INVERTED PENDULUM

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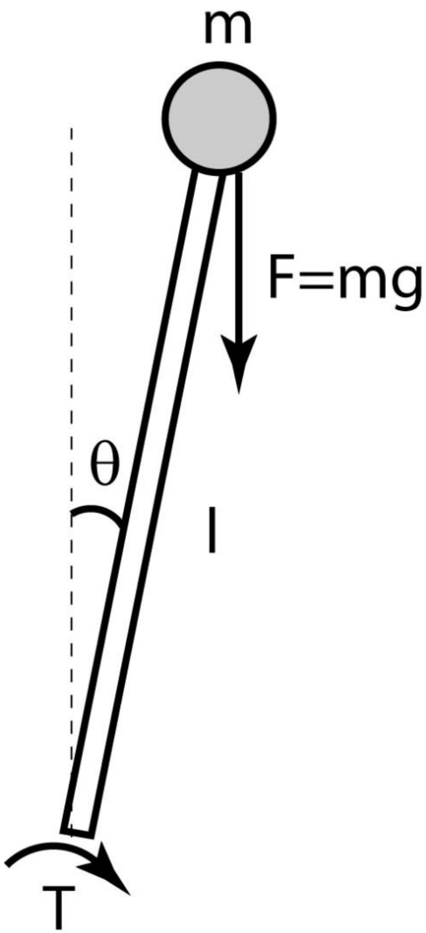
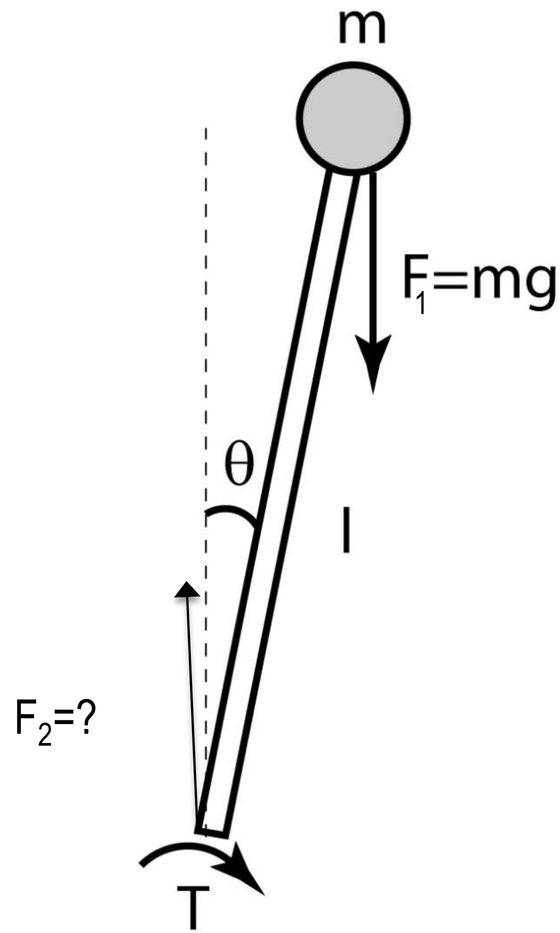
What does it take to oscillate a pendulum upside down?

INVERTED PENDULUM

NatSciDemos on Youtube

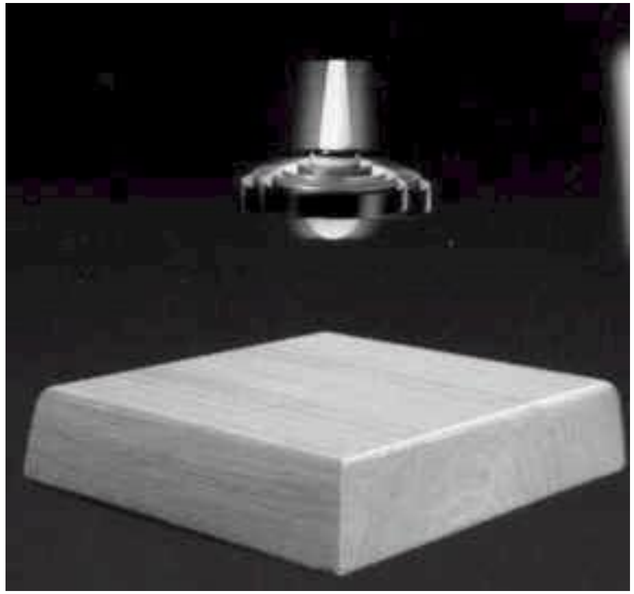
WHAT'S HAPPENING?





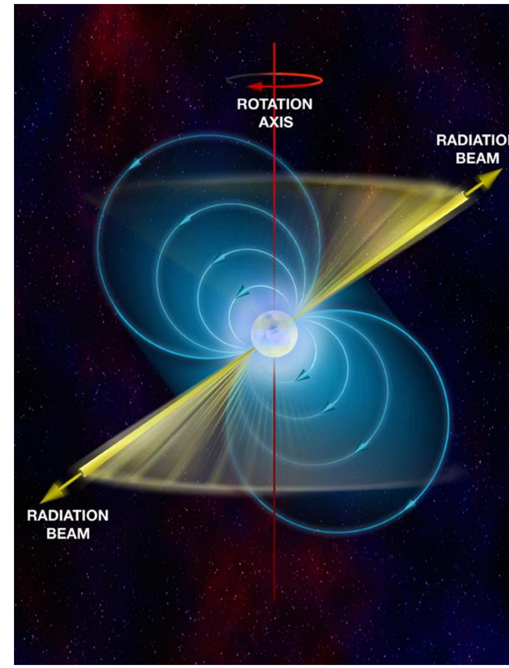
$F_1 - F_2 = 0$

Segway



Magnetic Levitron

Rotating Pulsar

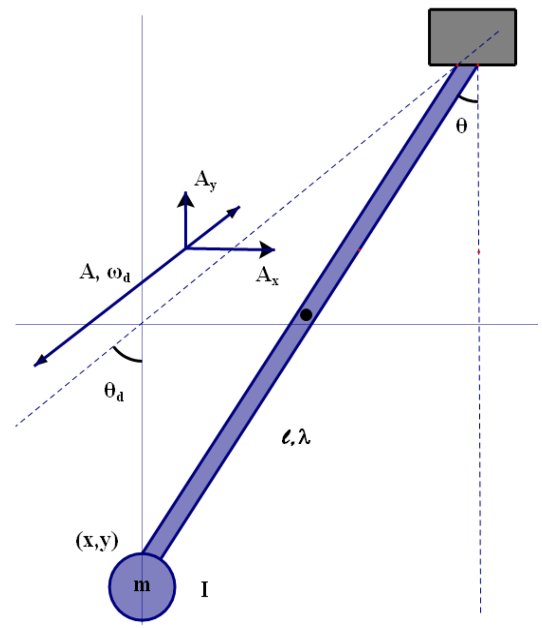


Applications

THE EQUATION OF MOTION

Position and Velocity

- The x and y coordinates:
  - $x = l\sin\theta + \ A_x \sin(\omega_d t)$
  - $y = lcos\theta + A_y cos(\omega_d t)$



- The velocity
  - $v_x = \dot{x} = l\dot{\theta}\cos\theta + A_x\omega_d \cos(\omega_d t)$
  - $v_y = \dot{y} = -l\dot{\theta}\sin\theta - A_y\omega_\mathrm{d}\sin(\omega_d t)$

THE EQUATION OF MOTION

Kinetic and Potential Energy

- Kinetic Energy

$$T = \underbrace{\frac{1}{2} I \dot{\theta}^2}_{\text{Rotational Kinetic Energy}} + \underbrace{\frac{1}{2} m\left(l^2 \dot{\theta}^2 \cos^2\theta + A_x^2 \omega_d^2 \cos^2(\omega_d t) + 2A_x \omega_d \cos(\omega_d t)\cos\theta\right)}_{\text{Translational Kinetic Energy for x}}$$

$$+ \underbrace{\frac{1}{2} m\left(l^2 \dot{\theta}^2 \sin^2\theta + A_y^2 \omega_d^2 \sin^2(\omega_d t) + 2A_y \omega_d \sin(\omega_d t)\sin\theta\right)}_{\text{Translational Kinetic Energy for y}}$$

- Potential Energy

$$V = mgy = mg(l\cos\theta + A_y\cos(\omega_d t))$$

THE EQUATION OF MOTION

Euler-Lagrange Equations

- The Lagrangian is $\mathcal{L} = \mathrm{T} - \mathrm{V}$

- The equation of motion is obtained from the Euler-Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\ddot{\theta} + \frac{g}{\lambda} \sin\theta + \frac{A\omega_d^2}{\lambda}\left(\cos(\theta_d)\cos(\omega_d t)\sin(\theta) - \sin(\theta_d)\sin(\omega_d t)\cos(\theta)\right) = 0$$

THE EQUATION OF MOTION

The Dimensionless Form

- Introduce non-dimensional time τ and dimensionless parameters γ and α where:

$$\tau = \omega_d t \qquad \gamma = \frac{\omega_o^2}{\omega_d^2} \qquad \alpha = \frac{A}{\lambda}$$

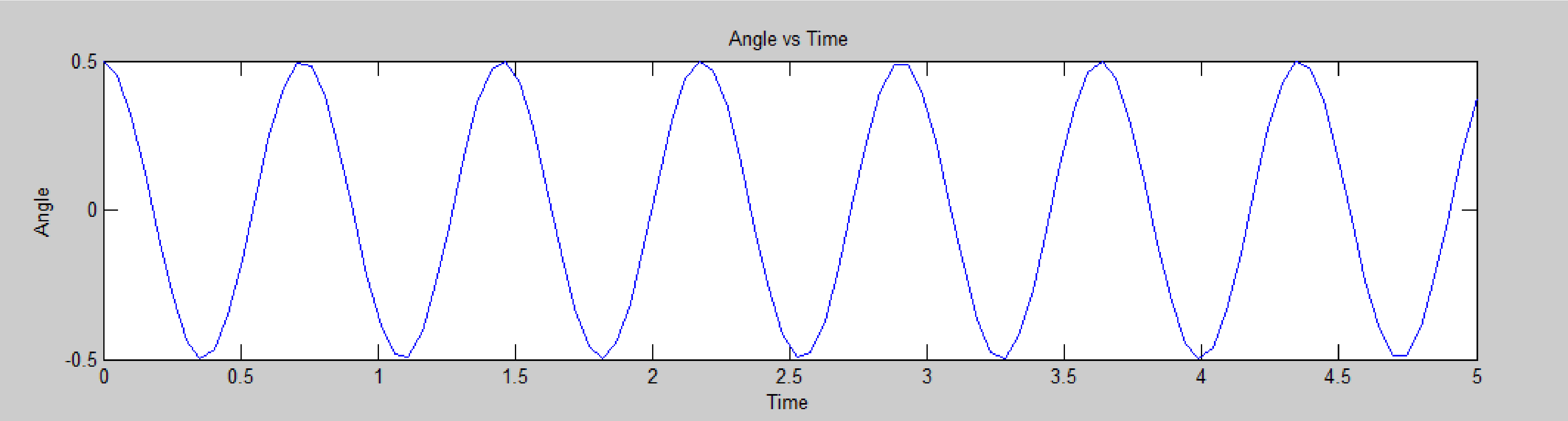
$$D(\theta, \tau) = \cos\theta_d \sin\theta \cos\tau - \sin\theta_d \cos\theta \sin\tau$$

- The dimensionless equation of motion:

$$\boldsymbol{\frac{d^2\theta}{d\tau^2} + \gamma \sin\theta + \alpha D(\theta, \tau) = 0}$$

NUMERICAL ANALYSIS

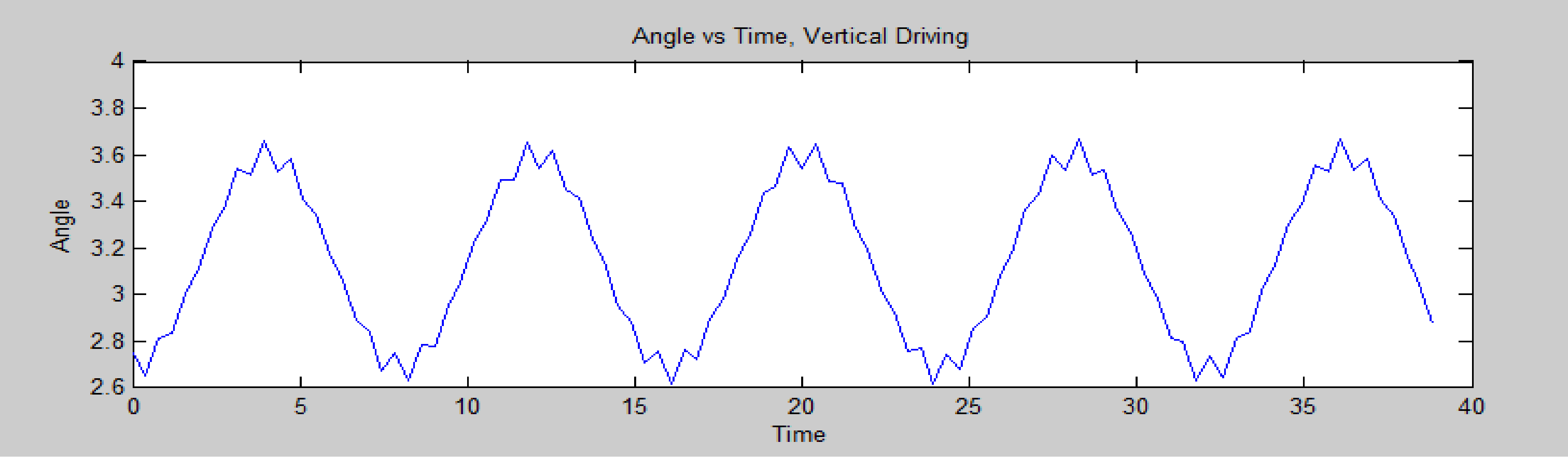
GENERIC PENDULUM



Initial angle: 0.5 Radians

Stability about: 0 Radians (Straight down)

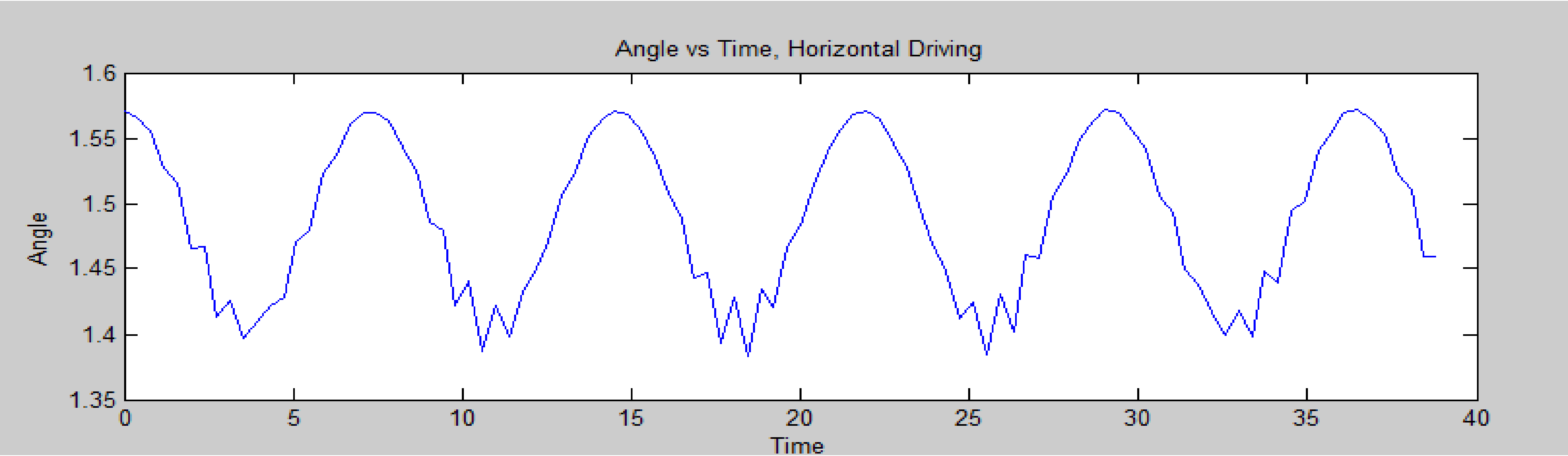
NUMERICAL ANALYSIS
VERTICALLY DRIVEN JIGSAW



Initial angle: (7/8)π (Slightly left of vertical)
Drive angle: 0 (Vertical drive)
Stability about: π (Straight up)

NUMERICAL ANALYSIS

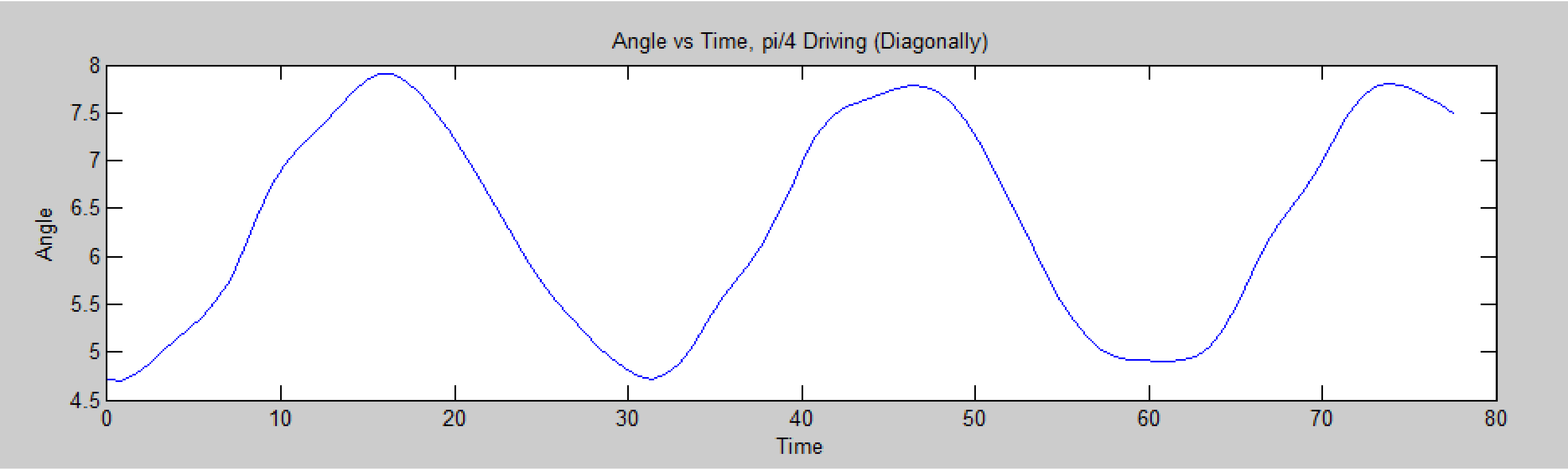
HORIZONTALLY DRIVEN JIGSAW



Initial angle: (1/2)π (Horizontally left)
Drive angle: -(1/2)π (Driven from the right)
Stability about: Approx. 1.49 Radians (An angle below the left horizontal)

NUMERICAL ANALYSIS

DIAGONALLY DRIVEN JIGSAW



Initial angle: -(1/2)π (Horizontally right)
Drive angle: -(1/4)π (Driven from the bottom right)
Stability about: 2π (straight down, with odd behavior)

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

CONCEPT

- Separate the *slow* and *fast* components

SLOW		*FAST*	
force of gravity	$F(\theta)$	driving force	$f(\theta,t)$
pendulum's rotation	$\Phi(t)$	rapid driving oscillations	$\xi(t)$

- Assume the fast components have:
  - ***MUCH higher frequency*** *and*
  - ***relatively low amplitude***

- The *effective potential* U_{eff} is the average potential energy associated with $\Phi(t)$

Fascinating. Let's apply it to our pendulum …

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

FINDING U_{EFF}

1. Relate the fast and slow components:

$$I_o\ddot{\theta} = I_o(\ddot{\phi} + \ddot{\xi}) = \big(F(\theta) + f(\theta, t)\big)$$

2. First-order approximation discarding insignificant terms:

$$I_o(\ddot{\phi} + \ddot{\xi}) \cong F(\phi) + \frac{dF}{d\theta}(\phi)\xi + f(\phi, t) + \frac{df}{d\theta}(\phi, t)\xi \longrightarrow I_o\ddot{\xi} \cong f(\phi, t)$$

3. Substitute back into and time-average the equation from 1.:

$$I_o\ddot{\phi} \cong F(\phi) + \langle\frac{df}{d\theta}(\phi, t)\xi\rangle$$

4. Use the definition of the effective potential:

$$\ddot{\phi} \cong -\frac{1}{I_o}\frac{dU_{eff}}{d\phi}$$

5. To solve for the effective potential: $U_{eff} = U_o - \int \left(\langle\frac{df}{d\theta}(\phi, t)\xi\rangle\right) d\phi$

ZZZZZZZZZzzzzzzzzzz ....

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

FINDING U_{EFF} (CONT.)

6. Write the effective potential in the terms from our governing equation:

$$F(\theta) = -I_o\gamma\sin\theta \qquad\qquad f(\theta,t) = -I_o\alpha\, D(\theta,t)$$

$$U_o = -\int F(\theta)d\theta = I_o\gamma\cos\theta$$

$$U_{eff} = U_o + \frac{\langle f^2\rangle}{2I_o}$$

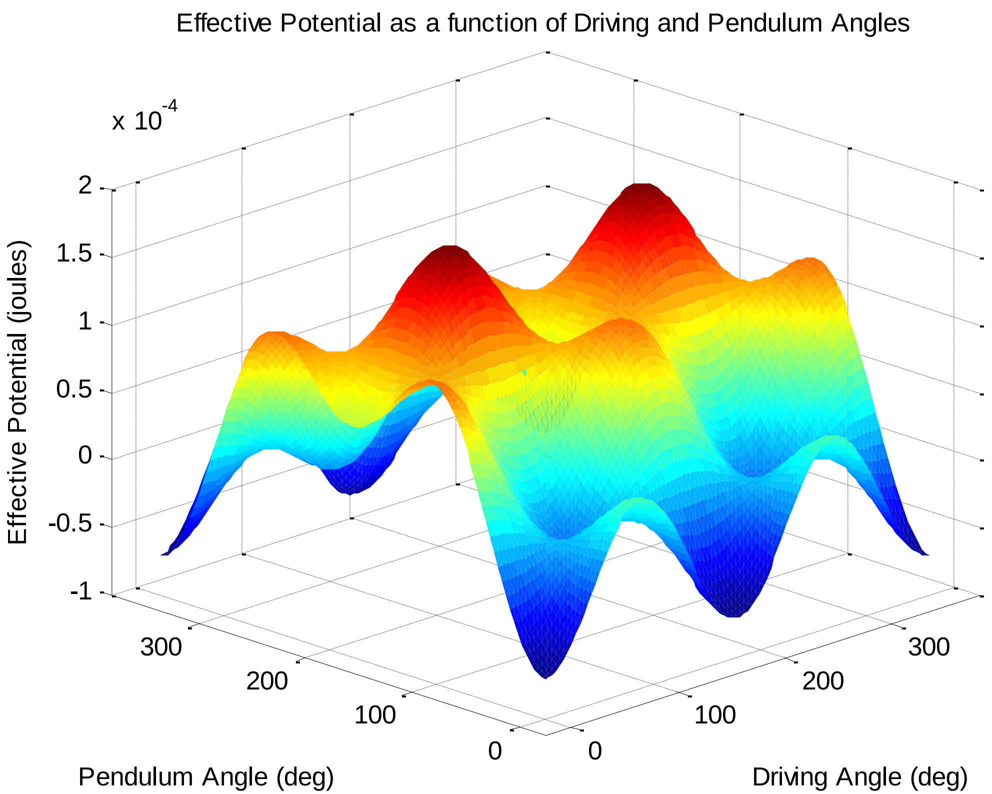
7. Finally, simplify to get:

$$\boldsymbol{U_{eff} = I_o\left(-\gamma\cos\theta + \frac{\alpha^2}{4}\left(\cos^2\theta_d\sin^2\theta + \sin^2\theta_d\cos^2\theta\right)\right)}$$

Huh … What in the world does this mean …

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

STABILITY ANALYSIS



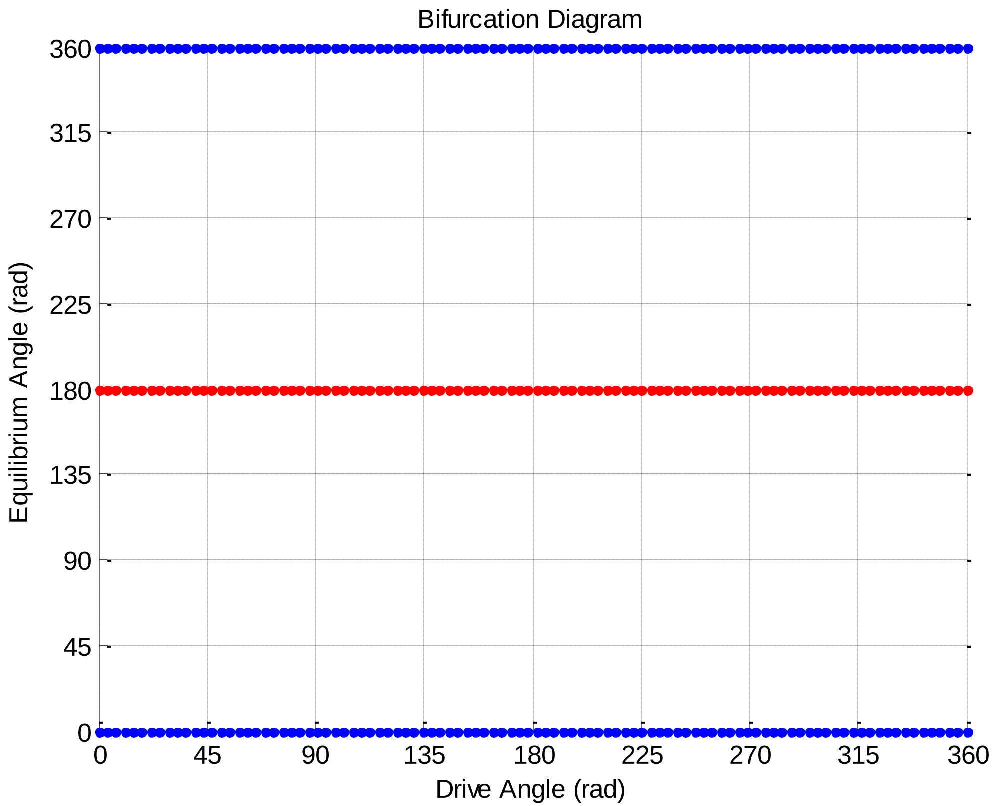
No Driving:
$I_o = 0.443$
$\gamma = 1$
$\alpha = 0$

With Driving:
$I_o = 0.443$
$\gamma = 1.6\text{x}10^{-5}$
$\alpha = 0.01$

Cool. Let's apply derivative tests for stability …

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

BIFURCATION DIAGRAM



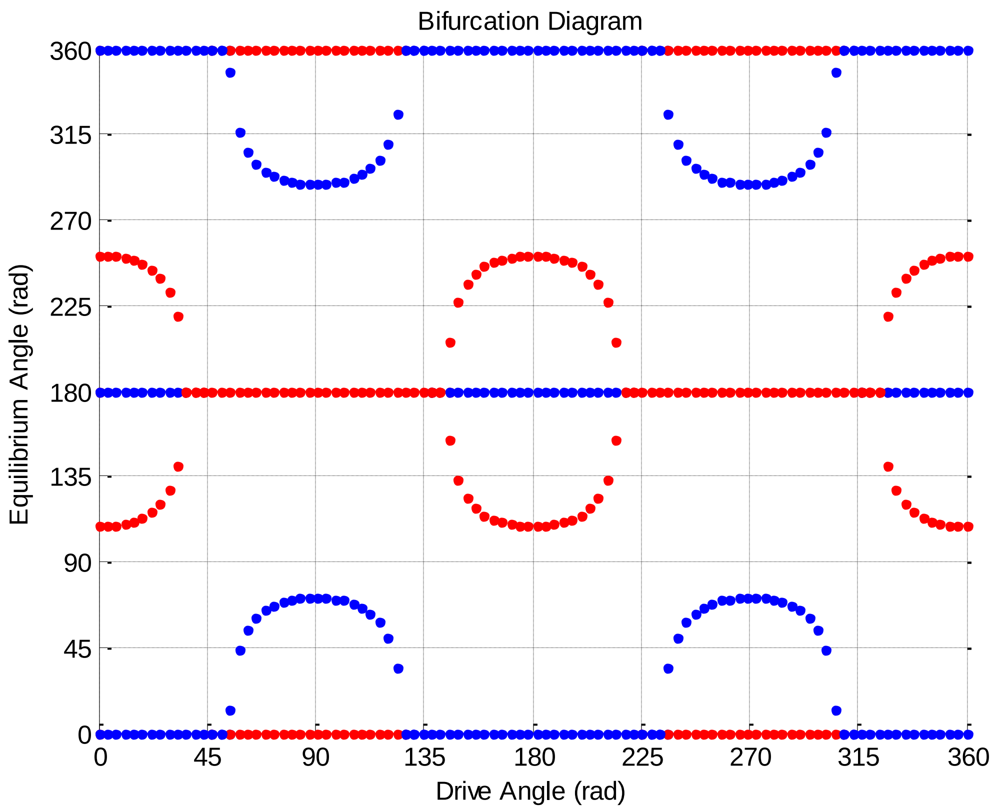
No Driving

- stable
- unstable

Boring. Let's start some driving …

AVERAGING & EFFECTIVE POTENTIAL U_{EFF}

BIFURCATION DIAGRAM



$\gamma = 1.6 \times 10^{-5}$

- stable
- unstable

Neat-O!

FUTURE WORK

- Dynamic Manifold
- Experimentation