Importance Sampling of Gordon–Mollenauer Soliton Phase Noise in Optical Fibers

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***Abstract*—We develop an importance sampling method to perform the direct numerical computation of the probability density function of the random optical soliton phase under the influence of both amplifier spontaneous emission noise and nonlinear conversion of amplitude to phase fluctuations owing to the Kerr effect, or nonlinear phase noise.**

***Index Terms*—Differential phase-shift keying (DPSK), nonlinear optics, optical fiber communication, optical solitons.**

AT PRESENT, there is a great interest in exploiting advanced modulation formats such as differential phase-shift keying (DPSK) for improving the performance of long-haul fiber-optics transmission systems [1]. As is well known, in DPSK transmissions, the information is coded in the phase difference and not in the amplitude modulation of the signal. In particular, the format return-to-zero (RZ)-DPSK offers the possibility of minimizing detrimental nonlinear effects resulting from intensity variation in the signal, while keeping the high signal-to-noise margins of the RZ modulation [2]. In a long distance transmission using a PSK format, it is essential to evaluate the statistical properties of the signal phase evolution when a signal pulse propagates under the action of amplified spontaneous emission (ASE) noise and fiber nonlinearity, which result in a nonlinear or Gordon–Mollenauer phase noise [1]–[3].

In order to arrive at a theoretical prediction of the system bit-error rate (BER) for systems employing DPSK, it would be necessary to evaluate the full probability density function (pdf) of the signal phase difference between adjacent bits. [4]–[6]. In the presence of nonlinearity, this distribution cannot be approximated by a Gaussian in the low BER ($\leq 10^{-9}$) region of interest for applications. This also means that the familiar quality (or Q) factor, as it is often estimated from a relatively small number of numerical simulations, has in general no direct link with the BER in the case of DPSK transmission systems. On the other hand, the exhaustive numerical simulation of the rare events leading to such small BERs from the statistics of the phase of signal pulses would imply the simulation of the propagation of hundreds of billions of different pulses, which is not a practical strategy even with present day advances in computing power.

Indeed, several authors have recently revealed the inaccuracy of the Gaussian approximation for the soliton phase (sp)-pdf and obtained its approximate analytical expression [4]–[6]. The theoretical sp-pdf exhibits an asymmetry with respect to the mean value that appears to be in qualitative agreement with the experiments [1], [4]. One procedure to fill the gap between theoretical methodologies and full numerical simulations is to apply biased Monte Carlo (MC) algorithms to evoke rare events in a controlled way. One possible solution is the multicanonical method (MMC) [7] that has been applied for DPSK in [8]. MMC is a powerful iterative method with the advantage of not requiring any priori knowledge of the rare events responsible for the shape of pdf tails. Another technique, falling in the category of biased MC methods, is known as importance sampling (IS) [9]. The main advantage of the IS method is that sampling of the rare events from an importance distribution is based on physical intuition (rather than a mechanical search as in MMC). Hence, MMC and IS provide two complementary techniques to sample the rare statistical events. The IS algorithm was earlier adapted to the statistical analysis of the amplitude and timing jitter of optical solitons in the presence of ASE noise and PMD [9], [10]. The basic idea of the IS method is to evoke the rare events that lead to errors and subsequently weigh such events by means of appropriate likelihood ratios (see e.g., [10]). The numerically generated amplifier noise is the practical vehicle to guide such biased statistics. Recently, the phase statistics of optical soliton pulses was analyzed by means of a numerical method based on both IS and soliton perturbation theory of the inverse scattering transform (see [11]). In that paper, the practical consequences on non-Gaussian statistics of the soliton phase on the BER of DPSK systems were discussed.

In this work, we shall develop a different IS-based numerical technique that is based on the so-called root-mean-square method [12]. The important merit of our method is that it can be applied to all practical cases of present installed fiber-optics communication systems where nonsoliton modulation formats (e.g., RZ or dispersion-managed soliton) are employed. Moreover, in this work, we analyze the role of signal-to-noise ratio and nonlinear phase shift on the symmetry of the pulse phase pdf, in excellent agreement with the predictions of the analytical theory [8], [9].

Our IS algorithm is based on the calculation of a signal-dependent special function that biases the amplifier noise. We modify the mean value of the Gaussian statistical process

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in order to excite rare and large deviations in the soliton phase. Let us define the root-mean-square soliton phase as $\Phi = tan^{-1}\{iM/N\}$ with

$$M = \int_{-\infty}^{+\infty} q|q|^2 dt - c.c.; \quad N = \int_{-\infty}^{+\infty} q|q|^2 dt + c.c. \quad (1)$$

Whenever one adds a complex noise field component Δn to the soliton pulse q so that $q' = q + \Delta n$, the phase associated with q' reads as $\phi' = \phi + \Delta\phi$, where

$$\Delta\phi = i\frac{(N\Delta M - M\Delta N)}{M^2 - N^2} \equiv R \quad (2)$$

$\Delta M = \langle A|\Delta n^*\rangle + \langle B|\Delta n\rangle - c.c, \Delta N = \langle A|\Delta n^*\rangle + \langle B|\Delta n\rangle + c.c$, and $A = 2|q|^2$, $B = q^2$, whereas the scalar product is defined as $\langle f|g\rangle = \int_{-\infty}^{+\infty} f(t)g(t)^* dt$.

Equation (1) directly links the noise perturbation to the corresponding perturbation of the soliton root-mean-square phase. For a given amount of phase shift $\Delta\phi$, the most likely noise is the one of least energy: such constrained minimization problem is solved in terms of the auxiliary function L and the Lagrange multiplier λ, where

$$L = \int_{-\infty}^{+\infty} |\Delta n|^2 dt + \lambda(R - \Delta\phi). \quad (3)$$

By computing the relevant functional derivatives, one obtains the following relation between the most likely biasing ASE noise $\Delta n(t)$ at a given amplifier site and the corresponding soliton phase shift $\Delta\phi(q(t))$

$$\Delta n = \Delta\phi \frac{\{N(B-A) - M(B+A)\}}{(N-M)H_1 - (N+M)H_2} \quad (4)$$

with $H_1 = \langle A|g^*\rangle + \langle B|g\rangle$, $H_2 = \langle A|g\rangle + \langle B^*|g^*\rangle$, $g = i\{N(B-A) - M(A+B)\}/(M^2 - N^2)$, and $\Delta n = -\lambda g$ as solution of (3).

As a result, the total ASE noise at a given amplifier site reads as $y(t) = x(t) + \Delta n(t)$, where $x(y)$ is a complex zero-average (biased) Gaussian uncorrelated random process and $\Delta n(t)$ is the deterministic function given by (4). Finally, the likelihood factor associated with such a biased ASE noise sequence is given by the ratio $R_L = P(x + \Delta n)/P(x)$ [9], [10]. In the present numerical simulations, we shall assume for simplicity that the overall phase offset $\Delta\phi_T$ at the end of a link of N_A amplifiers is divided into equal contributions at each amplifier of index k, of amplitude $\Delta\phi_k = \Delta\phi_T/N_A$. For long distances, however, this assumption may be incorrect (see [10] for the computation of amplitude and timing jitter of solitons): the relative contribution of each amplifier span to the overall phase shift will then be related to the specific properties (i.e., power, time width) of the transmitted pulse. According to the discussion of [4], by using a preliminary noiseless simulation, one can use the propagated signal at a given amplifier site for numerically evaluating the relative contribution of that span to the soliton phase shift.

To test our algorithm, we considered a soliton transmission system with constant fiber dispersion of -0.2 ps^2/km and with a self-phase modulation coefficient of 2 W^{-1}km^{-1}. Under these conditions, one can transmit a first-order path-average soliton with a peak power of 1 mW. With an amplifier spacing of 50 km and distributed fiber losses of 0.2 dB/km, the input pulse should be enhanced by a factor of 1.59 (see [2]) to keep the path average soliton peak power equal to 1 mW. The analytical approach of Mecozzi [4], which is valid in the limit of a very large number of amplifiers, points out the key parameters that control the behavior of the soliton phase statistics, that is the optical signal-to-noise ratio (OSNR) ρ and the nonlinear phase shift $\phi_{\rm NL}$ (see [4]).

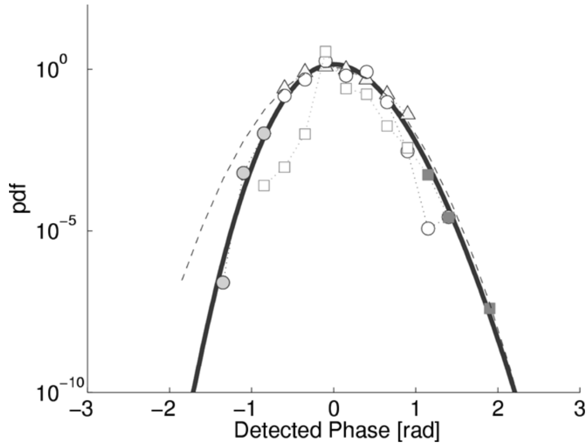

Fig. 1. Filled marks: numerical sp-pdf at 3000 km versus analytical theory (solid curve). Unbiased samples (triangles), biased samples (circles and squares). Empty marks denote unreliable samples. Thin dashed curve: Gaussian approximation. Amplifier noise figure is 5 dB and OSNR is 19.5 dB.

In order to validate our numerical method, let us compare our results against the analytical predictions of [4] for different sets of ρ and $\phi_{\rm NL}$. As we shall see, we could confirm that the nonlinear interaction of soliton and noise causes an asymmetrical deviation of the sp-pdf from a reference Gaussian distribution: such a deviation is a function of ρ and $\phi_{\rm NL}$. Indeed, for a fixed $\phi_{\rm NL}$, the symmetry of the sp-pdf increases (close to its Gaussian approximation) as the OSNR ρ grows larger. On the other hand, for a fixed level of OSNR, the shape asymmetry of the sp-pdf is more pronounced as $\phi_{\rm NL}$ grows larger, that is for longer propagation distances.

We first numerically studied the soliton phase histogram by using unbiased ASE noise samples at each amplifier site (not shown here). We verified that the standard deviation of the observed phase noise (0.31 rads after 3000 km) was in agreement with the predictions of the soliton perturbation theory [2].

In order to display the tails of the sp-pdf, it is necessary to use a log scale: in Fig. 1, we show the sp-pdf for a transmission distance of 3000 km and 5 dB of amplifier noise figure. The thin dashed curve is a fitting Gaussian function with the same mean value and variance as the unbiased numerical results. In Fig. 1, we use different symbols for unbiased and biased levels: triangles for the nonbiased set of simulations, filled circles and filled squares for two sets of biased simulations. Empty circles and squares show the unreliable part of IS results (see also [9] and [10]). Choosing the bias levels was the result of a trial and error search for a tradeoff between numerical accuracy and computational efficiency. For a fixed bias level, the results are obtained

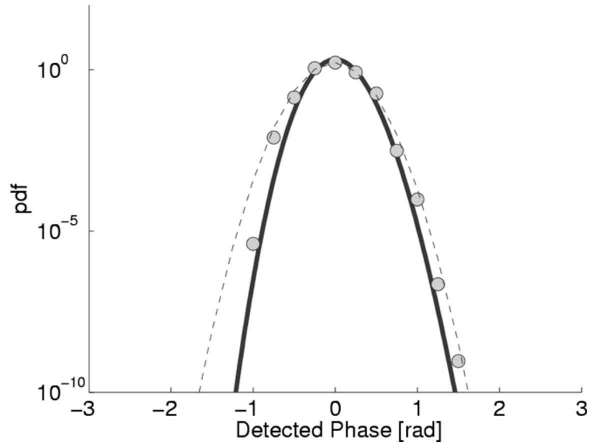

Fig. 2. Circles: numerical sp-pdf at 3000 km versus analytical theory (solid curve). Thin dashed curve: Gaussian approximation. Amplifier noise figure is 2 dB, OSNR is 22.5 dB.

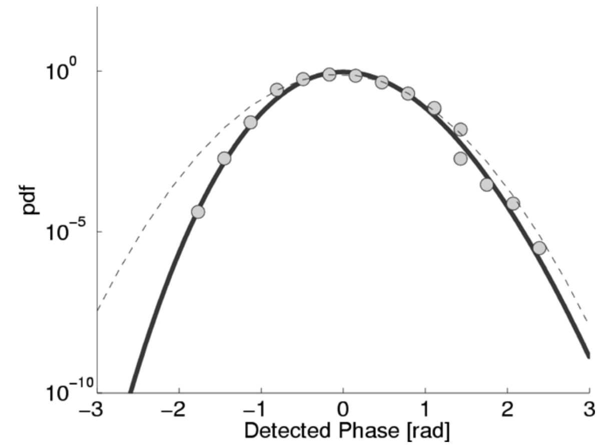

Fig. 3. Same as in Fig. 2, with 5-dB amplifier noise figure and 23.4-dB OSNR.

by classifying the phase occurrences of 500 000 runs and with a frequency sampling of 1024 points of fast Fourier transform (no multiple IS is used). The thick solid line was obtained by using the analytical sp-pdf of [4] (with $\rho = 19.5$ dB). The remarkable agreement between the numerical and the analytical pdfs clearly shows that the soliton phase statistics is not Gaussian. Indeed, the sp-pdf exhibits an asymmetry with respect to its mean value.

The analytical theory further predicts that the asymmetry of the sp-pdf should be reduced if one increases the OSNR. In order to verify this, we performed a simulation using lumped amplifiers with a "virtual" noise figure reduced down to 2 dB.[1] This increases the OSNR ρ by 3 dB while keeping $\phi_{\rm NL}$ a constant. Our results are shown in Fig. 2: as can be seen, in this case, the asymmetry of the sp-pdf is reduced. At the same time, of course, the sp-pdf variance is also less pronounced as it is proportional to the amplifier's noise figure. Note that if one increases the OSNR by using a higher soliton peak power instead of a lower amplifier noise, the asymmetry may be still significant, owing to the comparatively larger value of $\phi_{\rm NL}$. This point is illustrated by Fig. 3, where we increased the soliton peak power to 2.5 mW by means of changing the fiber dispersion to -0.5 ps^2/nm, while keeping a 5-dB noise figure.

In summary, we developed a numerical method in the class of the IS algorithms to perform a direct computation of the random optical soliton phase statistical distribution under the action of Gordon–Mollenauer or nonlinear phase noise. We confirmed the validity of our method by comparing its results with an analytic approximation, and we have shown that the sp-pdf departs from a simple Gaussian as the propagation distance grows larger (for a fixed level of OSNR). We envisage that the computational method developed here will be an important tool to evaluate the performance of fiber-optics telecommunication systems whenever they are limited by nonlinear phase noise, in situations where no analytical theory is available such as for dispersion and nonlinearity (e.g., with distributed Raman amplification) managed transmissions.

[1]Although for lumped optical amplifiers the quantum limit to their noise figure is 3 dB, in practice when adopting a distributed amplification scheme such as Raman amplification, one may represent the situation though an equivalent lumped amplifier with a noise figure below 3 dB.

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