eigenfunctions of $\hat{L}_{\eta_\beta}$,

$$\begin{pmatrix}\tilde{\Psi}_{\beta 1}(x)\\ \tilde{\Psi}^*_{\beta 1}(x)\end{pmatrix}=\tilde{c}_0 f_0(x)+\tilde{c}_1 f_1(x)+\tilde{c}_2 f_2(x)+\tilde{c}_3 f_3(x)$$
$$+\int_{-\infty}^{+\infty}\frac{dk}{2\pi}[c_k\varphi_k(x)+c_k^*\overline{\varphi_k}(x)]. \tag{2.14}$$

Expansion of the right-hand side of Eq. (2.10) over these eigenfunctions is

$$\frac{id_3\eta_\beta^4}{(1+3d_3\beta)^{3/2}}\tilde{\Psi}'''_{\beta 0}(x)\begin{pmatrix}1\\1\end{pmatrix}$$
$$=\frac{d_3\eta_\beta^2}{4(1+3d_3\beta)^{3/2}}\int_{-\infty}^{+\infty}dk\left[\frac{k(k/\eta_\beta+i)^2\varphi_k(x)}{\cosh[(\pi k)/(2\eta_\beta)]}\right.$$
$$\left.-\frac{k(k/\eta_\beta-i)^2\overline{\varphi_k}(x)}{\cosh[(\pi k)/(2\eta_\beta)]}\right]+\frac{id_3\eta_\beta^4}{(1+3d_3\beta)^{3/2}}f_1(x). \tag{2.15}$$

Substituting Eqs. (2.14) and (2.15) into Eq. (2.10), and also using relations (2.12) and (2.13), one obtains

$$\int_{-\infty}^{+\infty}\frac{dk}{2\pi}[(k^2+\eta_\beta^2)c_k\varphi_k(x)-(k^2+\eta_\beta^2)c_k^*\overline{\varphi_k}(x)]$$
$$-2\eta_\beta^2\tilde{c}_3 f_0(x)-2\eta_\beta^2\tilde{c}_2 f_1(x)$$
$$=\frac{d_3\eta_\beta^2}{4(1+3d_3\beta)^{3/2}}\int_{-\infty}^{+\infty}dk\left[\frac{k(k/\eta_\beta+i)^2\varphi_k(x)}{\cosh[(\pi k)/(2\eta_\beta)]}\right.$$
$$\left.-\frac{k(k/\eta_\beta-i)^2\overline{\varphi_k}(x)}{\cosh[(\pi k)/(2\eta_\beta)]}\right]+\frac{id_3\eta_\beta^4}{(1+3d_3\beta)^{3/2}}f_1(x). \tag{2.16}$$

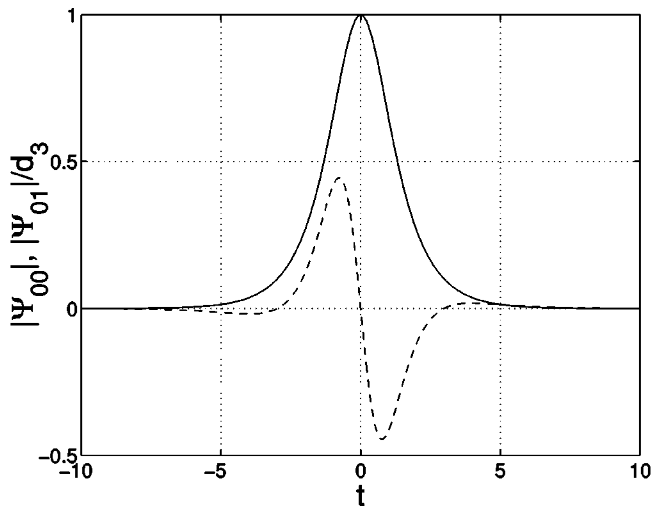

FIG. 2. Ideal ($d_3=0$) soliton solution in the $\beta=0$ channel, $|\Psi_{00}|=\tilde{\Psi}_{00}$, solid line, and the principal part of the respective first order (in d_3) correction, $|\Psi_{01}|/d_3=-i\tilde{\Psi}_{01}/d_3$, dashed line, are shown as functions of the time, *t*.

Projecting the last equation on the eigenfunctions of $\hat{L}_{\eta_\beta}$, one gets the following expressions for the expansion coefficients

$$\tilde{c}_2=-\frac{id_3\eta_\beta^2}{2(1+3d_3\beta)^{3/2}},\quad \tilde{c}_3=0 \tag{2.17}$$

and

$$c_k=\frac{\pi d_3 k(k/\eta_\beta+i)}{2(1+3d_3\beta)^{3/2}(k/\eta_\beta-i)\cosh[(\pi k)/(2\eta_\beta)]}. \tag{2.18}$$

Using Eqs. (A5) and (A6) for the continuous spectrum eigenfunctions, one can simplify the expression for $\tilde{\Psi}_{\beta1}(x)$,

$$\begin{pmatrix}\tilde{\Psi}_{\beta1}(x)\\ \tilde{\Psi}^*_{\beta1}(x)\end{pmatrix}=\tilde{c}_0 f_0(x)+\tilde{c}_1 f_1(x)+\tilde{c}_2 f_2(x)-\frac{id_3\eta_\beta^2}{2(1+3d_3\beta)^{3/2}}I(x)\begin{pmatrix}1\\-1\end{pmatrix}, \tag{2.19}$$

where the function $I(x)$ is defined by

$$I\equiv\int_{-\infty}^{+\infty}dk\,\frac{k[k\cos(kx)\tanh(x)+\frac{1}{2}(k^2-1)\sin(kx)]}{(k^2+1)\cosh[(\pi k)/2]}. \tag{2.20}$$

For illustrative purposes we show $\tilde{\Psi}_{00}$ and $-i\tilde{\Psi}_{01}/d_3$ together in Fig. 2.

Comparing Eqs. (2.17) and (2.20) with Eq. (A15), one concludes that one effect of third order dispersion is an $O(d_3)$ frequency change, which is not accompanied by a corresponding change in the group velocity. Such an effect was observed earlier by other authors who studied the evolution of an initially ideal soliton under the influence of third order dispersion [9,10]. Even though the correction $\tilde{\Psi}_{\beta1}(x)$ gets contributions from the continuous spectrum eigenfunctions, the existence of the double poles at $k=\pm i$ in the integrand appearing on the rhs of Eq. (2.20) guarantees that $\tilde{\Psi}_{\beta1}(x)$ is localized, i.e., it decays exponentially in x for $x \gg 1$. In addition, linear stability of the stationary solution $\tilde{\Psi}_{\beta0}+\tilde{\Psi}_{\beta1}$, was proved and numerically confirmed in Ref. [8]. Notice that the coefficients $\tilde{c}_0$ and $\tilde{c}_1$ in Eq. (2.19), which correspond to $O(d_3)$ corrections to the soliton's phase and position, remain arbitrary. For convenience we shall choose $\tilde{c}_0=\tilde{c}_1=0$, thus obtaining that $\tilde{\Psi}_{\beta1}(x)$ is odd in x and is purely imaginary. One should also point out that the perturbation theory presented here is valid for any value of β, provided $d_3 \ll 1$ and $1+\beta d_3 \gg d_3$. The last inequality is the condition that the effect of the second order dispersion coefficient in channel β essentially exceeds the effect of the third order dispersion coefficient. Finally, by a similar method one can obtain the $O(d_3^2)$ and higher order *z*-independent corrections to the stationary solution found here [8].

III. TWO-SOLITON COLLISION ACCOUNTING FOR THIRD ORDER DISPERSION

A. General formulation of the perturbation method

We now turn to calculation of the effects caused by collision of two solitons from different frequency channels. We assume that the initial condition for the solitons, far away from the collision region, is given by a sum of the stationary single-soliton solutions found in the preceding section. Since these solutions are stable, even if the initial shape of a pulse is not precisely of the stationary form it evolves to it after a transient. Thus, no radiation and no change in the soliton parameters are caused by the free propagation of these stationary pulses before the collision. We develop a double perturbation theory with the two small parameters d_3 and $1/\beta$. For simplicity, and without any loss of generality, one of the two channels is chosen as a reference one with $\beta=0$. We assume that for the second channel $|\beta| \gg 1$. (In practice, this condition is satisfied very well even for neighboring channels.) As it is shown in Appendix B, expansion of the exact two-soliton solution of the ideal NLSE [Eq. (2.1) with $d_3=0$] in a series in $1/\beta$ is given by

$$\psi_{two}=\psi_0+\psi_\beta+\phi_0+\phi_\beta+\phi_{-\beta}+\phi_{2\beta}+O(1/\beta^3), \tag{3.1}$$

where ψ_0 and ψ_β are $O(1)$ single-soliton solutions of the ideal NLSE described by Eq. (2.2). The terms ϕ_0 and ϕ_β are corrections of the leading order $1/\beta$ to these single-soliton solutions in channels 0 and β, respectively. The terms $\phi_{-\beta}$ and $\phi_{2\beta}$ oscillate with frequencies $-\beta$ and 2β, respectively. These terms are of leading order $1/\beta^2$, and are exponentially small outside of the collision region. Furthermore, outside the collision region ϕ_0 and ϕ_β generate only small constant corrections to the parameters of ψ_0 and ψ_β, so that ψ_{two} reduces to a sum of the two ideal single-soliton solutions.

This last result remains valid in any order in $1/\beta$. We also show in Appendix B that for $|\beta|\gg 1$ the only effects of collision between two ideal solitons are $O(1/\beta)$ change of phase and $O(1/\beta^2)$ position shift (time retardation).

It is therefore natural to look for a two-soliton solution of Eq. (2.1) with $d_3\neq 0$ in the following form:

$$\Psi_{two}=\Psi_0+\Psi_\beta+\Phi. \tag{3.2}$$

Here Ψ_0 and Ψ_β are stationary single-soliton solutions of Eq. (2.1) with $d_3\neq 0$ in channels 0 and β, respectively, and Φ is a small z-dependent correction to these solutions, arising from the collision only. Substituting Ψ_{two} into Eq. (2.1), one arrives at

$$\begin{aligned} i\partial_z\Phi+\partial_t^2\Phi+4|\Psi_0+\Psi_\beta|^2\Phi+2(\Psi_0+\Psi_\beta)^2\Phi^* &\\ =-4|\Psi_\beta|^2\Psi_0-4|\Psi_0|^2\Psi_\beta-2\Psi_0^*\Psi_\beta^2-2\Psi_\beta^*\Psi_0^2 &\\ -4(\Psi_0+\Psi_\beta)|\Phi|^2-2(\Psi_0^*+\Psi_\beta^*)\Phi^2-2|\Phi|^2\Phi &\\ +id_3\partial_t^3\Phi. & \end{aligned} \tag{3.3}$$

Let us calculate, for example, the collision induced correction Φ_0 to the stationary pulse in the reference channel. Then, the total pulse in the reference channel is given by $\Psi_0^t=\Psi_0+\Phi_0$. Calculation of the correction Φ_β to the stationary pulse in the β channel is similar. By analogy with Eq. (3.1), one substitutes

$$\Phi=\Phi_0+\Phi_\beta+\Phi_{-\beta}+\Phi_{2\beta}+\cdots, \tag{3.4}$$

where $\Phi_{-\beta}$ and $\Phi_{2\beta}$ correspond to terms in channels $-\beta$ and 2β, respectively, and the ellipses represents higher order terms in other channels. Since Φ_0 oscillates together with Ψ_0, and since $|\beta|\gg 1$, one can use resonant approximation, simply neglecting exponentially small contributions coming from terms rapidly oscillating with respect to t and z. Φ_0 dynamics is governed by

$$\begin{aligned} i\partial_z\Phi_0+\partial_t^2\Phi_0+4|\Psi_0|^2\Phi_0+2\Psi_0^2\Phi_0^* &\\ =-4|\Psi_\beta|^2\Psi_0-4|\Psi_\beta|^2\Phi_0-4\Psi_0(\Psi_\beta\Phi_\beta^*+\Psi_\beta^*\Phi_\beta) &\\ -4\Psi_0^*\Psi_\beta\Phi_{-\beta}-2\Psi_\beta^2\Phi_{2\beta}^*-4\Psi_0|\Phi_0|^2-2\Psi_0^*\Phi_0^2 &\\ -4\Psi_0|\Phi_\beta|^2-4\Phi_0(\Psi_\beta\Phi_\beta^*+\Psi_\beta^*\Phi_\beta) &\\ +id_3\partial_t^3\Phi_0. & \end{aligned} \tag{3.5}$$

In writing Eqs. (3.4) and (3.5) we neglected terms of order $1/\beta^3$ and higher. Indeed, such terms only contribute to $O(1/\beta^3)$ or higher order effects that already exist in collisions between ideal solitons. Substituting $\Psi_0(t,z)=\tilde\Psi_0(\tilde h_0)\exp(i\chi_0)$ and $\Phi_0(t,z)=\tilde\Phi_0(\tilde h_0)\exp(i\chi_0)$, where $\tilde h_0=\eta_0 t$ and $\chi_0=\alpha_0+\eta_0^2 z$ [see Eqs. (2.3)–(2.6) with $\beta=0$], one derives

$$\begin{aligned} \partial_z\tilde\Phi_0-i[(\partial_t^2-\eta_0^2)\tilde\Phi_0+4|\Psi_0|^2\tilde\Phi_0+2\tilde\Psi_0^2\tilde\Phi_0^*] &\\ =4i\Big[|\Psi_\beta|^2\tilde\Psi_0+|\Psi_\beta|^2\tilde\Phi_0+\tilde\Psi_0(\Psi_\beta\Phi_\beta^*+\Psi_\beta^*\Phi_\beta) &\\ +\tilde\Psi_0^*\Psi_\beta\Phi_{-\beta}+\frac{1}{2}\Psi_\beta^2\Phi_{2\beta}^*+\tilde\Psi_0|\Phi_0|^2+\frac{1}{2}\tilde\Psi_0^*\tilde\Phi_0^2 &\\ +\tilde\Psi_0|\Phi_\beta|^2+\tilde\Phi_0(\Psi_\beta\Phi_\beta^*+\Psi_\beta^*\Phi_\beta)\Big]+d_3\partial_t^3\tilde\Phi_0. & \end{aligned} \tag{3.6}$$

Three different regions in z are naturally separated. The first region is the small interval $[z_0-\tilde z/|\beta|,z_0+\tilde z/|\beta|]$ in the vicinity of the collision point z_0, where $|\beta|\gg\tilde z\gg 1$. In this region $\tilde\Phi_0$ acquires fast change with respect to z as a result of collision. Since $\Delta z\sim 1/\beta$, in the first order of the perturbation theory the $\partial_z\tilde\Phi_0$ and $|\Psi_\beta|^2\tilde\Psi_0$ terms give the leading contributions to Eq. (3.6), while the $\partial_t^2\tilde\Phi_0$ term can be neglected together with all other terms. In the next orders of the perturbation theory one should carefully consider contributions coming from terms such as $\partial_t^2\tilde\Phi_0$, $d_3\partial_t^3\tilde\Phi_0$, as well as contributions coming from interaction terms such as $|\Psi_\beta|^2\tilde\Psi_0$, $\tilde\Psi_0\Psi_\beta\Phi_\beta^*$, etc. In the precollision region $z<z_0-\tilde z/|\beta|$ and in the postcollision region $z>z_0+\tilde z/|\beta|$, the strength of the interaction between the two solitons is exponentially small, so that the term $|\Psi_\beta|^2\tilde\Psi_0$ and all other interaction terms can be neglected. Thus, these two regions correspond to the free propagation of the stationary pulse before and after the collision. Indeed, the description of the propagation of Φ_0 given by Eq. (3.6) is exactly equivalent to the description given by substituting the complete *single-soliton* solution $\Psi_0^t=\Psi_0+\Phi_0$ into Eq. (2.1).

It follows, from what is explained above, that in order to obtain Φ_0 one has to solve and subsequently match three Cauchy problems imposed at $z=-\infty$, $z=z_0-\tilde z/|\beta|$ and $z=z_0+\tilde z/|\beta|$, respectively. Matching means that the initial conditions at $z=z_0-\tilde z/|\beta|$, and $z=z_0+\tilde z/|\beta|$ are obtained from the solutions found in the $(-\infty,z_0-\tilde z/|\beta|]$ and the $[z_0-\tilde z/\beta,z_0+\tilde z/|\beta|]$ regions, respectively. As we shall see later, the results of these calculations are not sensitive to the specific value of the collision region cutoff parameter, $\tilde z$.

The correction $\tilde\Phi_0$ and the complete solution $\tilde\Psi_0^t=\tilde\Psi_0+\tilde\Phi_0$ are obtained in the form of a perturbation series. That is, in the collision region one substitutes

$$\tilde\Phi_0(\tilde h_0,z)=\tilde\Phi_{01}(\tilde h_0,z)+\tilde\Phi_{02}(\tilde h_0,z)+\cdots, \tag{3.7}$$

$$\tilde\Psi_0(\tilde h_0)=\tilde\Psi_{00}(\tilde h_0)+\tilde\Psi_{01}(\tilde h_0)+\cdots, \tag{3.8}$$

and

$$\Psi_\beta(\tilde h_\beta,z)=e^{i\chi_\beta}[\tilde\Psi_{\beta 0}(\tilde h_\beta)+\tilde\Psi_{\beta 1}(\tilde h_\beta)+\cdots] \tag{3.9}$$

into Eq. (3.6), and linearizes the result with respect to the two small parameters d_3 and $1/\beta$. Index notations introduced in Eqs. (3.7)–(3.9) are as follows. $\tilde{\Phi}_{01}$ and $\tilde{\Phi}_{02}$ account for the first and second combined order corrections with respect to both small parameters d_3 and $1/\beta$. The stationary zero order terms $\tilde{\Psi}_{00}$ and $\tilde{\Psi}_{\beta 0}$ in the expansions (3.8) and (3.9) correspond to the bare (ideal) solitons given in Eq. (2.9) with $\beta=0$ and generic β, respectively. The stationary correction terms $\tilde{\Psi}_{01}$ and $\tilde{\Psi}_{\beta 1}$, which are given by Eqs. (2.14), (2.17), and (2.18) with $\beta=0$ and general β, respectively, are $O(d_3)$. In the free propagation region, one substitutes $\Psi_0^t=\exp(i\chi_0)(\tilde{\Psi}_0+\tilde{\Phi}_0)$ into Eq. (2.1) and linearizes it with respect to d_3 and $1/\beta$.

Contributions to $\tilde{\Phi}$ of first, second, and third combined orders will be found as solutions of the respective linear inhomogeneous equations obtained by this perturbation method. To analyze the free (after collision) propagation of $\tilde{\Phi}$ one needs to project $\tilde{\Phi}$ onto the set of eigenfunctions of $\hat{L}_{\eta_0}$. This procedure allows one to easily separate the parts in $\tilde{\Phi}_0$, corresponding to changes in the soliton parameters, from the part corresponding to emitted radiation. The expansion of $\tilde{\Phi}_0$ in the series over the eigenfunctions of $\hat{L}_{\eta_0}$ is

$$\begin{pmatrix} \tilde{\Phi}_0(\tilde{h}_0) \\ \tilde{\Phi}_0^*(\tilde{h}_0) \end{pmatrix} = \sum_{i=0}^{3} \tilde{a}_i f_i(\tilde{h}_0) + \begin{pmatrix} \tilde{v}_0 \\ \tilde{v}_0{}^* \end{pmatrix}$$
$$= \sum_{i=0}^{3} \tilde{a}_i f_i(\tilde{h}_0) + \int_{-\infty}^{+\infty} \frac{dk}{2\pi} [a_k(z)\varphi_k(\tilde{h}_0) + a_k^*(z)\overline{\varphi_k}(\tilde{h}_0)], \tag{3.10}$$

where $\tilde{v}_0$ corresponds to the radiation (unlocalized) part of $\tilde{\Phi}_0$.

B. First order perturbation theory

Let us consider propagation of the soliton in the precollision region. We find it convenient to choose such $\tilde{\Phi}_{01}$ at $z=-\infty$ that coincides with the $O(1/\beta)$ term in the $z\ll z_0-1/|\beta|$ asymptotic form of the expansion of the ideal two-soliton solution (B1). Thus, $\tilde{\Phi}_{01}(\tilde{h}_0,-\infty)$ is proportional to the sixth term on the right-hand side of Eq. (B6),

$$\tilde{\Phi}_{01}(\tilde{h}_0,-\infty)=i\,\delta\alpha_{01}^{(0)in}\tilde{\Psi}_{00}, \tag{3.11}$$

where

$$\delta\alpha_{01}^{(0)in}=-\frac{2\,\eta_\beta(1+3d_3\beta)^{1/2}}{(1+3d_3\beta/2)|\beta|}. \tag{3.12}$$

In Eq. (3.12) and the following equations the superscript *in* stands for initial values of phase, position, etc., while the superscript *out* represents final values of the same parameters. The $O(d_3)$ part of the initial condition is taken to be $\tilde{\Psi}_{01}(\tilde{h}_0)$.

Let us now substitute $\Psi_0^t=\exp(i\chi_0)(\tilde{\Psi}_{00}+\tilde{\Psi}_{01}+\tilde{\Phi}_{01})$ into Eq. (2.1). The $O(d_3)$ part of the resulting equation is simply Eq. (2.10), whose solution is $\tilde{\Psi}_{01}(\tilde{h}_0)$. To analyze the $O(1/\beta)$ part of the equation, one defines

$$\tilde{\chi}_0^{in}=\alpha_0+\delta\alpha_{01}^{(0)in}+\eta_0^2 z, \tag{3.13}$$

where $\delta\alpha_{01}^{(0)in}$ is given by Eq. (3.12). We then substitute $\tilde{\Psi}_{00}\exp(i\tilde{\chi}_0^{in})$ instead of $\exp(i\chi_0)(\tilde{\Psi}_{00}+\tilde{\Phi}_{01})$ into Eq. (2.1). Clearly, $\tilde{\Psi}_{00}\exp(i\tilde{\chi}_0^{in})$ is a solution of the equation up to order $1/\beta$, which obeys the initial condition (3.11). Thus, one finds

$$\tilde{\Phi}_{01}(\tilde{h}_0,z\leqslant z_0-\tilde{z}/|\beta|)=i\,\delta\alpha_{01}^{(0)in}\tilde{\Psi}_{00}. \tag{3.14}$$

In the collision region Eq. (3.6) reduces to

$$\partial_z\tilde{\Phi}_{01}=4i|\Psi_{\beta 0}|^2\tilde{\Psi}_{00}=\frac{4i\,\eta_0\,\eta_\beta^2}{\cosh(\tilde{h}_0)\cosh^2(\tilde{h}_\beta)}. \tag{3.15}$$

Integrating Eq. (3.15) over the collision region, one finds

$$\tilde{\Phi}_{01}(\tilde{h}_0,z_0+\tilde{z}/|\beta|)-\tilde{\Phi}_{01}(\tilde{h}_0,z_0-\tilde{z}/|\beta|) =\frac{4i\,\eta_0\,\eta_\beta^2}{\cosh(\tilde{h}_0)}\int_{z_0-\tilde{z}/|\beta|}^{z_0+\tilde{z}/|\beta|}\frac{dz'}{\cosh^2(\tilde{h}_\beta)}. \tag{3.16}$$

Since the integrand on the rhs of Eq. (3.16) is sharply peaked in the vicinity of the collision point z_0, the integration limits in Eq. (3.16) can be replaced by $-\infty$ and ∞, respectively. Performing the integration and using the initial condition (3.14) at $z=z_0-\tilde{z}/|\beta|$, one arrives at

$$\tilde{\Phi}_{01}(\tilde{h}_0,z_0+\tilde{z}/|\beta|)=i\,\delta\alpha_{01}^{(0)out}\tilde{\Psi}_{00}, \tag{3.17}$$

where

$$\delta\alpha_{01}^{(0)out}=-\,\delta\alpha_{01}^{(0)in}. \tag{3.18}$$

In the post-collision region one can show in a similar manner that

$$\tilde{\Phi}_{01}(\tilde{h}_0,z\geqslant z_0+\tilde{z}/|\beta|)=i\,\delta\alpha_{01}^{(0)out}\tilde{\Psi}_{00}. \tag{3.19}$$

Comparing Eqs. (3.19) and (3.14), we see that the only effect of the collision in the first order of the perturbation theory is a change of the soliton phase,

$$\Delta\alpha_{01}^{(0)}=\delta\alpha_{01}^{(0)out}-\delta\alpha_{01}^{(0)in}=\frac{4\,\eta_\beta(1+3d_3\beta)^{1/2}}{(1+3d_3\beta/2)|\beta|}. \tag{3.20}$$

Notice that Eq. (3.20) is also consistent with the result (B8) obtained in Appendix B, from the $1/\beta$ expansion of the exact two-soliton solution of Eq. (2.1) with $d_3=0$.

Until now we have only calculated the form of the leading contribution $\tilde{\Phi}_{01}$ outside of the collision region. However, calculation of higher order terms requires knowledge of the complete z dependence of $\tilde{\Phi}_{01}$. To achieve this aim let us integrate Eq. (3.15) from $-\infty$ to some general z. This integration yields

$$\tilde{\Phi}_{01}(t,z)=-\frac{2i\eta_0\eta_\beta(1+3d_3\beta)^{1/2}}{(1+3d_3\beta/2)\beta}\frac{\tanh(\tilde{h}_\beta)}{\cosh(\tilde{h}_0)}, \tag{3.21}$$

where we have used the initial condition (3.11).

C. Second order perturbation theory

1. $O(d_3^2)$

The complete solution up to second order is given by $\Psi_0^t=\exp(i\chi_0)(\tilde{\Psi}_{00}+\tilde{\Psi}_{01}+\tilde{\Psi}_{02}+\tilde{\Phi}_{01}+\tilde{\Phi}_{02})$, where the $O(d_3^2)$ initial condition is taken to be $\exp(i\chi_0)\tilde{\Psi}_{02}(\tilde{h}_0)$. This term remains z independent throughout the entire collision.

2. $O(1/\beta^2)$

The initial condition for the $O(1/\beta^2)$ term $\tilde{\Phi}_{02}^{(0)}$ is chosen to coincide with the $O(1/\beta^2)$ term in the $z\ll z_0-1/|\beta|$ asymptotic form of the expansion of the ideal two-soliton solution (B1),

$$\tilde{\Phi}_{02}^{(0)}(\tilde{h}_0,-\infty)=-\frac{1}{2}(\delta\alpha_{01}^{(0)in})^2\tilde{\Psi}_{00}-\eta_0\delta y_{02}^{(0)in}\tilde{\Psi}'_{00}, \tag{3.22}$$

where

$$\delta y_{02}^{(0)in}=\frac{2\eta_\beta(1+3d_3\beta)^{1/2}}{(1+3d_3\beta/2)^2\beta|\beta|}, \tag{3.23}$$

and $\tilde{\Psi}'_{00}=d\tilde{\Psi}_{00}/d\tilde{h}_0$. In the precollision region one substitutes into Eq. (2.1) a solution of the form $\exp(i\tilde{\chi}_0^{in})\tilde{\Psi}_{00}(\tilde{h}_0^{in})$ where

$$\tilde{h}_0^{in}=\eta_0(t-\delta y_{02}^{(0)in}). \tag{3.24}$$

Evidently, $\exp(i\tilde{\chi}_0^{in})\tilde{\Psi}_{00}(\tilde{h}_0^{in})$ is a solution of the equation up to order $1/\beta^2$ that satisfies the initial condition (3.22). Thus,

$$\tilde{\Phi}_{02}^{(0)}(\tilde{h}_0,z\leqslant z_0-\tilde{z}/|\beta|)=-\frac{1}{2}(\delta\alpha_{01}^{(0)in})^2\tilde{\Psi}_{00}-\eta_0\delta y_{02}^{(0)in}\tilde{\Psi}'_{00}. \tag{3.25}$$

In the collision region the $O(1/\beta^2)$ part of Eq. (3.6) is

$$\partial_z\tilde{\Phi}_{02}^{(0)}-i[(\partial_t^2-\eta_0^2)\tilde{\Phi}_{01}+4|\Psi_0|^2\tilde{\Phi}_{01}+2\tilde{\Psi}_0^2\tilde{\Phi}_{01}^*]$$
$$=4i|\Psi_{\beta0}|^2\tilde{\Phi}_{01}+4i\tilde{\Psi}_{00}(\Psi_{\beta0}\Phi_{\beta1}^*+\Psi_{\beta0}^*\Phi_{\beta1}). \tag{3.26}$$

Using Eq. (3.21) for $\tilde{\Phi}_{01}(t,z)$ [and a similar expression for $\tilde{\Phi}_{\beta1}(t,z)$] one can show that the only change in the soliton's parameters comes from the term $i\partial_t^2\tilde{\Phi}_{01}$. Equation (3.26) can then be reduced to

$$\partial_z\tilde{\Phi}_{02}^{(0)}=\frac{-4\eta_0^2\eta_\beta^2}{(1+3d_3\beta/2)\beta}\frac{\tanh(\tilde{h}_0)}{\cosh(\tilde{h}_0)\cosh^2(\tilde{h}_\beta)}. \tag{3.27}$$

Integrating over the collision region, and using the initial condition (3.25) at $z=z_0-\tilde{z}/|\beta|$, one derives

$$\tilde{\Phi}_{02}^{(0)}(\tilde{h}_0,z_0+\tilde{z}/|\beta|)=-\frac{1}{2}(\delta\alpha_{01}^{(0)in})^2\tilde{\Psi}_{00}-\eta_0\delta y_{02}^{(0)out}\tilde{\Psi}'_{00}, \tag{3.28}$$

where

$$\delta y_{02}^{(0)out}=-\delta y_{02}^{(0)in}. \tag{3.29}$$

In the post-collision region one can show by arguments similar to the ones used above that

$$\tilde{\Phi}_{02}^{(0)}(\tilde{h}_0,z\geqslant z_0+\tilde{z}/|\beta|)$$
$$=-\frac{1}{2}(\delta\alpha_{01}^{(0)in})^2\tilde{\Psi}_{00}-\eta_0\delta y_{02}^{(0)out}\tilde{\Psi}'_{00}. \tag{3.30}$$

Comparing Eq. (3.30) and Eq. (3.25) we see that the only effect of the collision in order $1/\beta^2$ is a position shift (time retardation) given by

$$\Delta y_{02}^{(0)}=\delta y_{02}^{(0)out}-\delta y_{02}^{(0)in}=-\frac{4\eta_\beta(1+3d_3\beta)^{1/2}}{(1+3d_3\beta/2)^2\beta|\beta|}. \tag{3.31}$$

Taking $d_3=0$ we see that Eq. (3.31) coincides with Eq. (B9) obtained from the $1/\beta$ expansion of the two-soliton solution of the ideal NLSE. This is also the result obtained by Mollenauer *et al.* in Refs. [13] and [14] for the collision of ideal solitons.

3. $O(d_3/\beta)$

For convenience, the initial condition for the $O(d_3/\beta)$ correction term $\tilde{\Phi}_{02}^{(1)}$ is taken to be

$$\tilde{\Phi}_{02}^{(1)}(\tilde{h}_0,-\infty)=i\delta\alpha_{01}^{(0)in}\tilde{\Psi}_{01}(\tilde{h}_0). \tag{3.32}$$

To find $\tilde{\Phi}_{02}^{(1)}$ in the precollision region let us substitute $\exp(i\tilde{\chi}_0^{in})(\tilde{\Psi}_{00}+\tilde{\Psi}_{01})$ into Eq. (2.1). Obviously, $\exp(i\tilde{\chi}_0^{in})\tilde{\Psi}_{01}$ is a solution of the resulting equation (up to order d_3/β) that obeys the initial condition (3.32). One finds

$$\tilde{\Phi}_{02}^{(1)}(\tilde{h}_0,z\leqslant z_0-\tilde{z}/|\beta|)=i\delta\alpha_{01}^{(0)in}\tilde{\Psi}_{01}(\tilde{h}_0). \tag{3.33}$$

The $O(d_3/\beta)$ form of Eq. (3.6) in the collision region is

$$\partial_z\tilde{\Phi}_{02}^{(1)}=4i|\Psi_{\beta0}|^2\tilde{\Psi}_{01}+4i\tilde{\Psi}_{\beta0}(\tilde{\Psi}_{\beta1}+\tilde{\Psi}_{\beta1}^*)\tilde{\Psi}_{00}. \tag{3.34}$$

Fixing $\tilde{c}_0=\tilde{c}_1=0$ in Eq. (2.14), one gets that $\tilde{\Psi}_{\beta1}$ is pure imaginary so that the second term on the rhs of Eq. (3.34) is identically zero for any value of z. Notice that even for an arbitrary choice of these coefficients the real part of $\tilde{\Psi}_{\beta1}$ is an odd function of z and this term does not give any contribution when integrated over the collision region. Integrating Eq. (3.34) over the collision region while using the initial condition (3.33) at $z=z_0-\tilde{z}/|\beta|$, one obtains

$$\tilde{\Phi}_{02}^{(1)}(t,z_0+\tilde{z}/|\beta|)=i\,\delta\alpha_{01}^{(0)out}\tilde{\Psi}_{01}(\tilde{h}_0), \qquad (3.35)$$

where $\delta\alpha_{01}^{(0)out}$ is given by Eq. (3.18).

In the post-collision region one obtains

$$\tilde{\Phi}_{02}^{(1)}(t,z\geqslant z_0+\tilde{z}/|\beta|)=i\,\delta\alpha_{01}^{(0)out}\tilde{\Psi}_{01}(\tilde{h}_0). \qquad (3.36)$$

Comparing Eqs. (3.36) and (3.33) we see that the only effect of the collision in order d_3/β is an $O(1/\beta)$ change of phase, given by Eq. (3.20), on top of the $O(d_3)$ stationary solution $\tilde{\Psi}_{01}$.

D. Third order perturbation theory

The complete solution up to the third order of the perturbation theory is given by

$$\Psi_0^t=\exp(i\chi_0)(\tilde{\Psi}_{00}+\tilde{\Psi}_{01}+\tilde{\Psi}_{02}+\tilde{\Psi}_{03}+\tilde{\Phi}_{01}+\tilde{\Phi}_{02}+\tilde{\Phi}_{03}).$$

The $O(d_3^3)$ correction is given by $\tilde{\Psi}_{03}$. This term remains z independent throughout the collision dynamics. From the discussion of the $O(1/\beta)$ and $O(1/\beta^2)$ corrections, one should expect that the $O(1/\beta^3)$ term will only describe effects that already exist in the ideal soliton collision. In particular, it will not contain any contributions to radiation emission, amplitude change, or frequency change. Furthermore, one can show that the only contribution to the $O(d_3^2/\beta)$ term will describe a change of phase on top of the stationary solution. Therefore, the only new nontrivial effect, i.e., emission of radiation, can come from the $O(d_3/\beta^2)$ term.

To analyze the $O(d_3/\beta^2)$ correction, we first write it in the form

$$\tilde{\Phi}_{03}^{(1)}=\tilde{\Phi}_{03}^{(1)NR}+\tilde{v}_{03}, \qquad (3.37)$$

where $\tilde{v}_{03}$ is the leading, $O(d_3/\beta^2)$, contribution to radiation emitted due to the collision and $\tilde{\Phi}_{03}^{(1)NR}$ corresponds to the nonradiative part. The initial condition for $\tilde{\Phi}_{03}^{(1)}$ is taken to be

$$\tilde{\Phi}_{03}^{(1)}(\tilde{h}_0,-\infty)=\tilde{\Phi}_{03}^{(1)NR}(\tilde{h}_0,-\infty)=i\,\delta\alpha_{03}^{(1)in}\tilde{\Psi}_{00}(\tilde{h}_0) -\frac{1}{2}(\delta\alpha_{01}^{(0)in})^2\tilde{\Psi}_{01}(\tilde{h}_0)-\eta_0\delta y_{02}^{(0)in}\tilde{\Psi}_{01}'(\tilde{h}_0), \qquad (3.38)$$

from which it follows that

$$\tilde{v}_{03}(\tilde{h}_0,-\infty)=\tilde{v}_{03}^*(\tilde{h}_0,-\infty)=0, \qquad (3.39)$$

i.e., the initial condition contains no radiation. In Eq. (3.38), $\delta\alpha_{03}^{(1)in}$ is an initial $O(d_3/\beta^2)$ contribution to the phase.

In the precollision region one substitutes into Eq. (2.1) a solution of the form $\exp(i\hat{\chi}_0^{in})[\tilde{\Psi}_{00}(\tilde{h}_0)+\tilde{\Psi}_{01}(\tilde{h}_0^{in})+\tilde{v}_{03}(\tilde{h}_0)]$, where

$$\hat{\chi}_0^{in}=\alpha_0+\delta\alpha_{01}^{(0)in}+\delta\alpha_{03}^{(0)in}+\eta_0^2 z. \qquad (3.40)$$

Linearizing the resulting equation with respect to d_3 and $1/\beta$ one obtains a linear partial differential equation (PDE), which is automatically separated into two parts: one for $\tilde{\Phi}_{03}^{(1)NR}$ and the other for $\tilde{v}_{03}$, which can be written as

$$(\partial_z-i\hat{L}_{\eta_0})\begin{pmatrix}\tilde{v}_{03}\\ \tilde{v}_{03}^*\end{pmatrix}=0. \qquad (3.41)$$

To find $\tilde{\Phi}_{03}^{(1)NR}$ we notice that $\exp(i\hat{\chi}_0^{in})[\tilde{\Psi}_{00}(\tilde{h}_0)+\tilde{\Psi}_{01}(\tilde{h}_0^{in})]$ is an $O(d_3/\beta^2)$ stationary solution of Eq. (2.1) which obeys the initial condition (3.38). Thus,

$$\tilde{\Phi}_{03}^{(1)NR}(\tilde{h}_0,z\leqslant z_0-\tilde{z}/|\beta|) =i\,\delta\alpha_{03}^{(1)in}\tilde{\Psi}_{00}(\tilde{h}_0)-\frac{1}{2}(\delta\alpha_{01}^{(0)in})^2\tilde{\Psi}_{01}(\tilde{h}_0) -\eta_0\delta y_{02}^{(0)in}\tilde{\Psi}_{01}'(\tilde{h}_0). \qquad (3.42)$$

The only solution of Eq. (3.41), which satisfies the initial condition (3.39), is the trivial solution

$$\tilde{v}_{03}(\tilde{h}_0,z\leqslant z_0-\tilde{z}/|\beta|)=\tilde{v}_{03}^*(\tilde{h}_0,z\leqslant z_0-\tilde{z}/|\beta|)=0. \qquad (3.43)$$

To obtain the $O(d_3/\beta^2)$ correction in the collision region one considers all the $O(d_3/\beta)$ terms entering Eq. (3.6). Then one obtains

$$\begin{aligned}\partial_z\tilde{\Phi}_{03}^{(1)}-i[(\partial_t^2-\eta_0^2)\tilde{\Phi}_{02}^{(1)}+4|\Psi_{00}|^2\tilde{\Phi}_{02}^{(1)}+2\tilde{\Psi}_{00}^2\tilde{\Phi}_{02}^{(1)*}]\\ =4i|\Psi_{\beta 0}|^2\tilde{\Phi}_{02}^{(1)}+4i\tilde{\Psi}_{\beta 0}(\tilde{\Psi}_{\beta 1}+\tilde{\Psi}_{\beta 1}^*)\tilde{\Phi}_{01}\\ +4i\tilde{\Psi}_{00}\tilde{\Psi}_{01}(\tilde{\Phi}_{01}+\tilde{\Phi}_{01}^*)+4i\tilde{\Psi}_{00}\tilde{\Psi}_{01}^*\tilde{\Phi}_{01}+4i\tilde{\Psi}_{\beta 0}\\ \times(\tilde{\Phi}_{\beta 2}^{(1)}+\tilde{\Phi}_{\beta 2}^{(1)*})\tilde{\Psi}_{00}+4i\tilde{\Psi}_{\beta 0}(\tilde{\Phi}_{\beta 1}^*+\tilde{\Phi}_{\beta 1})\tilde{\Psi}_{01}\\ +d_3\partial_t^3\tilde{\Phi}_{01}+4i(\tilde{\Psi}_{\beta 1}\tilde{\Phi}_{\beta 1}^*+\tilde{\Psi}_{\beta 1}^*\tilde{\Phi}_{B1})\tilde{\Psi}_{00}.\end{aligned} \qquad (3.44)$$

For the choice $\tilde{c}_0=\tilde{c}_1=0$ in Eq. (2.14) one can show that the only terms in Eq. (3.44) contributing to the integral over the collision region are $d_3\partial_t^3\tilde{\Phi}_{01}$ and $i\partial_t^2\tilde{\Phi}_{02}^{(1)}$. Thus, Eq. (3.44) turns into

$$\partial_z\tilde{\Phi}_{03}^{(1)}-i\partial_t^2\tilde{\Phi}_{02}^{(1)}=d_3\partial_t^3\tilde{\Phi}_{01}. \qquad (3.45)$$

To solve this equation with the initial conditions (3.42) and (3.43) one first writes

$$\tilde{\Phi}_{03}^{(1)}=\tilde{\Phi}_{03}^{(1)NR1}+\tilde{\Phi}_{03}^{(1)R}, \qquad (3.46)$$

where

$$\partial_z \tilde{\Phi}_{03}^{(1)NR1} - i\partial_t^2 \tilde{\Phi}_{02}^{(1)} = 0 \tag{3.47}$$

and

$$\partial_z \tilde{\Phi}_{03}^{(1)R} = d_3 \partial_t^3 \tilde{\Phi}_{01}. \tag{3.48}$$

The term $\tilde{\Phi}_{03}^{(1)R} = \tilde{v}_{03} + \tilde{\Phi}_{03}^{(1)NR2}$ stands for the part of $\tilde{\Phi}_{03}^{(1)}$ that contributes to radiation emission, and $\tilde{\Phi}_{03}^{(1)NR} = \tilde{\Phi}_{03}^{(1)NR1} + \tilde{\Phi}_{03}^{(1)NR2}$ is its nonradiative part. The term $i\partial_t^2 \tilde{\Phi}_{02}^{(1)}$ in Eq. (3.47) leads only to a position shift equal to $\eta_0 \Delta y_{02}^{(0)}$, with $\Delta y_{02}^{(0)}$ given by Eq. (3.31), on top of the stationary $O(d_3)$ single-soliton solution $\tilde{\Psi}_{01}$. Therefore, the only term in Eq. (3.45) that gives a qualitatively new contribution (radiative term) is $d_3 \partial_t^3 \tilde{\Phi}_{01}$.

Using Eq. (3.21) for $\tilde{\Phi}_{01}$ one can rewrite Eq. (3.48) in the following form

$$\partial_z \tilde{\Phi}_{03}^{(1)R} = \frac{-6i\eta_0^3 \eta_\beta^2 d_3}{(1+3d_3\beta/2)\beta} \left[\frac{1}{\cosh(\tilde{h}_0)} - \frac{2}{\cosh^3(\tilde{h}_0)} \right] \times \frac{1}{\cosh^2(\tilde{h}_\beta)}. \tag{3.49}$$

Integrating Eq. (3.49) over the collision region with the initial conditions (3.42) and (3.43), one obtains

$$\tilde{\Phi}_{03}^{(1)R}(t, z_0 + \tilde{z}/|\beta|) = -iB\partial_{\tilde{h}_0}^2 \tilde{\Psi}_{00}(\tilde{h}_0) = -iB\left[\frac{1}{\cosh(\tilde{h}_0)} - \frac{2}{\cosh^3(\tilde{h}_0)} \right], \tag{3.50}$$

where the coefficient B is defined by

$$B = \frac{6\eta_0^3 \eta_\beta (1+3d_3\beta)^{1/2} d_3}{(1+3d_3\beta/2)^2 \beta|\beta|}. \tag{3.51}$$

Notice that the term $\tilde{\Phi}_{03}^{(1)R}$ is of the same form that would be generated by a variation with respect to z in the second order dispersion coefficient: $\Delta d \propto B$. It follows that the source term that gives the leading contribution to the collision-induced radiation emission can be equivalently described as a fast change in the second order dispersion coefficient occurring over the collision region. One can write Eq. (3.50) in the following form

$$\begin{pmatrix} \tilde{\Phi}_{03}^{(1)R}(t, z_0 + \tilde{z}/|\beta|) \\ \tilde{\Phi}_{03}^{(1)R*}(t, z_0 + \tilde{z}/|\beta|) \end{pmatrix} = -iB\left[\frac{1}{\cosh(\tilde{h}_0)} - \frac{2}{\cosh^3(\tilde{h}_0)} \right] \times \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.52}$$

Expanding the rhs of Eq. (3.52) in terms of the eigenfunctions of the operator $\hat{L}_{\eta_0}$, one arrives at

$$-iB\left[\frac{1}{\cosh(\tilde{h}_0)} - \frac{2}{\cosh^3(\tilde{h}_0)} \right] \begin{pmatrix} 1 \\ -1 \end{pmatrix} = iBf(\tilde{h}_0) + \begin{pmatrix} \tilde{v}_{03}(t, z_0 + \tilde{z}/|\beta|) \\ \tilde{v}_{03}^*(t, z_0 + \tilde{z}/|\beta|) \end{pmatrix}, \tag{3.53}$$

where the leading contribution to the emitted radiation at $z = z_0 + \tilde{z}/|\beta|$ is given by

$$\begin{pmatrix} \tilde{v}_{03}(t, z_0 + \tilde{z}/|\beta|) \\ \tilde{v}_{03}^*(t, z_0 + \tilde{z}/|\beta|) \end{pmatrix} = -2iB \frac{\tanh^2(\tilde{h}_0)}{\cosh(\tilde{h}_0)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = B\int_{-\infty}^{+\infty} ds [a_s(z_0 + \tilde{z}/|\beta|) f_s(\tilde{h}_0) + a_s^*(z_0 + \tilde{z}/|\beta|) \overline{f_s}(\tilde{h}_0)].$$

The expansion coefficients $a_s(z_0 + \tilde{z}/|\beta|)$ appearing in Eq. (3.54) are given by

$$a_s(z_0 + \tilde{z}/|\beta|) = \frac{-i(s+i)^2}{4\cosh(\pi s/2)}, \tag{3.54}$$

where $s = k/\eta_0$. From Eq. (3.54) it follows that $\tilde{\Phi}_{03}^{(1)NR2}(\tilde{h}_0, z_0 + \tilde{z}/|\beta|) = iB\tilde{\Psi}_{00}(\tilde{h}_0)$. This result combined with the result obtained from integration of Eq. (3.46) over the collision region gives

$$\tilde{\Phi}_{03}^{(1)NR}(\tilde{h}_0, z_0 + \tilde{z}/|\beta|) = i\delta\alpha_{03}^{(1)out} \tilde{\Psi}_{00}(\tilde{h}_0) - \frac{1}{2}(\delta\alpha_{01}^{(0)in})^2 \tilde{\Psi}_{01}(\tilde{h}_0) - \eta_0 \delta y_{02}^{(0)out} \tilde{\Psi}'_{01}(\tilde{h}_0), \tag{3.55}$$

where $\delta\alpha_{03}^{(1)out}$ is an $O(d_3/\beta^2)$ correction to the phase. From Eqs. (3.52), (3.53), and (3.55) one finds that the projections of $\tilde{\Phi}_{03}^{(1)}(\tilde{h}_0, z_0 + \tilde{z}/|\beta|)$ on the eigenmodes f_2 and f_3 are zero. This means that the soliton phase velocity and amplitude do not change at this order of the theory. The result for the soliton amplitude is consistent with the conservation law for the total energy, which requires $\eta = 1 + O(d_3^2/\beta^4)$ for both solitons. (See Ref. [11] where a similar situation was discussed.)

In the post-collision region one substitutes into Eq. (2.1) a solution of the form $\exp(i\hat{\chi}_0^{out})[\tilde{\Psi}_{00}(\tilde{h}_0)+\tilde{\Psi}_{01}(\tilde{h}_0^{out})+\tilde{v}_{03}(\tilde{h}_0)]$, where $\hat{\chi}_0^{out}$ is given by

$$\hat{\chi}_0^{out}=\alpha_0+\delta\alpha_{01}^{(0)out}+\delta\alpha_{03}^{(1)out}+\eta_0^2 z, \tag{3.56}$$

and $\tilde{h}_0^{out}=\eta_0(t-\delta y_{02}^{(0)out})$. By the same arguments given for the free propagation region before the collision, the linear PDE separates into two equations: one for $\tilde{\Phi}_{03}^{(1)NR}$, and the other for $\tilde{v}_{03}$, which is Eq. (3.41). Using the initial condition (3.55) one finds

$$\tilde{\Phi}_{03}^{(1)NR}(\tilde{h}_0, z\geqslant z_0+\tilde{z}/|\beta|) = i\,\delta\alpha_{03}^{(1)out}\tilde{\Psi}_{00}(\tilde{h}_0) - \frac{1}{2}(\delta\alpha_{01}^{(0)in})^2\tilde{\Psi}_{01}(\tilde{h}_0)-\eta_0\delta y_{02}^{(0)out}\tilde{\Psi}'_{01}(\tilde{h}_0). \tag{3.57}$$

One expresses $\tilde{v}_{03}$ via

$$\begin{pmatrix}\tilde{v}_{03}(t,z)\\ \tilde{v}_{03}^{*}(t,z)\end{pmatrix}=B\int_{-\infty}^{+\infty}ds[a_s(z)f_s(\tilde{h}_0)+a_s^{*}(z)\overline{f_s}(\tilde{h}_0)], \tag{3.58}$$

and calculates the dynamics of the coefficients $a_k(z)$. Projecting Eq. (3.41) on the eigenfunctions of $\hat{L}_{\eta_0}$ one gets

$$\partial_z a_k(z)-i(k^2+\eta_0^2)a_k(z)=0. \tag{3.59}$$

Integrating this last equation over z and changing from a_k to a_s one gets

$$a_s(z\geqslant z_0+\tilde{z}/\beta)=a_s(z_0+\tilde{z}/\beta)\exp[i\,\eta_0^2(s^2+1)(z-z_0)]. \tag{3.60}$$

Equations (3.54) and (3.60) describe the dynamics of the term $\tilde{v}_{03}$, which is the leading contribution responsible for radiation. The dependence of $|\tilde{v}_{03}|/B$ on time t for four values of z, z_0 (i.e., immediately after collision), $z=z_0+1$, $z=z_0+2$, and $z=z_0+5$, is shown in Fig. 3.

Since $\tilde{v}_{03}\sim O(d_3/\beta^2)$, the leading contribution to the radiation intensity emitted due to the collision is of order d_3^2/β^4. This contribution is given by

$$\mathcal{E}_{06}^{R}=\int_{-\infty}^{\infty}dt\{|\tilde{v}_{03}(t,z)|^2+[\text{cross terms, oscillating with } z]\}, \tag{3.61}$$

where cross terms include mixed products, such as $\tilde{v}_{04}^{(1)}\tilde{\Phi}_{01}$, $\tilde{v}_{05}^{(1)}\tilde{\Psi}_{01}$, etc., which necessarily oscillate with z. The contributions to $\mathcal{E}_{06}^{R}$ consist of a constant term originating from integration over t of $|\tilde{v}_{03}|^2$ as well as z-dependent terms originating from integration over t of both $|\tilde{v}_{03}|^2$ and the cross terms. Since the total energy of the soliton is conserved after the collision, the z dependence of the emitted radiation leads to an $O(d_3^2/\beta^4)$ z dependence of the soliton's amplitude. It is possible to show that for $z\gg z_0+1$, all z dependent contributions to $\mathcal{E}_{06}^{R}$ decay algebraically with $(z-z_0)$. Thus, far away from the collision region the only nonvanishing contribution to $\mathcal{E}_{06}^{R}$ is

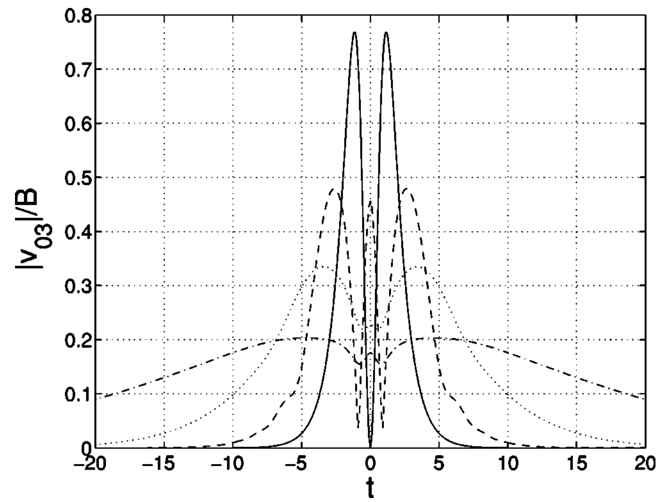

FIG. 3. Absolute value of the radiation profile function normalized to B, i.e., $|v_{03}|/B=|\tilde{v}_{03}|/B$, is shown as a function of t, for four values of z: $z=z_0$ (solid), $z=z_0+1$ (dashed), $z=z_0+2$ (dotted), and $z=z_0+5$ (dash-dotted).

$$\mathcal{E}_{06}^{R}(z\gg z_0+1)=\frac{2\pi B^2}{\eta_0}\int_{-\infty}^{\infty}ds|a_s|^2=\frac{16B^2}{15\eta_0}. \tag{3.62}$$

The details of this calculation are presented in Appendix C, which also contains an analysis of the $z\gg z_0+1$ asymptotic behavior of the z-dependent contributions originating from integration over t of all other terms in $|\tilde{v}_{03}|^2$.

For $t\gg 1$ and $(z-z_0)\gg 1$, that is far from the soliton and far from the collision region, Eq. (3.41) for $\tilde{v}_{03}$ reduces to the linear wave equation (for v_{03})

$$i\partial_z v_{03}+\partial_t^2 v_{03}=0. \tag{3.63}$$

Taking the limit $t\gg 1$ and $(z-z_0)\gg 1$ in expression (3.10) for $\tilde{v}_{03}(t,z)$, while using Eqs. (3.54) and Eq. (3.60), one derives

$$v_{03}(t,z)=\frac{-B}{4\eta_0[i(z-z_0)]^{1/2}}\left[1-\frac{it}{\eta_0(z-z_0)}\right]\times\exp\left(\frac{it^2}{4(z-z_0)}\right)+O\left(\frac{1}{(z-z_0)^{3/2}}\right). \tag{3.64}$$

It is also easy to check that the function given by Eq. (3.64) satisfies the linear wave equation (3.63).

IV. DISCUSSION AND CONCLUSIONS

We start this last section by presenting some estimations for the value and intensity of the inelastic collision effects.

We use the results (3.62) and (3.51) to obtain specific predictions for an optical fiber setup with distributed amplification compensating losses or with lumped amplification and dispersion tapered fibers. Taking $\eta_0=1$ and requiring that the widths of the colliding solitons are equal (bit rates should be the same in all the channels) one obtains $\eta_\beta=(1+3d_3\beta)^{1/2}$. Then, for the values specified in Sec. II one derives, $\mathcal{E}^R_{06}\approx 4.8\times 10^{-6}$, for the fraction of the total radiation emitted by the soliton in the reference channel relative to the total energy of the bare (ideal) soliton. In addition, neglecting the decrease in the soliton amplitude, the total energy emitted by the reference channel soliton as a result of many collisions with solitons from the β channel grows linearly with the number of collisions. Thus, for the parameters introduced above the average distance passed by the soliton until it experiences 2×10^4 collisions and loses about 10% of its energy is approximately 2500 km.

Let us continue discussion of the results. We have already seen that the source term that gives the leading contribution into the collision-induced radiation emission has the form of an effective fast change in the second order dispersion coefficient. Another interesting feature of the collision is that the leading contributions to the observed effects come from terms in the equations that involve $\tilde{\Phi}_{01}$. Thus, the leading, $O(1/\beta)$, contribution to phase shift, which is due to the term $4i|\Psi_{\beta 0}|^2\tilde{\Psi}_{00}$, is simply given by $\tilde{\Phi}_{01}$. Then, the leading, $O(1/\beta^2)$, contribution to position shift is due to the $\partial_t^2\tilde{\Phi}_{01}$ term. Finally, the leading, $O(d_3/\beta^2)$, contribution to the radiation emission is due to the $d_3\partial_t^3\tilde{\Phi}_{01}$ term that does not exist in the ideal two-soliton collision problem. Another related feature of the problem is that the $O(d_3)$ stationary single soliton solution of Eq. (2.1) $\tilde{\Psi}_{01}$ behaves like an ideal soliton in the collision. It acquires an $O(1/\beta)$ phase shift due to the $4i|\Psi_{\beta 0}|^2\tilde{\Psi}_{01}$ term and an $O(1/\beta^2)$ position shift due to the $\partial_t^2\tilde{\Phi}^{(1)}_{02}$ term, but does not give a contribution to emission of radiation up to third order of the perturbation theory. One can expect that the leading contribution to radiation from $\tilde{\Psi}_{01}$ will come only in order d_3^2/β^2 from the term $d_3\partial_t^3\tilde{\Phi}^{(1)}_{02}$.

Even though the effect of a single collision is relatively small (of third order), the accumulated effect of multiple collisions of a single soliton in the reference channel with many solitons from different frequency channels can be very important. One obvious result of multiple collisions is the accumulated loss of energy that was already discussed above. Another effect, which might be much more severe, is the radiation-induced interaction between solitons propagating in the same frequency channel, due to multiple collisions with solitons from all other channels. To study this effect, one can consider the solitons in all other channels as a pseudorandom sequence of pulses. Then propagation of solitons in a given channel is described by a perturbed NLSE, in which the perturbative term has the form of the radiation source term appearing on the rhs of Eq. (3.50) multiplied by a z-dependent function that describes the quasirandom nature of the multiple collisions. One finds that this kind of perturbative correction is identical to the one following from the equation

$$i\partial_z\Psi+[1+\xi(z)]\partial_t^2\Psi+2\Psi|\Psi|^2=0, \tag{4.1}$$

which explains pulse propagation in fibers with weak disorder $\xi(z)$ in the second order dispersion coefficient. Let us consider, for example, the effect of a pseudorandom sequence of pulses from channel β on pulses in the zero frequency channel. In this case $\xi(z)$ can be written as

$$\xi(z)=\langle\xi(z)\rangle+\mu(z), \tag{4.2}$$

where the average $\langle\xi(z)\rangle$ gives an $O(d_3/\beta)$ constant correction to the second order dispersion coefficient. The term $\mu(z)$ in the last equation is a zero mean Gaussian random function, characterized by

$$\langle\mu(z)\mu(z')\rangle=D\delta(z-z'), \tag{4.3}$$

where the disorder strength D is given by

$$D\equiv[2q(1-q)\tilde{\beta}B^2]/T. \tag{4.4}$$

In Eq. (4.4) $B=B(\beta)$ is the interchannel interaction intensity defined in Eq. (3.51), $\tilde{\beta}=(1+3d_3\beta/2)\beta$, T is the size of a slot allocated for a soliton, and q is the average number of occupied slots in channel β. We assumed here that the typical distance z_* traveled by a zero channel soliton between any two subsequent collision events, $T/(2q\tilde{\beta})$, is short, so that the δ-correlated character of the effective disorder term ξ in z is justified. To account for the effect of many channels one should modify the definition of D introducing summation over allowed (and probably equidistant) β on the rhs of Eq. (4.4). This set of observations means that results obtained in Ref. [11] for the system given by Eqs. (4.1) and (4.3) directly apply to the study of soliton propagation under multiple interactions with solitons from other channels. In particular, one should expect the emergence of long range, but zero average, radiation-mediated intrachannel interaction, leading to soliton jitter.

Let us now make some general remarks. It is important to stress that the study presented in this paper suggests a general recipe for studying fast inelastic collisions between solitons (pulses). The first step is to obtain a stationary single-soliton solution of the perturbed NLSE. For a variety of problems relevant for nonlinear fiber optics such stationary solutions exist and are stable, at least in some range of parameters. This solution is then used as an initial condition in the collision problem. Using the double perturbation theory presented here, one can understand all of the effects of collisions. Fast soliton collisions in the presence of Raman scattering is one such interesting as yet unexplored problem. (Raman scattering effects should be significant for propagation and interaction of very short pulses.) One can also apply this perturbation method to study fast non-ideal collisions of soliton-type solutions of equations other then the NLSE.

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APPENDIX A: KAUP'S PERTURBATION THEORY

In this appendix we give a summary of the theory derived by Kaup [7] for perturbations near an ideal soliton. Substituting

$$\psi=[\cosh^{-1}(t)+v]\exp(iz+i\alpha),$$

into the ideal NLSE and expanding the result over v one finds

$$i\partial_z\begin{pmatrix} v \\ v^* \end{pmatrix}+\hat{L}\begin{pmatrix} v \\ v^* \end{pmatrix}=0, \tag{A1}$$

where the operator $\hat{L}$ is

$$\hat{L}=(\partial_t^2-1)\hat{\sigma}_3+\frac{2}{\cosh^2[t]}(2\hat{\sigma}_3+i\hat{\sigma}_2), \tag{A2}$$

and the standard notations for the Pauli matrices, $\hat{\sigma}_{1,2,3}$, are used. $\hat{L}$ satisfies the following set of relations

$$\hat{\sigma}_1\hat{L}\hat{\sigma}_1=-\hat{L}^*, \quad \hat{L}^+=\hat{\sigma}_3\hat{L}\hat{\sigma}_3. \tag{A3}$$

The eigenset of the operator $\hat{L}$ solves

$$\hat{L}f=\lambda f, \tag{A4}$$

where f is an eigenfunction correspondent to the eigenvalue λ. A general solution of Eq. (A4) is

$$f_k=\exp[ikt]\left\{1-\frac{2ik\exp[-t]}{(k+i)^2\cosh[t]}\right\}\begin{pmatrix} 0 \\ 1 \end{pmatrix}+\frac{\exp[ikt]}{(k+i)^2\cosh^2[t]}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_k=k^2+1, \tag{A5}$$

where k runs from $-\infty$ to $+\infty$. According to Eq. (A3), $\overline{f_k}\equiv\hat{\sigma}_1 f_k^*$ are the other eigenfunctions of $\hat{L}$,

$$\overline{f_k}=\exp[-ikt]\left\{1+\frac{2ik\exp[-t]}{(k-i)^2\cosh[t]}\right\}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\frac{\exp[-ikt]}{(k-i)^2\cosh^2[t]}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_k=-(k^2+1). \tag{A6}$$

The eigenset of $\hat{L}$ also contains the following marginally stable modes:

$$f_0=\frac{1}{\cosh[t]}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_0=0;$$

$$f_1=\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{\tanh[t]}{\cosh[t]}, \quad \lambda_1=0. \tag{A7}$$

The existence of double poles at $k=\pm i$ means that two more functions must be added to the eigenset for completeness,

$$f_2=\frac{t}{\cosh[t]}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \hat{L}f_2=-2f_1; \tag{A8}$$

$$f_3=\frac{t\tanh[t]-1}{\cosh(t)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{L}f_3=-2f_0. \tag{A9}$$

$f_k^+\hat{\sigma}_3, \overline{f_k^+}\hat{\sigma}_3$ are the left eigenfunctions of $\hat{L}$, which satisfy

$$\int_{-\infty}^{+\infty}dt\overline{f_k^+}\hat{\sigma}_3\overline{f_q}=2\pi\delta(k-q), \tag{A10}$$

$$\int_{-\infty}^{+\infty}dt f_k^+\hat{\sigma}_3 f_q=-2\pi\delta(k-q), \tag{A11}$$

$$\int_{-\infty}^{+\infty}dt f_2^+\hat{\sigma}_3 f_1=2, \quad \int_{-\infty}^{+\infty}dt f_0^+\hat{\sigma}_3 f_3=-2. \tag{A12}$$

Let us obtain relations between infinitesimal changes in the four parameters of the soliton and the four eigenmodes of $\hat{L}$. For this purpose, consider the most general form of the single soliton solution of the ideal NLSE in the reference channel

$$\psi_s=\frac{\eta\exp(i\alpha+i\eta^2 z)}{\cosh(\tilde{x})}, \tag{A13}$$

where $\tilde{x}=\eta(t-y)$. Let us denote

$$\tilde{\psi}_s=\frac{\eta}{\cosh(\tilde{x})}, \tag{A14}$$

and calculate the infinitesimal changes $\delta\tilde{\psi}_s$ originating from infinitesimal changes in α, y, β, and η. For $\delta\alpha\ll 1$ and $\delta y\ll 1$ one obtains

$$\begin{pmatrix} \delta\tilde{\psi}_s \\ \delta\tilde{\psi}_s^* \end{pmatrix}_{\delta\alpha}=i\eta\delta\alpha f_0(\tilde{x}) \tag{A15}$$

and

$$\begin{pmatrix} \delta\tilde{\psi}_s \\ \delta\tilde{\psi}_s^* \end{pmatrix}_{\delta y}=\eta^2\delta y f_1(\tilde{x}), \tag{A16}$$

respectively. These two relations are used throughout the text to identify small changes in phase and position induced by the collision. For the sake of completeness we also give here the corresponding relations for $\delta\beta\ll 1$ and $\delta\eta\ll 1$. These relations are

$$\begin{pmatrix} \delta\tilde{\psi}_s \\ \delta\tilde{\psi}_s^* \end{pmatrix}_{\delta\beta} = i\,\delta\beta f_2(\tilde{x}) + 2\,\eta z\,\delta\beta f_1(\tilde{x}) \tag{A17}$$

and

$$\begin{pmatrix} \delta\tilde{\psi}_s \\ \delta\tilde{\psi}_s^* \end{pmatrix}_{\delta\eta} = -\delta\eta f_3(\tilde{x}) + 2i\,\eta^2 z\,\delta\eta f_0(\tilde{x}), \tag{A18}$$

respectively.

APPENDIX B: ASYMPTOTICS OF THE EXACT TWO-SOLITON SOLUTION AT $1/\beta \ll 1$

In this appendix we discuss the asymptotics of the exact two-soliton solution for the ideal NLSE. This analysis is used as the starting point for the derivation of the double perturbation method presented in Sec. III. It also serves as a benchmark for calculations performed within the framework of this double perturbation theory.

The exact two-soliton solution of the ideal NLSE is given by [15]

$$\psi_{two} = \frac{\eta_1 \exp(i\chi_1)\{[\eta_1^2 - \eta_2^2 + \beta_{12}^2]\cosh h_2 + 2i\,\eta_2\beta_{12}\sinh h_2\}}{[\eta_1^2 + \eta_2^2 + \beta_{12}^2]\cosh h_1 \cosh h_2 - 2\,\eta_1\eta_2[\cos\chi_{12} + \sinh h_1 \sinh h_2]} + \frac{\eta_2 \exp(i\chi_2)\{[\eta_2^2 - \eta_1^2 + \beta_{12}^2]\cosh h_1 - 2i\,\eta_1\beta_{12}\sinh(h_1)\}}{[\eta_1^2 + \eta_2^2 + \beta_{12}^2]\cosh h_1 \cosh h_2 - 2\,\eta_1\eta_2[\cos\chi_{12} + \sinh h_1 \sinh h_2]}, \tag{B1}$$

$$\chi_{12} \equiv \chi_1 - \chi_2, \quad \beta_{12} \equiv \beta_1 - \beta_2, \quad h_j \equiv \eta_j(t - y_j - 2\beta_j z),$$

$$\chi_j \equiv \alpha_j + \beta_j(t - y_j) + (\eta_j^2 - \beta_j^2)z, \tag{B2}$$

where $j = 1,2$ correspond to the two different frequency channels. The collision of the solitons occurs at

$$z_0 = -\frac{t_1 - t_2}{2(\beta_1 - \beta_2)}. \tag{B3}$$

Taking the limit $1/|\beta_1 - \beta_2| \ll 1$ in Eq. (B1), we obtain

$$\begin{aligned}\psi_{two} = {}& \frac{\eta_1 \exp(i\chi_1)}{\cosh(h_1)}\left[1 + \frac{2i\,\eta_2 \tanh(h_2)}{(\beta_1 - \beta_2)} - \frac{2\,\eta_2^2 - 2\,\eta_1\eta_2 \tanh(h_1)\tanh(h_2) - \eta_2^2 \cosh^{-2}(h_2)}{(\beta_1 - \beta_2)^2}\right] \\ & + \frac{\eta_2 \exp(i\chi_2)}{\cosh(h_2)}\left[1 - \frac{2i\,\eta_1 \tanh(h_1)}{(\beta_1 - \beta_2)} - \frac{2\,\eta_1^2 - 2\,\eta_1\eta_2 \tanh(h_1)\tanh(h_2) - \eta_1^2 \cosh^{-2}(h_1)}{(\beta_1 - \beta_2)^2}\right] \\ & + \frac{\eta_1^2\eta_2 \exp[i(2\chi_1 - \chi_2)]}{(\beta_1 - \beta_2)^2 \cosh^2(h_1)\cosh(h_2)} + \frac{\eta_1\eta_2^2 \exp[i(2\chi_2 - \chi_1)]}{(\beta_1 - \beta_2)^2 \cosh(h_1)\cosh^2(h_2)} + O\left(\frac{1}{|\beta_1 - \beta_2|^3}\right).\end{aligned} \tag{B4}$$

Notice that the last two terms on the rhs of Eq. (B4) correspond to oscillations in channels $2\beta_1 - \beta_2$ and $2\beta_2 - \beta_1$, respectively. Thus, the two-soliton solution ψ_{two} can be written in the form

$$\psi_{two} = \psi_{\beta_1} + \psi_{\beta_2} + \phi_{\beta_1} + \phi_{\beta_2} + \phi_{2\beta_1 - \beta_2} + \phi_{2\beta_2 - \beta_1} + O(1/|\beta_1 - \beta_2|^3), \tag{B5}$$

where ψ_{β_1} and ψ_{β_2} are the $O(1)$ single-soliton solutions of the ideal NLSE in channels β_1 and β_2, respectively, which are given by Eq. (2.2). The terms ϕ_{β_1} and ϕ_{β_2} are corrections of leading order $1/|\beta_1 - \beta_2|$ to these single-soliton solutions in channels β_1 and β_2, respectively. The terms $\phi_{2\beta_1 - \beta_2}$ and $\phi_{2\beta_2 - \beta_1}$ are corrections of order $1/|\beta_1 - \beta_2|^2$ in channels $2\beta_1 - \beta_2$ and $2\beta_2 - \beta_1$, respectively. Provided $|\beta_1 - \beta_2| \gg 1$, expression (B4) for ψ_{two} is valid for any value of z, including the collision region.

Let us consider the asymptotic behavior of solution (B4) far away from the collision point, that is at $|z - z_0| \gg 1/|\beta_1 - \beta_2|$. In the region $z \ll z_0 - 1/|\beta_1 - \beta_2|$, which corresponds to the situation before the collision, expression (B4) takes the following form:

$$\begin{aligned}\psi_{two} \simeq {}& \frac{\eta_1 \exp(i\chi_1)}{\cosh(h_1)}\left[1 \mp \frac{2i\,\eta_2}{(\beta_1 - \beta_2)} - \frac{2\,\eta_2^2 \pm 2\,\eta_1\eta_2 \tanh(h_1)}{(\beta_1 - \beta_2)^2}\right] + \frac{\eta_2 \exp(i\chi_2)}{\cosh(h_2)} \\ & \times\left[1 \mp \frac{2i\,\eta_1}{(\beta_1 - \beta_2)} - \frac{2\,\eta_1^2 \mp 2\,\eta_1\eta_2 \tanh(h_2)}{(\beta_1 - \beta_2)^2}\right], \\ & z \ll z_0 - 1|\beta_1 - \beta_2|,\end{aligned} \tag{B6}$$

where the upper and lower plus/minus signs correspond to $\beta_1-\beta_2>0$ and $\beta_1-\beta_2<0$, respectively. In the same manner, in the region $z \gg z_0+1/|\beta_1-\beta_2|$, i.e., after the collision, one obtains

$$\psi_{two} \simeq \frac{\eta_1 \exp(i\chi_1)}{\cosh(h_1)}\left[1 \pm \frac{2i\eta_2}{(\beta_1-\beta_2)} - \frac{2\eta_2^2 \mp 2\eta_1\eta_2 \tanh(h_1)}{(\beta_1-\beta_2)^2}\right] + \frac{\eta_2 \exp(i\chi_2)}{\cosh(h_2)} \times \left[1 \pm \frac{2i\eta_1}{(\beta_1-\beta_2)} - \frac{2\eta_1^2 \pm 2\eta_1\eta_2 \tanh(h_2)}{(\beta_1-\beta_2)^2}\right], \quad z \gg z_0+1/|\beta_1-\beta_2|. \tag{B7}$$

An immediate consequence of Eqs. (B6) and (B7) is that outside the collision region ψ_{two} can be written as a sum of two single-soliton solutions. In fact, using Eq. (B1) one can show that the last statement is valid for any value of $|\beta_1-\beta_2|>0$. Moreover, comparing these last two expressions while using Eqs. (A15) and (A16), one finds that the only effects of a collision between two ideal solitons up to order $1/|\beta_1-\beta_2|^2$ is an $O(1/|\beta_1-\beta_2|)$ phase shift given by

$$\Delta\alpha_1 = \frac{4\eta_2}{|\beta_1-\beta_2|}, \quad \Delta\alpha_2 = \frac{4\eta_1}{|\beta_1-\beta_2|}, \tag{B8}$$

and an $O(1/|\beta_1-\beta_2|^2)$ position shift given by

$$\Delta y_1 = \pm\frac{4\eta_2}{(\beta_1-\beta_2)^2}, \quad \Delta y_2 = \mp\frac{4\eta_1}{(\beta_1-\beta_2)^2}. \tag{B9}$$

Notice that the phase shift does not depend on the sign of $(\beta_1-\beta_2)$, while the position shift does. Similar analysis can be carried out for Eq. (2.1) with $d_3 \neq 0$, but taking into account only terms of order 1 (with respect to d_3) in the exact two-soliton solution. The expressions for the phase shift and the position shift obtained in this manner coincide with the ones given by Eqs. (3.20) and (3.31).

APPENDIX C: TOTAL RADIATION EMITTED

We present here a detailed calculation of the leading order contribution $\mathcal{E}_{06}^R$ to the radiation emitted as the result of collision. As explained in Sec. III D the only nonvanishing contribution to $\mathcal{E}_{06}^R$ comes from the integral

$$I_R = \int_{-\infty}^{\infty} dt |\tilde{v}_{03}(t,z)|^2. \tag{C1}$$

Using Eq. (3.58) one finds

$$I_R = \frac{B^2}{\eta_0}\int_{-\infty}^{\infty} d\tilde{h}_0 \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dp [a_s a_p f_{s1} f_{p2} + a_s a_p^* f_{s1} \overline{f}_{p2} + a_s^* a_p \overline{f}_{p1} f_{s2} + a_s^* a_p^* \overline{f}_{s1} \overline{f}_{p2}]. \tag{C2}$$

Subsequent integration over $\tilde{h}_0$ results in

$$I_R = \frac{2\pi B^2}{\eta_0}\int_{-\infty}^{\infty} ds |a_s|^2 + \mathcal{F} + \mathcal{G}, \tag{C3}$$

where

$$\mathcal{F} = \frac{\pi B^2}{48\eta_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds dp \exp[i\eta_0^2(s^2-p^2)(z-z_0)] \times \frac{(s-p)[(s-p)^2+4]}{\cosh(\pi s/2)\cosh(\pi p/2)\sinh[\pi(s-p)/2]} \tag{C4}$$

and

$$\mathcal{G} = -\frac{\pi B^2}{48\eta_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds dp \cos[\eta_0^2(s^2+p^2+2)(z-z_0)] \times \frac{(s+p)(s^2+p^2-4sp-2)}{\cosh(\pi s/2)\cosh(\pi p/2)\sinh[\pi(s+p)/2]}. \tag{C5}$$

Consider $\mathcal{F}$ in the limit $z \gg z_0+1$. Since in this limit the exponential factor in the integrand on the rhs of Eq. (C4) is rapidly oscillating (both in s and p), the major contribution to this term comes from the region $|s| \ll 1$ *and* $|p| \ll 1$. Indeed, when $s \sim p$ (or $s \sim -p$) and neither s nor p are small this integral is exponentially suppressed because of the $1/\cosh(\pi s/2)$ and $1/\cosh(\pi p/2)$ factors. Then, expanding all the terms in the integrand, except those with oscillating exponent, around $(s,p)=(0,0)$, one finds that for $z \gg z_0+1$ the term $\mathcal{F}$ decays like $1/(z-z_0)$. Similarly, one can show that in the limit $z \gg z_0+1$ the term $\mathcal{G}$ is oscillating, and that the amplitude of the oscillations is bounded by an envelope decaying like $1/(z-z_0)$. Hence, the only nonvanishing contribution to I_R, and thus to $\mathcal{E}_{06}^R$, at $z \gg z_0+1$ is given by

$$\mathcal{E}_{06}^R(z \gg z_0+1) = \frac{2\pi B^2}{\eta_0}\int_{-\infty}^{\infty} ds |a_s|^2 = \frac{16B^2}{15\eta_0}, \tag{C6}$$

which is exactly Eq. (3.62). In a similar manner one can show that the oscillating cross terms appearing in the integrand on the rhs of Eq. (3.61) vanish for $z \gg z_0+1$.

[1] G.P. Agrawal, *Fiber-Optic Communication Systems* (Wiley, New York, 1997).

[2] G.P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, San Diego, 1995).

[3] A. Hasegawa and Y. Kodama, *Solitons in Optical Communications* (Clarendon Press, Oxford, 1995).

[4] V.I. Karpman and V.V. Solov'ev, Physica D **3**, 487 (1981).

[5] J.P. Gordon, Opt. Lett. **8**, 596 (1983).

[6] D. Anderson and M. Lisak, Opt. Lett. **11**, 174 (1986).
[7] D.J. Kaup, Phys. Rev. A **42**, 5689 (1990).
[8] A. Peleg and Y. Chung (unpublished).
[9] Y. Kodama, Phys. Lett. **107A**, 245 (1985).
[10] J.N. Elgin, Opt. Lett. **17**, 1409 (1992); Phys. Rev. A **47**, 4331 (1993); J.N. Elgin, T. Brabec, and S.M.J. Kelly, Opt. Commun. **114**, 321 (1995).
[11] M. Chertkov, Y. Chung, A. Dyachenko, I. Gabitov, I. Kolokolov, and V. Lebedev, Phys. Rev. E **67**, 036615 (2003).
[12] F.G. Omenetto, Y. Chung, D. Yarotski, T. Schaefer, I. Gabitov, and A.J. Taylor, Opt. Commun. **208**, 191 (2002).
[13] L.F. Mollenauer and P.V. Mamyshev, IEEE J. Quantum Electron. **34**, 2089 (1998).
[14] L.F. Mollenauer, J.P. Gordon, and P.V. Mamyshev, in *Optical Fiber Tellecomunications III*, edited by I.P. Kaminov and T.L. Koch (Academic Press, San Diego, 1997), Chap. 12, Sec. V B.
[15] V.E. Zakharov and A.B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].