Olber’s Paradox
Math 485 Final Report

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Abstract: The dark sky riddle, also known as Olber’s paradox, asks us “Why is the sky dark at night?” We look into previous solutions developed over the past few centuries and analyze why these solutions have been disproven. By examining the density of stars in concentric shells surrounding the planet, or understanding the redshift of cosmic radiation in a static universe, one can reason the different factors of luminosity hitting Earth. The Big-Bang Solution provides us with the most plausible answer to-date, but the riddle is still being analyzed by astronomers, physicists, and mathematicians. We propose to take one of these existing solutions and experimentally determine the accuracy of such using computer simulations.

I. Background/Brief History

In the sixteenth century, the riddle of why the sky is dark at night stumped mathematicians and astronomers. It has since played a large role in the understanding of the continuously expanding universe and complex solutions to the problem are still being created. The paradox is as follows: In the case that the universe is static and homogeneous, as well as populated by an infinite number of stars, any line of sight from the human eye should eventually intercept the surface of a star. Hence, the night sky should be completely illuminated, which contradicts the darkness and nonuniformity of the night.

Thomas Digges formulated this riddle in the 16th century and was discussed by 17th century astronomers Johannes Kepler, Otto von Guericke, Bernard de Fontenelle, and Christina Huygens. It was first discussed in print in 1722 by Edmund Halley whose resolution involved the apparent luminosity dropping off by a factor of ($1/r^4$). German astronomer Wilhelm Olbers noted Halley's error and gave a corrected discussion of the paradox in a paper published in 1823.

The darkness of the night sky represents one of the pieces of evidence of a dynamic universe, such as the Big Bang model. By invoking spacetime's expansion, the non-uniformity of brightness can be explained via redshift, which is the lengthening of microwave radiation background wavelengths to those outside the visible light spectrum, so it appears dark to the naked eye.

Various resolutions have been discussed in the past centuries. If the assumptions of infinite stars in a continuously expanding universe are assumed to be correct, then the simplest resolution is that the average lifetime of stars is too short for their light to have reached Earth yet for extremely distant stars. In another context, the universe is too young for light from distant regions to have reached the Earth.

II. Previous Solutions to the Problem

For Olber's Paradox, there are two ways to interpret the problem that come from two basic distinct assumptions. One assumption is that the universe has uniform distribution density of stars. The other assumption is simply the opposite: the universe does not have a uniform distribution of stars, even if it is infinite.

Let us first take a look at some of the approaches used to tackle the paradox under the assumption that the stars are uniformly distributed among the universe meaning that the sky should be covered by stars. The question that arises from this assumption is, ‘what happened to the missing starlight?’.

Edmund Halley suggested dividing the universe into concentric shells. If the stars have a uniform density distribution in the universe then it follows that the number of stars in each shell depends on the volume of the shell. The volume of a shell increases with the radius squared, hence there are r^2 as of many stars with increasing radius.

$$\Delta V = 4\pi r^2 \Delta r \tag{1}$$

However the intensity of light of each star decreases as $(1/r^2)$. So, each shell gives off the same intensity of light regardless of the distance. Then why is the night sky not covered by stars?

Jean-Pillipe Cheseaux adds to Halley’s by saying that interstellar medium absorbs the light from some of these stars and that is why we cannot see those stars. He called D the average distance between stars and therefore $V = D^3$ the average volume taken up by a star. Also, the surface area of a star $S = \pi R^2$. By dividing, Cheseaux calculated $\lambda = V/S$, where λ is the distance of the background stars. At this distance the stars form a continuous stellar background. He calculated λ to be $3 \cdot 10^{15}$ light years.

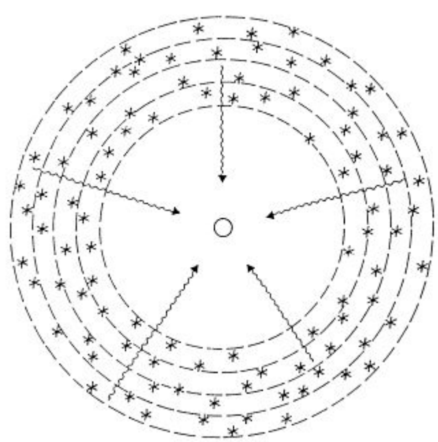


Figure 1: Halley’s concentric shells of equal thickness uniformly populated with stars [1]

Heinrich Olber applied Halley's and Cheseaux techniques, but with a different approach. He developed an expression for the fraction of the sky covered by stars. For a shell of radius q the total number of stars, N, is

$$N = 4\pi n q^2 dq \tag{2}$$

Where n is a unit star per volume ($n = 1/D^3$). Multiplying the number of stars in a shell by the fraction of area each star takes up on the surface area of the shell, we obtain the fraction of the shell covered by stars.

$$d\alpha = \left(\frac{S}{4\pi q^2}\right) 4\pi n q^{2dq} = Sndq = \frac{S}{D^3}dq = \frac{1}{\lambda}dq \tag{3}$$

By integrating the above equation from $q = 0$ (the observer) to some radius $q = r$, we obtain the fraction of the sky cover up to that radius. Here each shell adds to the total number of visible stars.

$$\alpha = \frac{r}{\lambda} \tag{4}$$

From the equation one notices that when $r = \lambda$, the fraction is unity meaning that 100% of the sky is covered in stars. If the universe is infinite any star beyond the radius r must not be visible due to occultation from stars in the foreground. Olber decides to account for stars occulating other stars by factoring in $exp(-q/\lambda)$ on each shell, now the fraction of sky covered by stars in each shell is

$$d\alpha = \frac{1}{\lambda}e^{-q/\lambda}dq \tag{5}$$

This makes sense because as the radius of the shell increases, one would expect less and less stars to be visible from each shell due to stars from smaller shells blocking or interfering with our view of stars from outer shells.
After integration up to $q = r$,

$$\alpha = 1 - e^{-r/\lambda} \tag{6}$$

$\alpha = 1$ when $r >> \lambda$ (infinite universe).

Aside from occultation, Olber took absorption of starlight into account as well. He factors in $exp(-q/\lambda)$. Now the fraction of sky covered by stars in each shell is

$$d\alpha = (e^{-q/\lambda}) \cdot (e^{(-q/\mu)\cdot(1/\lambda)})dq \tag{7}$$

where μ is the absorption distance. This factor makes sense since we can expect the fraction of visible stars on each shell to decrease with an increased radius of shell due to more interstellar medium between shell and observer, hence more absorption of starlight. After integrating our final expression for fraction of visible stars on each shell from the observer to a radius $q = r$ we get,

$$\alpha = \frac{\mu}{\lambda+\mu}\left(1 - e^{-r(\mu+\lambda)/\lambda\mu}\right) \tag{8}$$

Notice that when r approaches infinity (infinite universe),

$$\alpha = \frac{\mu}{\lambda+\mu} \tag{9}$$

which is not 1, so therefore in an infinite universe the fraction of visible stars in the sky is not 100%. Solved?!

John Herschel said that the absorption for missing starlight should not be taken into account for the absorbing medium would just emit the radiation back out. So then in an infinite universe the sky should be covered by stars when looking at it, according to Olber. Why are there dark spots in the night sky if starlight is not absorbed? What happened to the missing stars? This is the second interpretation of the paradox. Herschel states that the stars are not uniformly distributed in the universe. Therefore a dark spot in the night sky is simply explained by the absence of stars in that particular line of sight.

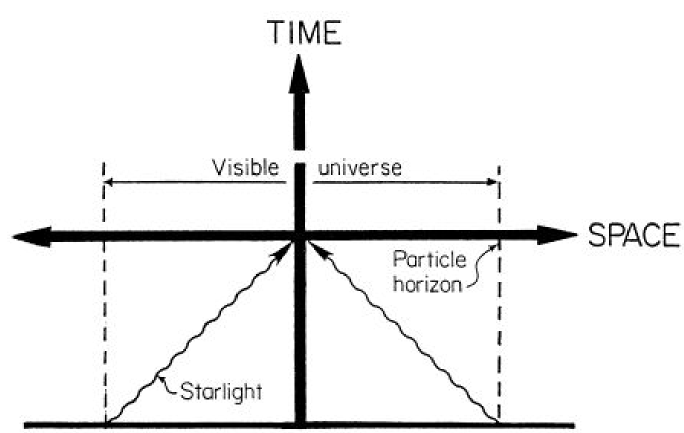


Figure 2: With the observer at the center, the visible universe at finite age is bounded by the particle horizon. The darkness of the sky is the result of the finite age of the luminous universe [1]

With the introduction and acceptance of the speed of light, the argument of the Visible Universe is borned. In a unbounded, static and finte age universe we say that the the visible universe is or radius $r = ct$, where c is the speed of light and t is the age of the universe. Therefore, stars that are further than r are not visible to us because their starlight has not reached us yet. The visible argument can have both interpretations of the paradox:

- If $\lambda < c \cdot t$, the sky is covered by stars. If so, what happened to the missing starlight (the first interpretation of paradox is correct)
- If $c \cdot t > \lambda$, stars do not cover the sky. If so, what happened to the missing stars (the second interpretation of the paradox is correct)

III. Evaluation of Previous Solutions

The simplicity of the question being asked in the dark night sky riddle belies the complexity of the form a plausible solution must take. Over the centuries many astronomers, physicists, and mathematicians have suggested seemingly reasonable solutions to Olbers' riddle only to find their solutions are demonstrably false.The difficulty in confirming a unitary solution to this puzzle arises from the current uncertainty in our knowledge of the universe. As astronomers uncover fundamental truths that govern the relationships between celestial bodies, we are better able to reject previous solutions to Olbers' question.

The first case we consider is that of Kepler's solution. Kepler's argument for a spatially finite universe might have been well founded when it was concluded in 1610 but current astronomical observations allow us to disregard this "solution" in a trivial manner. Certainly, if the universe was spatially finite then his conclusion might hold true but the qualifier has been shown to be false. The fact that the universe is expanding was theoretically proven by Alexander Friedmann in 1914 (Friedmann Aleksandr). Since the universe is expanding it cannot have a finite size, thus trivializing Kepler's position.

In 1823 Olbers performed calculations similar to Chaseux's work from 1744. While Olber made significant headway in the problem, notably by removing the need for uniform distribution of stars, his solution of a dust cloud absorbing the light from distant stars is definitively false (Harrison Edward). Olber failed to recognize, or perhaps was unaware of, the consequences of his solution. By the 1st law of thermodynamics:

$$\Delta U = Q - W \tag{10}$$

Where ΔU is the change in the internal energy, Q is the heat added to the system, and W is the work done by the system. Given Olbers' solution, the dust cloud must be absorbing the light and thermal radiation from infinite stars. Under the condition of absorbing infinite heat, the dust cloud itself would begin to glow radiating as intensely as the light it absorbed from distant stars. This is a direct consequence of the 1st law of thermodynamics.

It is Bondi's redshift solution that presents the first non-trivial discussion. Bondi correctly observed that the finite age of the universe as well as the luminous lifetime of stars can account for the thermodynamic disequilibrium that is present. Despite this, his work largely referenced a steady-state universe of infinite past and future with stars that exist in perpetuity. As discussed above, Bondi's solution leads to the conclusion:

$$\frac{u}{u^*} = \frac{D_H}{4\lambda + D_H} \approx 10^{-13} \tag{11}$$

So, the ratio of radiation density in interstellar space and radiation density at the sources is a constant term. What's more, expansion of the universe does not guarantee darkness, rather the relative magnitudes of λ and D_H are responsible for the darkness condition in the night sky. Namely darkness occurs when:

$$D_H \ll 4\lambda \tag{12}$$

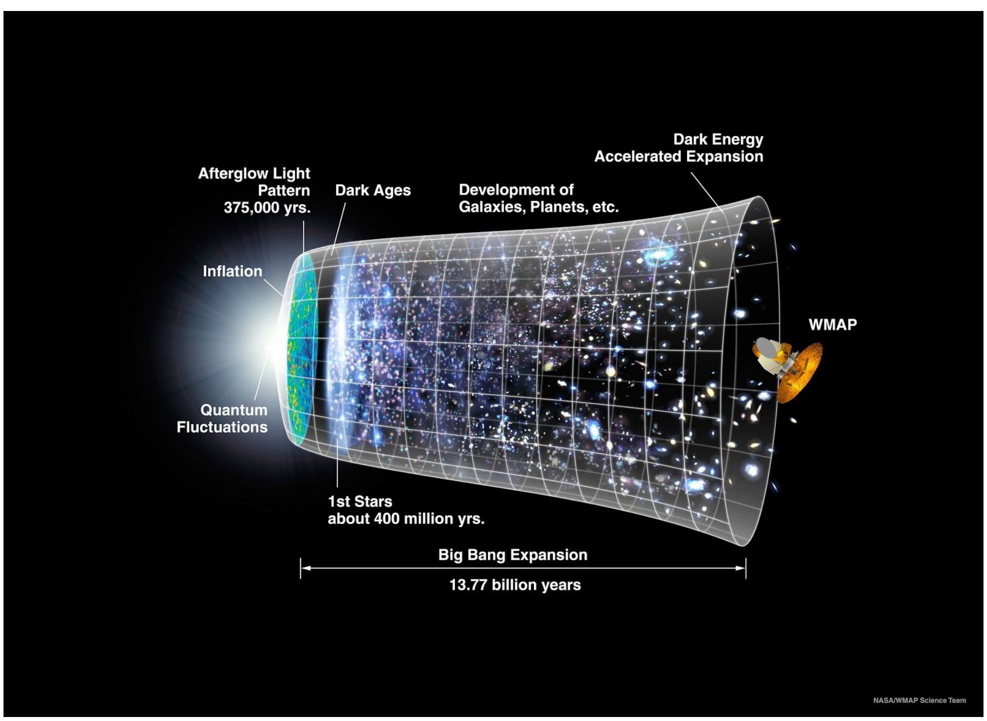
As a direct consequence of the finite speed of light, any observer looking up at the night sky must extend their vision backwards in space as well as time. The cone of light observed must then stretch from the observer a distance so that it lies tangential to the Hubble Sphere (the temporal edge of the observable universe). The lightcone never reaches a particle horizon, meaning that all celestial bodies trapped within its volume are witnessed by the observer. This leads us to the conclusion that all light from stars near the edge of the universe must be red-shifted out of the visible range of light. This is a plausible solution to Olbers' paradox but it fails to account for one important detail. Stars radiate in more frequencies than just visible. If the visible wavelengths are red-shifted into infrared, then the ultraviolet wavelengths must be red-shifted into the visible.

Light from these stars would still be visible. After some finite time, the light from stars at the edge of the steady-state universe will reach the observer, causing the night sky to glow brightly.

While Bondi's solution holds for a steady state universe it fails to provide a complete solution to Olbers' riddle in our expanding universe. The Big-Bang Solution is presently the most complete response to Olbers' "Paradox." Outside the steady-state universe examined by Bondi, The Big-Bang solution maintains conservation of matter and energy. As evidenced above, the red-shift in an expanding universe is insufficient to account for a dark night sky. This can be witnessed using the resultant equation in II:

$$\alpha = \int_{t_1}^{t} \frac{n(t')SV(t')cdt'}{V(t)} exp(-\int_{t}^{t'} \frac{cdt''}{\lambda(t'')}) \qquad (13)$$

Where, assuming $\alpha = 1$, we find $u < u^*$ because of redshift. In the model of a big bang universe the finite speed of light combined with the fact that our universe is ~ 10- 15 billion years old results in the undeniable truth that the maximum distance from which light can be received is ~ 10- 15 billion light-years away. Galaxies beyond this distance might exist, though their light has yet to reach us. The second half of the big bang solution is found in the finite lifetime of stars. Eventually, stars dim. This effect is observed more succinctly in nearby galaxies as a direct consequence of the shorter light-travel time. The sum of these effects leave us with the conclusion that the night sky will never be completely filled with stars. Either, light from distant stars has yet to reach us, or those close to us has grown cold and dark.



IV. Analysis of a Plausible Modern Solution

While the Big-Bang Solution presents the most widely agreed upon plausible conclusion to Olbers' Paradox there still exist a variety of solutions that potentially explain the phenomenon. One, less intuitive, answer to the riddle is a consequence of vacuum decay in separate universes.

As explored in Frieden's paper, "Spontaneous Formation of Universes from Vacuum via Information-induced Holograms" (Frieden, B. R., & Gatenby) one can use Fisher Information to derive the existence of the 26 fundamental constants of the universe. In this paper he postulates that minimizing the loss of Fisher information, I, can be achieved using Cramer-Rao:

$$e^2 = I^{-1} \tag{14}$$

Where e^2 is the mean squared error attainable in measuring a signal value. He continues by commenting that the Big bang is usually considered a large scale version of the "emergence from false vacuum" effect. In the case of the Big bang, however, the scales of time and space are undefined, this dilemma can be rectified through maximizing the Fisher information resulting in these criteria.

Frieden's theory continues into the discussion of a multiverse, based on the Lorentzian Wormhole-based model, in which there exists a finite region bounded by a universe which does not realize the minimum energy of a true vacuum. This may result in the case where an "umbilical pathway" may grow and create a new "baby" universe.

Frieden goes on to prove, using Fisher information, that this wormhole must deliver the 26 universal constants as well as the Higgs field to the new universe via a high intensity hologram. We also consider this new universe to be a false vacuum so that it does not collapse on itself. In this scenario, a huge amount of energy is "lost" from one universe and imparted to another and could be a solution to the thermal disequilibrium first remarked upon by Kepler, Olber, and Bondi. This modern solution to Olbers' "Paradox" is one of many that are currently discussed. The surprising complexity of this "simple" question becomes more apparent as we discuss the accuracy of various historical solutions. Kepler, Olber, and numerous physicists have devised seemingly plausible solutions only to have them refuted as astronomical discoveries are made. The current generation of solutions (those of Bondi, Frieden, and the Big Bang solution) may yet prove to be false as astronomy develops even further. Only through a more intimate understanding of our universe can Olbers' "Paradox" truly be resolved.

V. Self-Avoiding Random Walk

We chose to dive deeper into the self-avoiding random walk model, which provides a plausible solution to the riddle as well as an interesting way to model the distribution of stars in the universe. Let us assume we are dealing with an infinite random walk in which each step is a unit length and each vertex is a sphere with radius a. The radius should be chosen to accurately reflect the ratio between the radius of a star and the distance between two nearby stars. It is also important to note that this random walk is self-avoiding due to the fact that two stars cannot occupy the same region of space. The question we ask is, "if an observer is placed at one of these spheres and looks around, would he/she see overlapping spheres in every direction (a bright night sky) or would only a certain percentage of the viewing area be covered by these spheres (a dark night sky)?"

Let W be a self-avoiding random walk with vertices $X_0, X_1, X_{-1}, X_2, X_{-2}, \ldots$ and so on, such that X_k and X_{k+1} are adjacent to each other. If we place the observer at the origin, the nearest star can be represented by X_0. We know that the radius of gyration of a self-avoiding random walk of length n is of the order of n^{μ} with $\mu \approx .588$, so we can then assume the mean distance between X_0 and X_k is of the order k^{μ}. We can also assume the distribution of X_k can be approximated by a classical Gaussian distribution and the density function approximated by the function

$$f(X_k) \approx \left(\frac{1}{\sqrt{2\pi}\sigma_k}\right)^3 exp\left(-\frac{|X_k|^2}{2\sigma_k^2}\right) \tag{15}$$

The observed area of the star at X_k from the observer's standpoint is given by the equation $\frac{4\pi a_k^2}{|X_k|^2}$ where a_k is the radius of the star at position X_k. We can then let $d_0 = |X_0|$ be the distance between the observer and the nearest star. We can then find the mean contribution of X_k to the total observed area as seen by the observer by integrating the density function over infinite space.

$$\begin{aligned} &\iint\int_{|X_k| \geq d_0} \frac{4\pi a_k^2}{|X_k|^2}\left(\frac{1}{\sqrt{2\pi}\sigma_k}\right)^3 exp\left(-\frac{|X_k|^2}{2\sigma_k^2}\right)dX_k \\ &= \frac{4\pi^2 a_k{}^2}{\sigma_k^2} \end{aligned} \tag{16}$$

We can then find the mean of the total observed area by summing the contributions of the observed area from all the stars and find that it is bounded above by

$$\sum_{k=\pm1,\pm2,\ldots} \frac{4\pi^2 a_k{}^2}{\sigma_k^2} \tag{17}$$

Because each a_k is very small (in terms of astronomical units) and $\sigma_k{}^2$ is of the order $|k|^{2\mu}$ then the above series is convergent. If we were to substitute a_k by the largest known radius of a star, and σ_k by $b|k|^{\mu}$ where b > 0 is a constant determined by the random walk model, the series can be bounded by

$$\frac{10\pi a^2}{.17b^2} = \frac{10^3\pi a^2}{.17} \tag{18}$$

After we substitute the largest known radius of a star, we find that the observed area of the night sky populated by light is extremely small. In conclusion, if the stars were to follow a distribution similar to a self-avoiding random walk model of infinite length, an observer at the origin would only see a very small percentage of the night sky lit up.

VI. Results

To assess the validity of the Self Avoiding Random Walk model we choose to simulate the process in MATLAB and compare the results to experimental observations of the night sky. Our metric for comparison was the percentage of the night sky that is illuminated by stars, α.

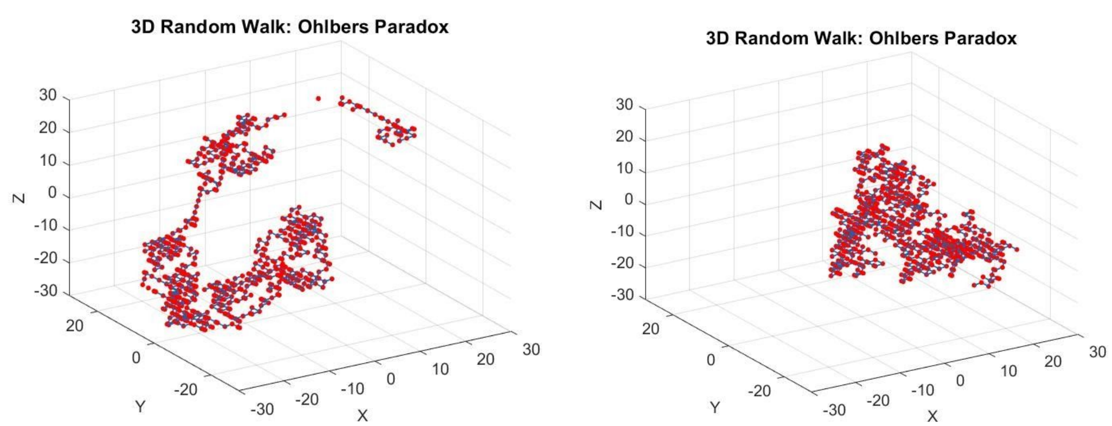


Figure 3: Two different paths generated from the same MATLAB code defining a self avoiding random walk where the distance between stars is given by one parsec

In order to analytically assess the random walk solution, we began by defining an upper bound to the series from equation (17) using an expression by Diao in his paper from 2007 [4]:

$$\alpha \leq \frac{10^3 \pi a^2}{.17} \tag{19}$$

To calculate the upper bound in equation (19) we used the radius of the largest currently observed star: IRAS 05280-6910. We compare the values calculated by us to the value calculated by Diao in 2007. Though the two answers differ this is simply because Diao chose a smaller stellar radius to use in his calculations as IRAS 05280-6910 had yet to be observed. As part of our comparison we include the experimental solution of the percentage of the illuminated night sky. This experimental answer is found by finding the mean free path of the universe and extending it across the night sky [5]:

Diao's Upper Bound	Our Upper Bound	Experimental Solution
$1.51 * 10^{-4}$ %	$2.84 * 10^{-3}$ %	10^{-14} - 10^{-17}%

Table 1: Upper bounds to the percentage of the night sky illuminated by stars calculated by us and Diao [4] as well as the observed solution of the same phenomena.

Through the disparity in orders of magnitude between the upper bounds calculated and the experimental solution, it is clear that the upper bound found using the self avoiding random walk model is not a viable solution to Olbers' Paradox. The reasoning behind this disparity though, is quite simple. The self avoiding random walk model is uniquely general in its context. The upper bound solution implies a steady-state static universe of infinite expanse. Through astronomical observations we are aware that these implications are not true. The universe, as we observe it, is finite spatially and temporally. As a result of this finiteness astronomers define a Hubble Sphere that bounds the observable size of the universe. In order to better fit our calculations to the observable universe we constrain equation (17) by the Hubble Sphere and numerically solve for the percentage of the night sky that is illuminated. This yields the following results:

Method	Percentage of Night Sky Covered
Self Avoiding Walk	$2.44*10^{-13}$ %
Olbers	10^{-13} %
Bondi	10^{-13} %
Big Bang	10^{-13} %
Experimental Solution	10^{-14} - 10^{-17} %

Table 2: Values for the percentage of the night sky illuminated by stars using different methods. The values for Olbers, Bondi, and Big Bang were taken from [1]. The experimental solution was found using [5]. Though the order of magnitude found in both Olbers' and Bondi's solutions is equivalent to that found by the Big Bang solution, they are colored red as the reasoning behind these answers is blatantly false in our observed universe as discussed previously.

Clearly, the introduction of a Hubble Sphere to the Self Avoiding Random Walk model significantly improved the accuracy of the results found. Notably, the order of magnitude using this method agrees with the order of magnitude found using the Big Bang solution, the most widely agreed upon plausible solution to the paradox.

VII. Conclusion

Olbers’ “Paradox” has been the subject of study by astronomers, physicists, and mathematicians for hundreds of years. Over the course of its analysis various plausible solutions have been presented only to be refuted later as new data comes to light. The current leading theory is that of the Big-Bang solution which claims that light from distant stars has yet to reach us, and by the time it does the light from the closest stars will have gone dark. While the Big-Bang solution is the most widely agreed upon theory, it has yet to be confirmed. Many astronomers, physicists, and mathematicians are still analyzing this peculiar phenomenon and presenting independent solutions to solve Olbers’ question.

Olber’s Paradox is used as a starting argument for the question of the universe being infinite in age or space and whether it is static or expanding. Self Avoiding Random Walk is an attractive approach to the paradox because these questions don’t factor in. Only a fraction of the night sky will be covered regardless of the age of the universe or its expansion. However, a static infinite universe with a non-uniform distribution is assumed by this model, leaving the only question to be what the distribution of stars is. Moreover, it is a safe bet to say that a uniform distribution can be ruled out due to modern data making this model a safe bet.

The solution we assessed, the Self Avoiding Random Walk, is a plausible solution to Olbers’ Paradox so long as we constrain the expression with a hubble sphere. Under just this one constraint we find that the order of magnitude for the percentage of the night sky illuminated by starlight is of the same order as the Big-Bang solution. It is likely that further constraining the expression for the Random Walk method with other observed phenomena in the universe will decrease the order of magnitude to one more closely resembling the observed portion of the night sky that is illuminated. Introducing black holes, red-shift, and diffraction about solar bodies would be a key step towards improving the accuracy of this model.

References:

[1] Friedmann, Aleksandr. “125. On the Curvature of Space.” *A Source Book in Astronomy and Astrophysics, 1900–1975*, 1922, pp. 838–843., doi:10.4159/harvard.9780674366688.c134.

[2] Harrison, Edward. “The Dark Night-Sky Riddle, ‘Olbers's Paradox.’” *Symposium - International Astronomical Union*, vol. 139, 1990, pp. 3–17., doi:10.1017/s0074180900240369.

[3] Frieden, B. R., & Gatenby, R. A. (2019). Spontaneous Formation of Universes from Vacuum via Information-induced Holograms. *arXiv preprint arXiv:1909.11435*.

[4] Diao, Y., and A. Stasiak. “Self-Avoiding Random Walks and Olbers' Paradox.” International Journal of Contemporary Mathematical Sciences, vol. 2, 2007, pp. 445–449., doi:10.12988/ijcms.2007.07038.

[5] SCHRÖDINGER, E. Mean Free Path of Protons in the Universe. Nature 141, 410 (1938). https://doi.org/10.1038/141410a0