Mysteries of the Earth-Moon System

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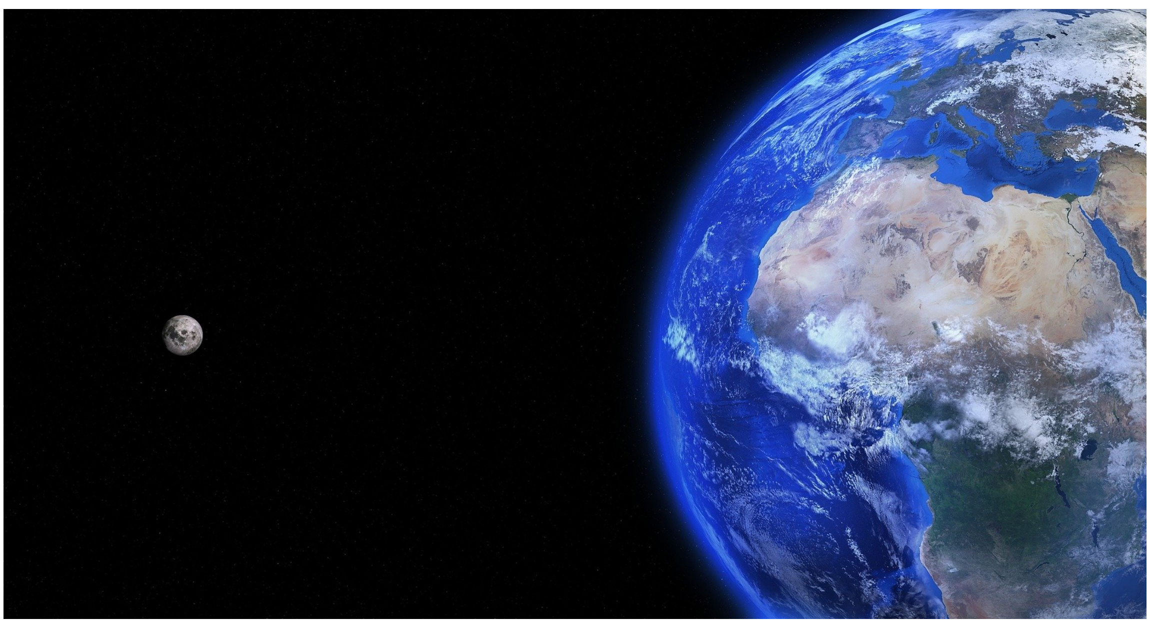

Mysteries of the Earth-Moon System
MATH 485 Final Report

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1 Introduction

This project was based on the mystery, or more appropriately, a counter-intuitive result lying within the Earth-Moon system. That being the existence of tidal friction within the system, which causes the system to lose energy, which means the Moon should be coming in closer to the Earth. However, experimental observations show that the Moon is in fact moving further and further away from the Earth as time goes on. As mentioned, the initial consideration of tidal friction would leads to the opposite conclusion than what has been experimentally observed, so this project delves into the theory behind why tidal friction is in fact the cause of the Moon moving away from the Earth.

To begin with, tidal friction is described by the equations for tidal torques, which are derived from the fact that there is an unequal gravitational pull on the different points of Earth. This unequal pull results in friction between the rotation of Earth and the oceans being pulled by the Moon. These equations are described further below but it must be noted that the forces exerted by the tidal torques require a very large time scale for effects to be observed, so these timescales must be taken into account within the equations. Further below, it is described how the system is broken down into three different timescale and how equations must be sequentially averaged from the short to the intermediate and then from the intermediate to the long timescales.

This paper heavily follows the research done by Goldreich [1] and by Touma and Wisdom [2] as they provide derivations and implementations for the aforementioned equations including tidal friction. Once the equations were understood, numerical analysis was done by implementing them into an algorithm in Python using two RK4 solvers in order to solve for parameters on the long timescale. These results, which are discussed below, show that with the inclusion of tidal friction, one would indeed expect that the Moon should be moving away from the Earth just as the results from [1] and [2] indicate as well.

2 Model

The following model was derived following the approach laid out by Goldreich [1]. The derivation follows a successive averaging, where motion in the system is described on increasingly larger time scales via integration. While the majority of the analysis is consistent with Goldreich, some assumptions are corrected following work done by Touma and Wisdom [2]. Where the analysis varies from Goldreich's, the changes are noted.

*This report was completed in collaboration with an appointed University of Arizona Math Department mentor. We would like to express our gratitude for her help and guidance.

2.1 Time

Motion in the Earth-Moon system can be described on three time scales, as described by both Goldreich [1] and Touma and Wisdom [2]. The short time scale, or orbital time scale, is used to describe the orbital periods of the bodies involved and corresponds to a day (the orbital period of the Earth around its axis), a month (the orbital period of the moon's orbit about the Earth), and a year (the orbital period of the Earth-Moon system about the sun). On the intermediate time scale, or precessional time scale, we see precessional motion of the bodies, which is the periodic change in the angles between orbital planes. Of most import for this model are the precessions of the angle between the lunar orbit plane and the Earth's equatorial plane, ϵ, the angle between the lunar orbit plane and the ecliptic, I, also referred to as the inclination of the lunar orbit, and third, the angle between the Earth's equatorial plane and the ecliptic, γ. The period's of these motions define the intermediate time scale and are reported by Touma and Wisdom [2] as 17.83 years, 34,245.8 years, and 74,590.96 years respectively. Lastly, the long time scale, or tidal time scale, is on a much larger order of about a billion years. On this time scale we see the effects of tidal friction on the system, and while orbital periods may remain constant on both the short and intermediate time scale, it is on the long time scale that we find large variations in their values and speculations on both the history and future of the system may be made.

2.2 Keplerian Elements

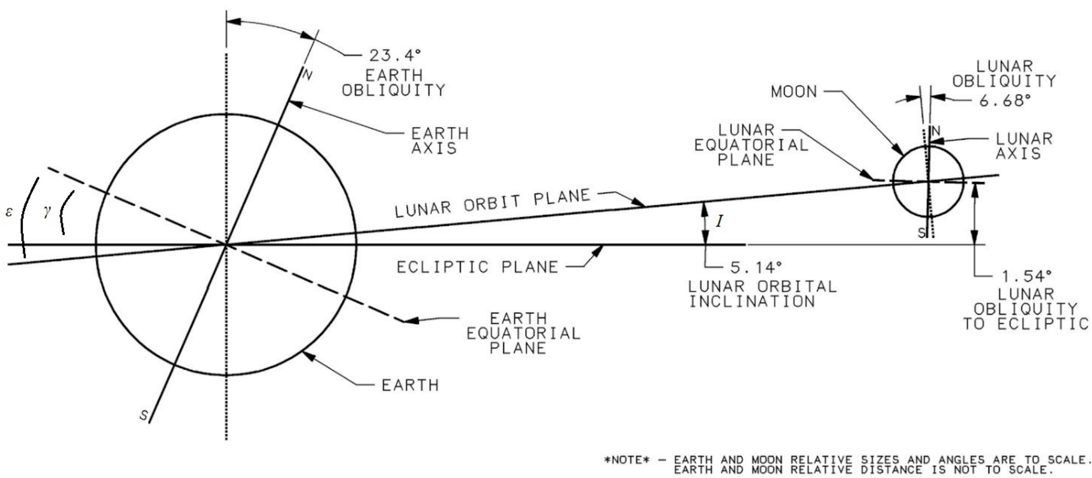

Figure 1: Earth-Moon Terminology Diagram.

To describe the orbit of a body a full set of Keplerian elements is required. For this project the set of elements for the lunar orbit with respect to both the ecliptic and the equatorial planes is needed. Some of the angles involved in this description were mentioned above, ϵ, I, and γ (see figure 1). Other than these angles we also need to define the angle in the orbit measured from the ascending node on the equatorial plane and the ecliptic plane, Φ and Φ', as well as the position in the orbit measured from the moon's ascending node on the ecliptic, u. Other than these angles, the other Keplerian elements for the lunar orbit are its mean motion, n_m, and its eccentricity, which following both Goldreich [1] and Touma and Wisdom [2] we have chosen to ignore in favor of a circular orbit with radius a, the semi-major axis of its orbit.

2.3 Motion on the Short and Intermediate Time Scales

Given the ultimate goal for deriving this model, describing the actual motion of the system on the short time scale is unimportant. Of importance however, are the effects of both the Earth's oblate figure (that the Earth is not a sphere, but squished at the poles, making the equatorial radius greater) and the sun on the Moon's orbit. These effects are referred to as disturbing potentials. Three disturbing potentials are included in the model, R_1, the disturbing potential of the Earth's figure on the lunar orbit, R_2, the disturbing potential of the Sun on the lunar orbit, and R_3, the disturbing potential of the Earth's figure on the Sun. First, we need to know the Earth's principal moments of inertia, which are defined as

$$C = I[1 + (2k_s R_e^5/9GI)\Omega_e^2] \tag{1}$$

and

$$A = I[1 - (k_s R_e^5/9GI)\Omega_e^2]. \tag{2}$$

where C is the moment of inertia about the Earth's rotational axis and A is the moment of inertia about an axis in the equatorial plane. $I = 9.72 \times 10^{37}\ m^2 kg$, is the moment of inertia of the equivalent sphere, $k_s = 0.947$ is the fluid Love number (where the Earth is considered a fluid for the purposes of this model), R_e is the radius of the Earth, G is the gravitational constant, and Ω_e is the angular velocity of the Earth (at present $\Omega_e = 7.29 \times 10^{-5}\ s^{-1}$, however, Ω_e will vary drastically on the long time scale).

The disturbing potential equations are first derived in terms on Legendre Polynomials and non-Keplerian elements. The conversion to Keplerian elements is then made prior to averaging. Since R_1 and R_3 are analogous except for their angles and masses, the derivation of R_3 is saved for after the averaging over the short time scale, where it is a trivial matter of substituting the correct values. The orbital eccentricities are also ignored, where the semi-major axes of the Moon and Earth's orbits are defined as a and a_s respectively and taken as the orbital radii. The masses of the bodies involved are also defined as m_e, m_m, and m_s, for the Earth, Moon, and Sun respectively. The disturbing potential equations can then be written in Keplerian elements as

$$R_1 = \frac{2}{3}\frac{\mu}{a^3} J R_e^2 \left[\frac{1}{2} - \frac{3}{4} sin^2\epsilon + \frac{3}{4} sin^2\epsilon\ cos2\Phi\right] \tag{3}$$

and

$$R_2 = \mu\left(\frac{m_s}{m_e + m_m}\right)\frac{a^2}{a_s^3}\left[\frac{1}{4} - \frac{3}{8}\beta^2 + \left(\frac{3}{4} - \frac{3}{8}\beta^2\right)cos2(\Phi' - u) + \frac{3}{8}\beta^2(cos2\Phi' + cos2u)\right] \tag{4}$$

where $J = 3/2(C - A)/m_e R_e^2$, $\mu = G(m_e + m_m)$, and $\beta = sinI$.

These equations are then averaged over one orbital period and all terms with a cosine vanish. This leaves the secular parts of the disturbing potentials (secular refers to the fact that these parts are not averaged to zero over one period of the short time scale, and are thus the parts of the disturbing potentials which carry an impact on the precessional motion of the system). As mentioned, R_3 is then derived by replacing μ, a and ϵ in R_1 with Gm_e, a_s and γ respectively.

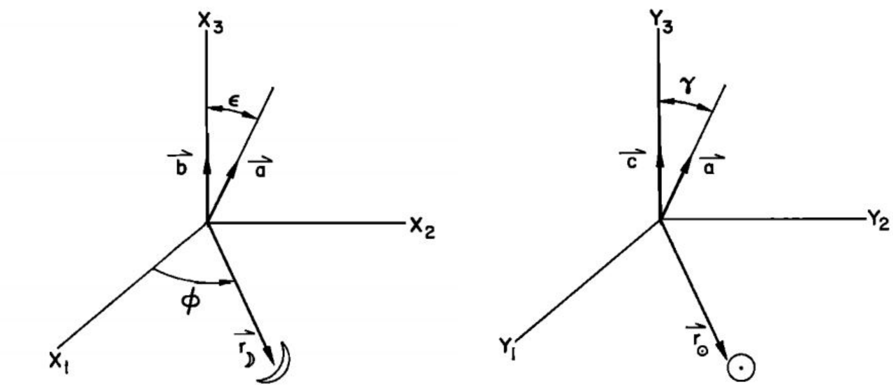

Figure 2: This figure shows the coordinate systems used to describe the Moon's motion (left) and the Sun's motion (right) from [4].

These secular disturbing potentials can then be differentiated with respect to there Keplerian elements (ϵ, I, and γ) and multiplied by the appropriate mass to yield equations for the secular torques in the system (the torques which affect the precessional motion). Making a change in coordinates to a set of unit vectors normal to the equator (**a**), lunar orbit (**b**), and ecliptic planes (**c**), as shown in figures 2, the equations for secular torque can be written as

$$L_1 = -\frac{\mu m_m}{a^3} J R_e^2 (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \times \mathbf{b}), \tag{5}$$

$$L_2 = \frac{3}{4}\mu \left(\frac{m_s m_m}{m_e + m_m} \right) \frac{a^2}{a_s^3} (\mathbf{b} \cdot \mathbf{c})(\mathbf{b} \times \mathbf{c}), \tag{6}$$

and

$$L_3 = -\frac{G m_e m_s}{a_s^3} J R_e^2 (\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \times \mathbf{c}). \tag{7}$$

It is important to note that these equations are analogous to the disturbing potential equations, in that L_1 is the torque exerted on the lunar orbit by the Earth's figure, L_2 is the torque exerted on the lunar orbit by the Sun, and L_3 is the torque exerted on the Sun by the Earth's figure. The reverse torques are simply the opposite signs of these. Using this fact, three values are defined relating these torques which are used in further derivation. These are

$$L = \frac{\mu m}{a^3} J R_e^2, \quad K_1 = \frac{\mu m_s}{a_s^3} J R_e^2, \quad K_2 = \frac{3}{4}\mu \left(\frac{m_s m_m}{m_e + m_m} \right) \frac{a^2}{a_s^3}. \tag{8}$$

There are a total of four elements of the system who's changes in the long time scale are of particular interest, the angular momenta H and h of the Earth's spin and the lunar orbit, the total angular momentum normal to the ecliptic Λ, and the total potential energy χ. It is trivial to show that these values are otherwise constant on the short and intermediate time scale. The angular momentum of the Earth's spin can be used as a measure of the Earth's angular velocity and the angular momentum of the lunar orbit can be used to find the Earth-Moon distance. The values of Λ and χ are of importance in solving the precessional equations, which will be discussed shortly. Since the equations for these values are of more importance in the long time scale, they will be described in the next section. Here, we are only interested in two equations which relate these values which are important to the precessional equations.

Making another change in coordinates to a notation which relates the unit vectors to the Keplerian elements, we can use the secular torque equations, as well as equations for H, h, Λ, and χ, a set of four precessional equations can be derived. The new coordinates are $x = (\mathbf{a} \cdot \mathbf{c}) = cos\gamma$, $y = (\mathbf{b} \cdot \mathbf{c}) = cosI$, $z = (\mathbf{a} \cdot \mathbf{b}) = cos\epsilon$, and $w = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = sin\epsilon$. The precessional equations are then

$$\begin{aligned}
\frac{dx}{dt} =& \frac{L}{H} zw \\
\frac{dy}{dt} =& -\frac{L}{h} zw \\
\frac{dz}{dt} =& \left(\frac{K_2}{h} y - \frac{K_1}{H} x\right) w \\
\frac{dw}{dt} =& \frac{L}{H} z(yz - x) - \frac{L}{h} z(xz - y) + \left(\frac{K_2}{h} y - \frac{K_1}{H} x\right)(xy - z).
\end{aligned} \tag{9}$$

These equations are solved numerically as described in section 3. However, to solve these equations initial values are required. In order to find these initial values three equations can be derived to relate x, y, z, and w. They are

$$\begin{aligned}
\Lambda =& Hx + hy \\
\chi =& K_1 x^2 + K2y^2 + Lz^2 \\
w^2 =& 1 - x^2 - y^2 - z^2 + 2xyz.
\end{aligned} \tag{10}$$

Making the assumption $w = 0$, since the solutions of the precessional equations are periodic and w is used as a check for x, y, and z, a sixth order polynomial in z can be found. This polynomial has two real roots and one is chosen as the initial value of z, which is then used to find the initial values of x and y.

2.4 Motion on the Long Time Scale

As mentioned previously, motion in the long time scale is due to tidal friction in the system. Modeling tidal friction is a complex process and thus the results of MacDonald [3] will be assumed in the form they are derived by him. Both Goldreich [1] and Touma and Wisdom [2] detail these results and the equations presented here and the values of constants and variables are a conglomerate from all three sources.

First, we must define equations for the four variables we are interested in the evolution of, H, h, Λ, and χ. Equations governing these values were first derived on the intermediate time scale using the secular torque equations, and the derivation here simply augments (via addition) these equations to include tidal torques. Since the effects of secular torque are negligible on the long time scale, those terms vanish and we are left with the following equations

$$\begin{aligned}
\frac{dH}{dt} &= T_e \cdot \mathbf{a} \\
\frac{dh}{dt} &= T_m \cdot \mathbf{b} \\
\frac{d\Lambda}{dt} &= (T_e + T_m) \cdot \mathbf{c} \\
\frac{d\chi}{dt} &= \frac{2K_1}{H} x T_e \cdot \mathbf{c} + \frac{2K_2}{h} y (T_m \cdot \mathbf{c} + y T_m \cdot \mathbf{b}) + 2Lz\left(\frac{T_e \cdot \mathbf{b}}{H} + \frac{T_m \cdot \mathbf{a} - 4zT_m \cdot \mathbf{b}}{h}\right).
\end{aligned} \tag{11}$$

The values T_e and T_m are the equations for tidal friction, and are most conveniently defined in terms of the precessional variables x, y, z, and w. Defining unit vectors along the axes used in defining the vectors **a**, **b**, and **c**, shown in figure 2, we can show that either T_e or T_m can be defined as $T = T_1\mathbf{e_1} + T_2\mathbf{e_2} + T_3\mathbf{e_3} = T_1'\mathbf{f_1} + T_2'\mathbf{f_2} + T_3'\mathbf{f_3}$, where either the T or T' notations can be used with regards to convenience. Henceforth, for MacDonald tides, only the T terms will be used. Ultimately, using this notation, we can use the following to resolve the dot products in equations (11)

$$\begin{aligned}
T \cdot \mathbf{a} &= T_2(1 - z^2)^{1/2} + T_3 z \\
T \cdot \mathbf{b} &= T_3 \\
T \cdot \mathbf{c} &= T_1 \frac{w}{(1 - z^2)^{1/2}} + T_2 \frac{(x - yz)}{(1 - z^2)^{1/2}} + T_3 y.
\end{aligned} \tag{12}$$

The values of T_1, T_2, and T_3 are then taken from MacDonald assuming a circular orbit. In Goldreich's presentation of these equations he includes an incorrect factor of n, which is corrected by Touma and Wisdom and also removed here. In defining these equations we also need new quantities $\alpha = n_m/\Omega_e$, the ratio of the mean motion of the moon and the Earth's angular velocity, $A = \frac{3}{2} G m_m R_e^5 k_2$, $q^2 = (1 - z^2)/(1 + \alpha^2 - 2)$, and $q'^2 = 1 - q^2$. Here, $k_2 = 0.29$ is a dimensionless Love number which can be found by observing the Chandler wobble. A set of integrals known as the Complete Eliptic Integrals of the First and Second kind is also needed and are defined by $F(q)$ and $K(q)$ as follows.

$$\begin{aligned}
F(q) &= \int_0^{2\pi} \frac{d\phi}{(1 - q^2 sin^2\phi)^{1/2}} \\
K(q) &= \int_0^{2\pi} (1 - q^2 sin^2\phi)^{1/2} d\phi
\end{aligned} \tag{13}$$

In section 3 these integrals are solved using the built in Eliptic integrals in Python.

Using these new quantities, the tidal torque equations for T_1, T_2, and T_3 are then

$$\begin{aligned}
T_{1m} = -T_{1e} &= 0 \\
T_{2m} = -T_{2e} &= \frac{2m_m A}{\pi a^6} \frac{1}{q} [K(q) - q'^2 F(q)] sin2\delta \\
T_{3m} = -T_{3e} &= \frac{2m_m A}{\pi a^6} q' F(q) sin2\delta,
\end{aligned} \tag{14}$$

where δ is the phase lag due to the fact that "High Tide" doesn't occur directly under the Moon's orbit, but slightly in front of it in the direction of the Earth's rotation due to the friction between the Earth's rotation and its oceans.

Thus, the set of equations which must be solved is given by (11) which are expanded using the dot products (12) as well as the expressions (14). These equations along with the precessional equations (9) form the basis for the numerical analysis in the next section.

3 Numerical Analysis

To implement our results with the formulated model, a numerical approach was used within the Python programming language. Although the precessional equations within the short and intermediate time scales are capable of being solved analytically, the long time scale must be solved using a numerical approach. As a result of this, the short and intermediate time scales are solved numerically as they will already be present within our environment. The primary foundation of the model to solve the equations on the long time scale is an RK4 algorithm, with each step of one million years that conducts several required sub operations.

Figure 3 provides a flowgraph that illustrates our numerical formulation, and algorithm 1 provides a high level algorithmic overview of our Python implementation. As mentioned previously, unlike Goldreich's formulation with varying time step width to estimate error, we capitalize on the potential of modern computers by running at a small (relatively) step width of one million years. Using this configuration, the results took approximately 36 hours to complete running on a High Performance Computer node within the University of Arizona's ECE department. The results of these experiments are discussed next.

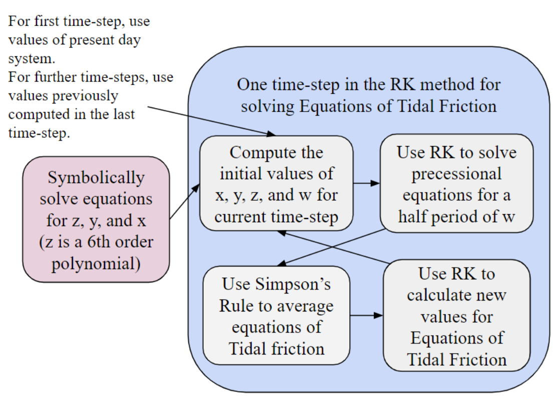

Figure 3: Numerical Method Flowgraph

Algorithm 1: Modified Runge Kutta for Earth-Moon Modeling

Result: a, H, Λ, χ values for each step of integration period
Calculate initial values of ω_E, a, H, Λ, χ;
Symbolically solve polynomials for z, y, x;
while *time* $<$ *steps * step size* **do**
  Calculate h, L, K_1, K_2;
  Substitute values for z polynomial and solve;
  Substitute values for y polynomial and solve;
  Substitute values for x polynomial and solve;
  while *sgn(prior w value) is sgn(current w value)* **do**
    Calculate 4 Runge Kutta terms for x, y, z, w;
    Append to solution list;
  end
  Average x, y, z, w over precessional period;
  Calculate $\Omega_E, \alpha, q, q', T_E, T_M$;
  Calculate 4 Runge Kutta terms for a, H, Λ, χ;
  Append to solution list;
end

3.1 Numerical Results

Here, we present the data obtained for the several key intermediate and long-scale quantities mentioned previously. Figure 4 depicts the solution for precessional variable z over the length of one period. Our data suggests the precessional period is approximately 17.7 years, which is consistent with related work [5]. This is the only solution presented for one precessional period as the other precessional quantities x and y have much longer periods than z, and therefore do not change much over one period of z.

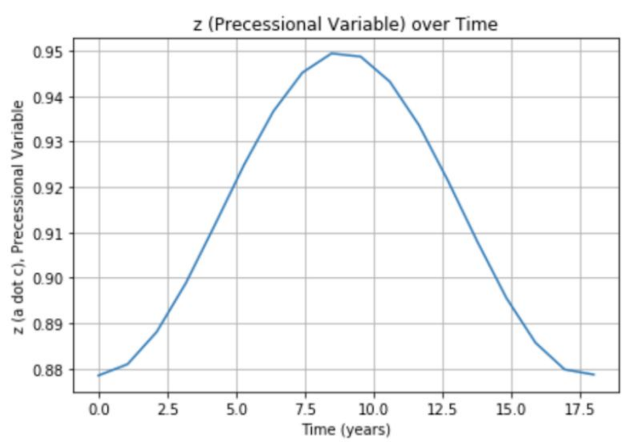

Figure 4: This plot shows the results for solving the precessional equations for the current day over one period z, which is the same as one period of w. This plot shows that the period of z is about 17.7 years.

Figures 5 and 6 present the projected future evolution of Earth's length of day and lunar distance over the next 10 billion years. While 10 billion years is unreasonable due to other dynamics in the Solar System such as the expansion of the Sun, solving over this longer timescale allows for trends within the model to be more clearly observed. The trends and magnitudes of these results are consistent with those of Goldreich, Touma, and Wisdom [1, 2] as when length of day is plotted against the Earth-Moon distance, the plot is concave up, but the amount of difference is approximately 15 hours at the 10 billion year mark.

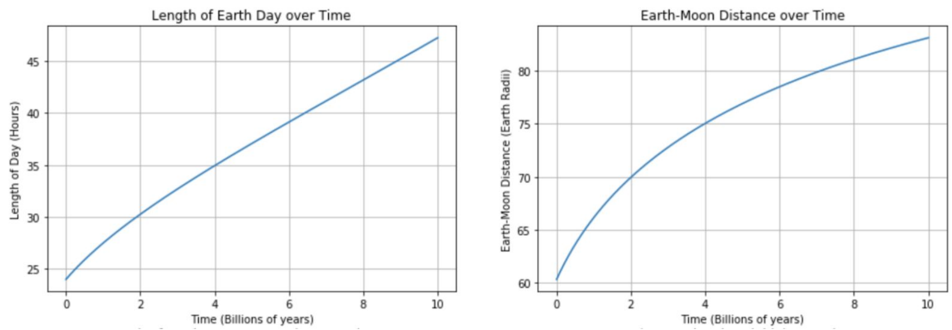

Figure 5: This figure shows the numerical results for the length of day and the Earth-Moon distance found by our model plotted against time.

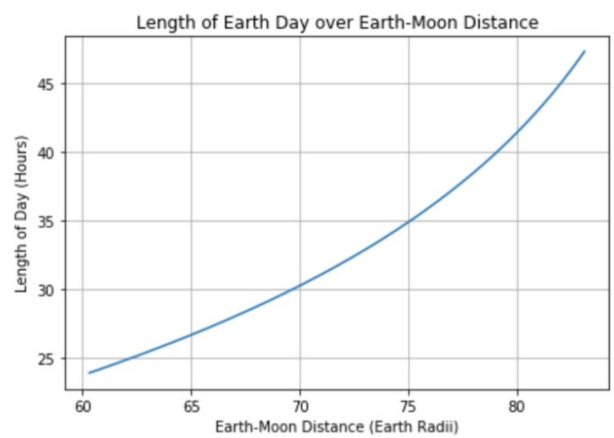

Figure 6: This figure shows the Length of Day plotted against the earth-Moon distance, which is an important plot in order to see that the length of day should be growing faster than the earth-Moon distance.

Finally, in Figure 7, we run our model backwards in time to estimate historical behavior of the Earth-Moon system. As the literature suggests [1, 2] the trends of our model are consistent with the expected past behavior of the planetary system as the Earth's spin speeds up as the model goes backwards in time.

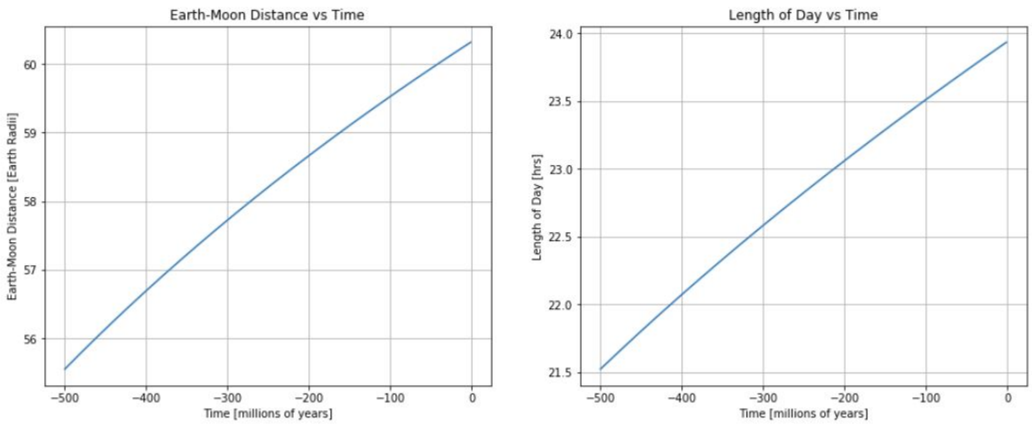

Figure 7: This figure shows solutions from the model going backwards in time, indicated by the negative values of time up to the present day at time 0.

4 Conclusion

The Moon-Earth system is incredibly complex and thereby impractical to calculate analytically entirely. The creation of our numerical model allows us and potentially future researchers to project the behavior of the Moon in its future, paving way for work in the area of tides, sedimentation rates, and more [3]. Prediction of Earth's future tide behavior, day length, and modeling the future behavior and environment of our solar system are all possibilities with studies within this space.

Some of the challenges we faced during the course of this modeling project included calculating system parameters both for initial values and within each time-step. Establishing and tweaking the flow of this process was critical to ensure the validity and accuracy of our model. In the end, we believe our goal to implement a model following the work of those formulated previously backed with modern computational capability was achieved.

Most importantly, the mystery of the Earth Moon system can finally be solved. Even though the transfer of energy due to tidal friction incurs some loss of total energy through heat dissipation from the Earth's oceans, we notice that the total amount of angular momentum within the system is conserved. Even when looking 10 billion years into the future or 750 million years into the past, our model suggests this quantity remained conserved. Alongside this fact, we know the Earth's rotation is slowing down and thereby decreasing in its angular momentum. As the masses of both the Earth and the Moon remain constant, this decrease must result in an increase of the angular momentum of the Moon, thereby expanding its orbit. This explanation is rather counter intuitive, and not immediately apparent, but is verified and consistent with the result of our model. With further research into this application space, perhaps many more related mysteries that surround our solar system and beyond can finally be solved.

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5 Appendix A: Variable Descriptions

Variable Symbol	Description
G	Gravitational Constant
m_e	Mass of the Earth
m_m	Mass of the Moon
m_s	Mass of the Sun
R_e	Radius of the Earth
R_M	Radius of the Moon
R_S	Radius of the Sun
I	Equivalent sphere moment of inertia of Earth
Ω_E	Earth's spin angular velocity
A	Principal moment of inertia about rotation axis
C	Principal moment of inertia about equatorial plane
k_s	Secular Love number
a	Semimajor axis of the Moon's orbit about the Earth
a_S	Semimajor axis of the Earth's orbit about the Sun
S	Angle between Earth-Moon and Earth-Sun center lines
θ	Latitude of the Moon's position
ϵ	Inclination of the Moon from the equator plane
Φ	Angle in the orbit measured from the ascending node on the equator plan
γ	Obliquity of the earth's equator to the ecliptic
u	Position in orbit as measured from the Moon's ascending node on the ecliptic
R_1	Disturbing potential felt by the Moon due to Earth
R_2	Disturbing potential felt by the Moon due to Sun
R_3	Disturbing potential felt by the Sun due to the Earth
$\overline{R_1}$	Time average of R_1 over one lunar orbit period
$\overline{R_2}$	Time average of R_2 over
$\overline{R_3}$	Time average of R_3 over
$\mathbf{a}$	Unit vector in direction normal to the equator plane
$\mathbf{b}$	Unit vector in direction normal to the lunar orbit plane
$\mathbf{c}$	Unit vector in direction normal to the ecliptic plane
L_1	Torque exerted by the Earth on the Moon
L_2	Torque exerted by the Earth on the Sun
L_3	Torque exerted by the Sun on the Moon
H	Scalar angular momenta of the Earth's spin
h	Scalar angular momenta of the lunar orbit
Λ	Component of total angular momentum in the Earth-Moon system that is normal to the ecliptic
χ	Sum of potential energies

Variable Symbol	Description
x	Used as for more compact notation, $x = \mathbf{a} \cdot \mathbf{c}$
y	Used as for more compact notation, $y = \mathbf{b} \cdot \mathbf{c}$
z	Used as for more compact notation, $z = \mathbf{a} \cdot \mathbf{b}$
w	Used as for more compact notation, $w = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})$
T_e	Tidal torque acting on the Earth
T_m	Tidal torque acting on the Moon
r	Radius of the Earth to the Moon
r_s	Radius of the Sun to the Earth