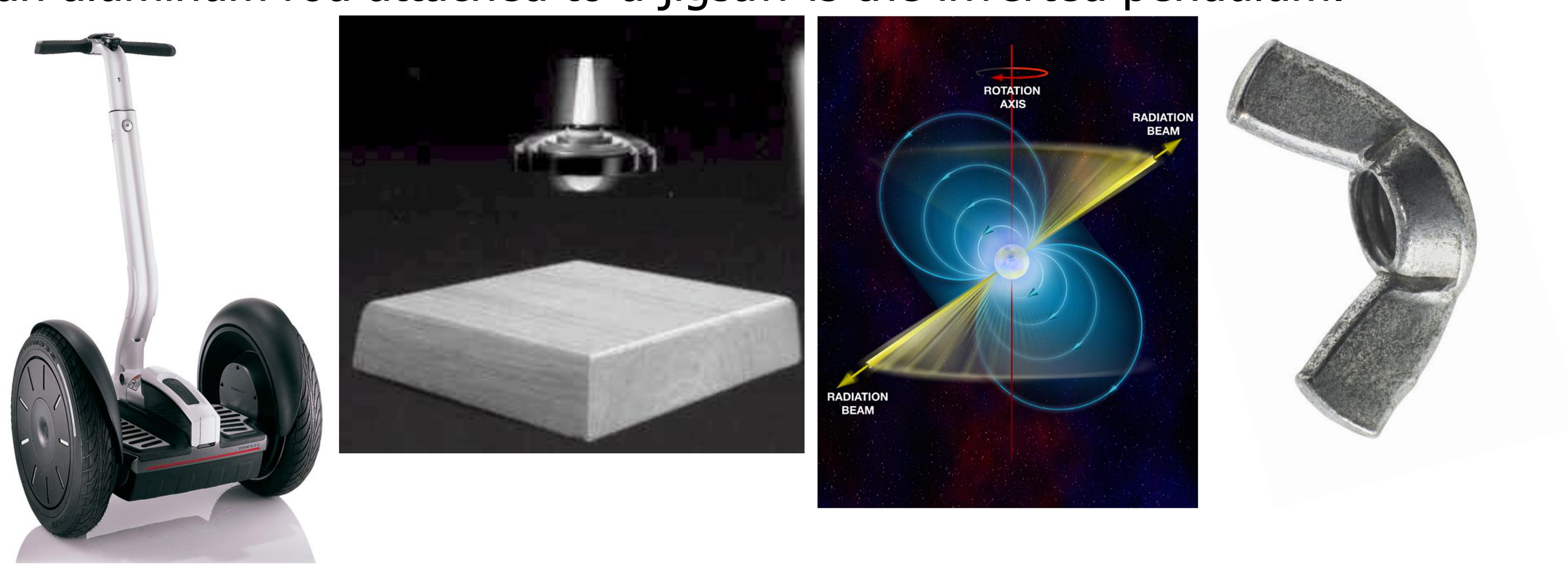
Stability of an Inverted Pendulum

Avery Hope D'Amelio, Cameron Warren, Jiaying Liu, Sean Ashley
The University of Arizona Math Department

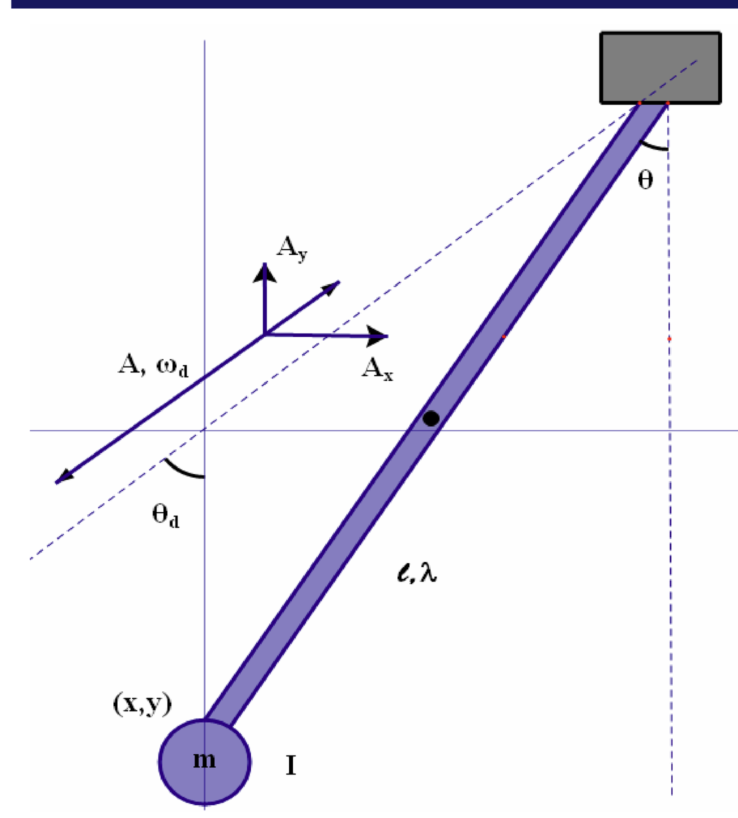
Introduction

The upwards position of a simple pendulum is an unstable point. However, if a high frequency oscillation is applied to the base, the vertical position becomes stable. The stabilization of the vertical position with high driving frequencies has a well known solution from stability analysis with an effective potential [3]. In our project, we studied the stability of the pendulum for arbitrary drive angles. The model is compared to actual experimental data from a physical pendulum where an aluminum rod attached to a jigsaw is the inverted pendulum.



The segway is an example of an inverted pendulum. The magnetic levitron, the pulsar, and the Dzhanibekov effect are examples of systems containing interactions of two scales of oscillation speeds.

Model



- t: Time
- g: Acceleration due to gravity
- θ: Angle of deviation of pendulum
- m: Mass of pendulum
- λ : Effective length of pendulum
- l: Distance from center of mass to base
- I_0: Rotational inertia about base
- ω_0: Natural frequency of pendulum
- θ_d: Drive angle of the base
- ω_d: Drive frequency
- x, y: Position of center of mass
- A_x, A_y: Driving amplitudes

Lagrangian:

$$\mathcal{L} = (Kinetic\ Energy) - (Potential\ Energy)$$

$$\mathcal{L} = \frac{1}{2} I_o \dot{\theta}^2 + \frac{1}{2} m A^2 \omega_d^2 \sin^2(\omega_d t) + m A l \dot{\theta} \omega_d (\cos\theta_d \sin\theta - \sin\theta_d \cos\theta) \sin\omega_d t + mg\big(l\cos\theta + A_y \cos(\omega_d t)\big)$$

Euler-Lagrange Equation: $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$$

Equation of Motion:

$$\ddot{\theta} + \frac{g}{\lambda} \sin\theta + \frac{A\omega_d^2}{\lambda} \sin(\theta - \theta_d) \cos(\omega_d t) = 0$$

Equation of Motion

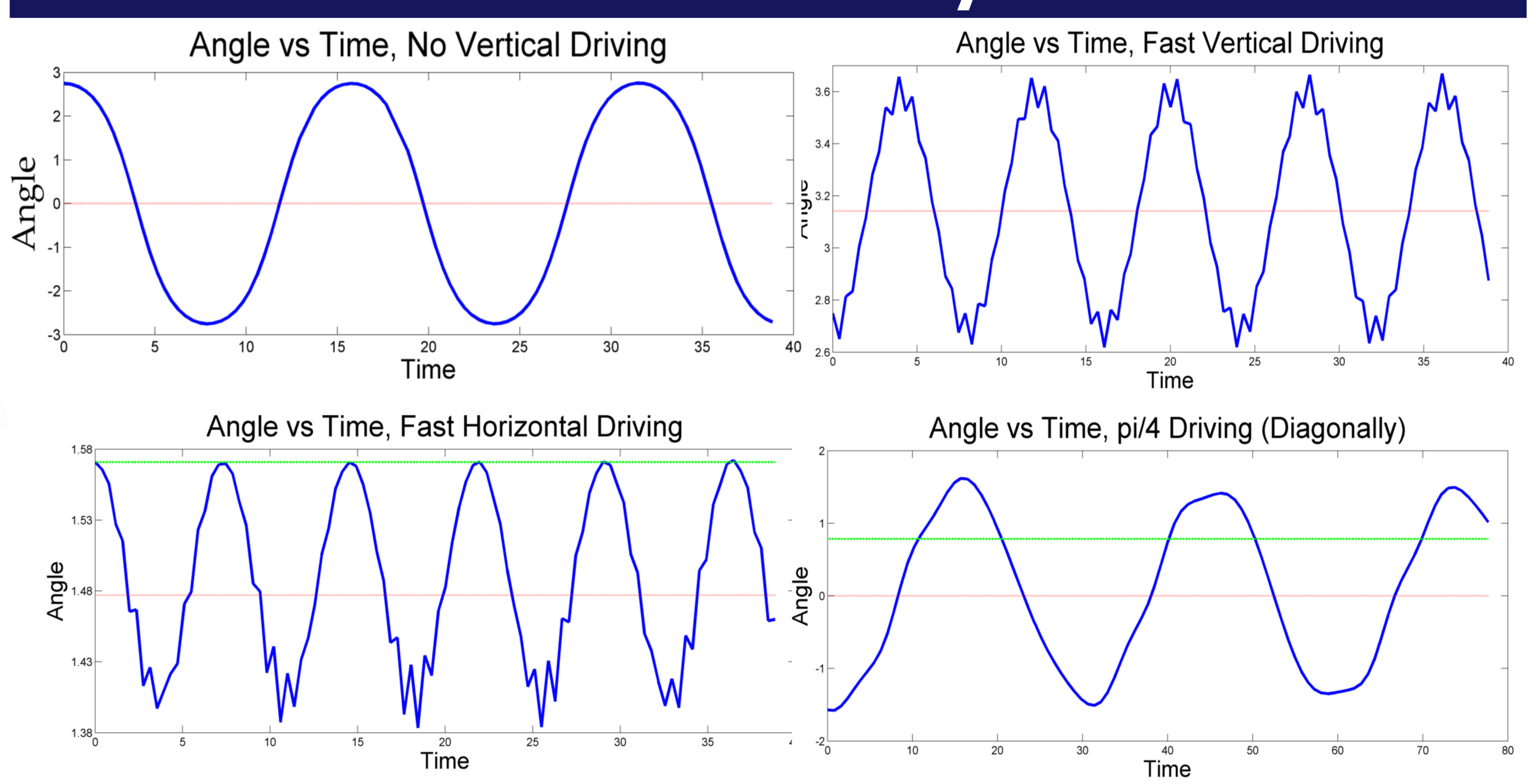
Dimensionless Parameters:

$$\tau = \omega_d t \qquad \gamma = \frac{\omega_o^2}{\omega_d^2} \qquad \alpha = \frac{A}{\lambda}$$

Dimensionless Equation of Motion:

$$\frac{d^2\theta}{d\tau^2} + \gamma \sin\theta + \alpha \sin(\theta - \theta_d) \cos\tau = 0$$

Numerical Analysis

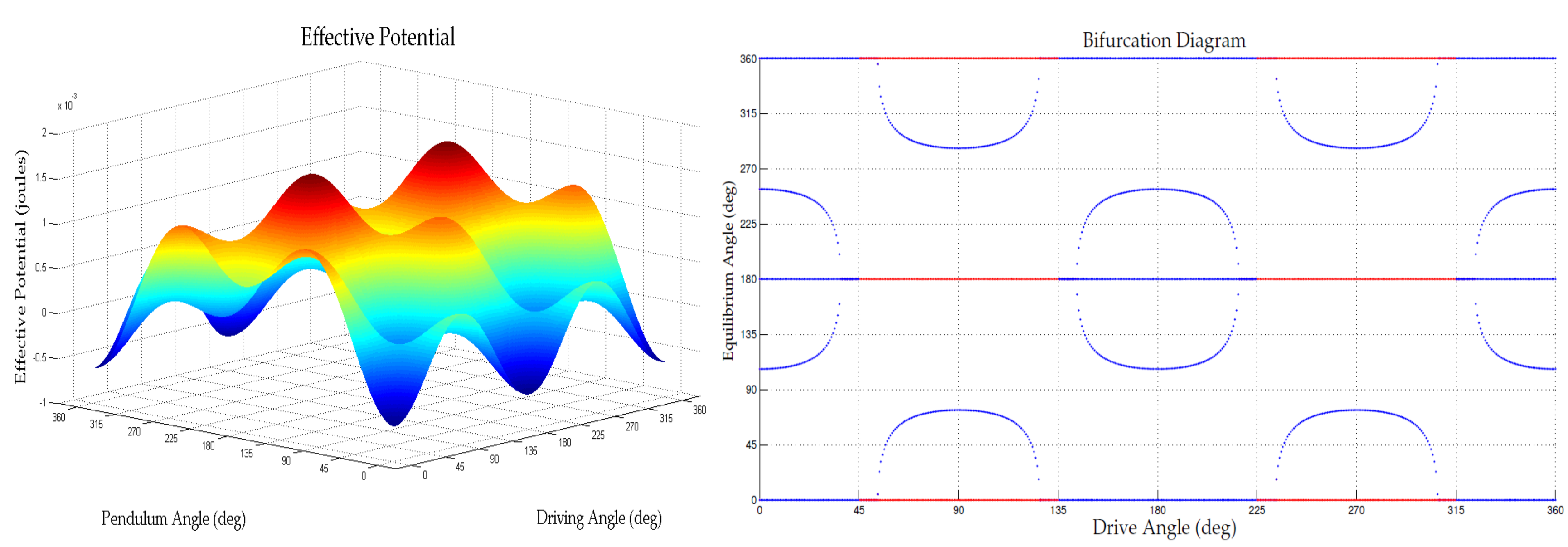


The red lines mark the numerical equilibrium angle while the green lines mark the driving angle.

Effective Potential

The real potential energy of the pendulum is time-dependent, so it is very complex to analyze. However, by averaging over the period of oscillation of the driving, an average or effective potential energy U_{eff}, is dependent on θ, is derived and provides simpler stability analysis for rapid driving oscillations and relatively small driving amplitudes.

Effective Potential: $$U_{eff} = I_o\left(-\gamma\cos\theta + \frac{\alpha^2}{4}(\cos^2\theta_d \sin^2\theta + \sin^2\theta_d \cos^2\theta)\right)$$

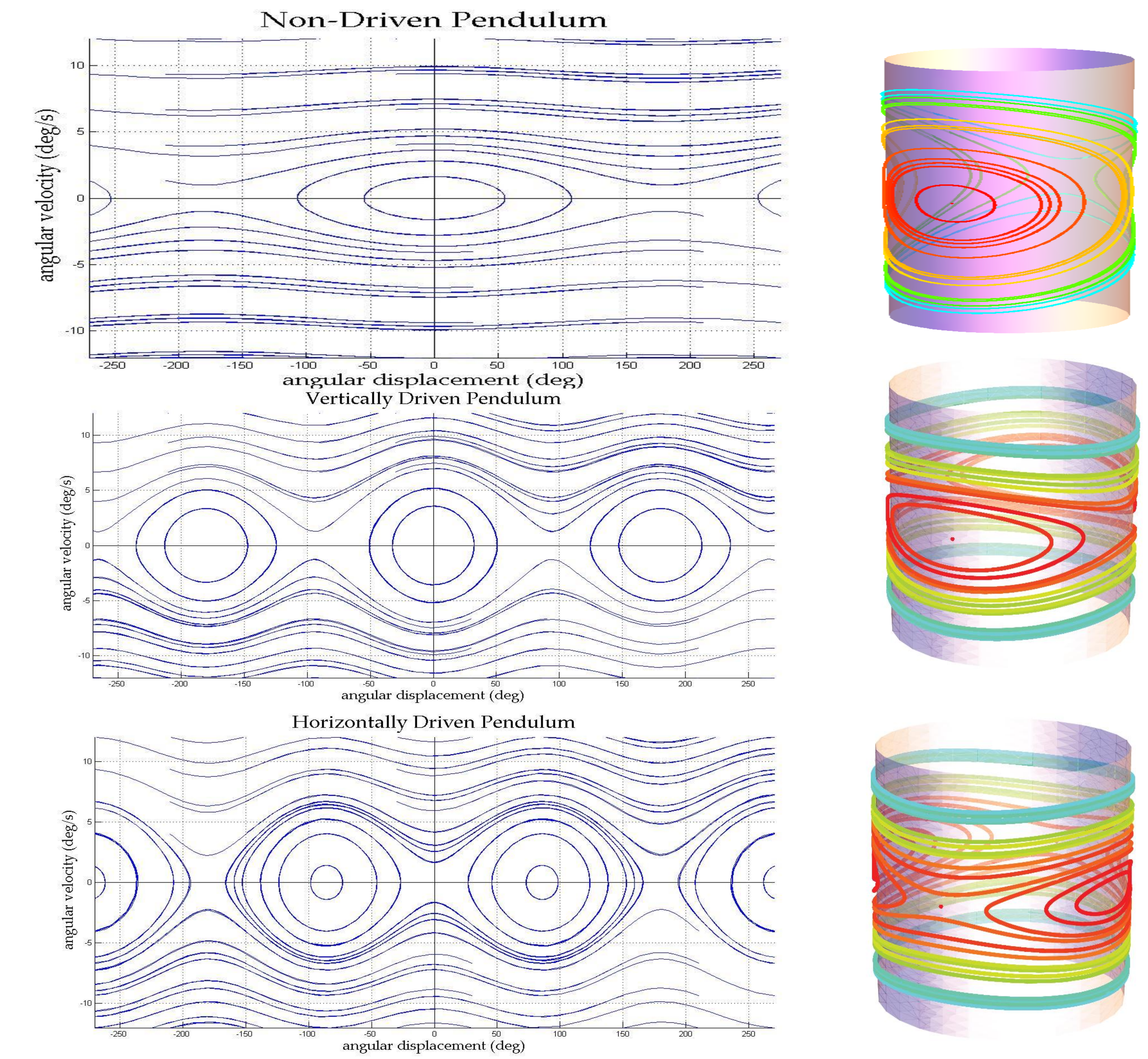


Effective Potential Surface Plot and the Bifurcation Diagram
(**blue – stable; red – unstable**)

The equilibrium angles are:

$$\theta_{eq} = 0 \qquad \theta_{eq} = \pi \qquad \theta_{eq} = \pm \arccos\left(\frac{-2\gamma}{\alpha^2 \cos 2\theta_d}\right)$$

Phase Portraits & Dynamic Manifolds



Experimental Data

Measurement (driving)	Theoretical	Experimental
Vertical Stability Angle	π (straight up)	π
Horizontal Stability Angle	1.477 radians	~1.405 radians
Diagonal Stability Angle	0 (straight down)	-
Vertical Frequency	314.16 dps	300.3 dps
Horizontal Frequency	314.16 dps	453 dps

References

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2. VanDalen J., Gordon. "The driven pendulum at arbitrary drive angles". *arvix.org.* Department of Physics at University of California and Embry-Riddle Aeronautical University. 2 Feb, 2008. Web. 24 Mar, 2013.
3. L.D. Landau, E.M. Lifshitz (1960). *Mechanics*. Vol. 1 (1st ed.).

Acknowledgments

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