Chapter 13: Complex Numbers

Sections 13.1 & 13.2

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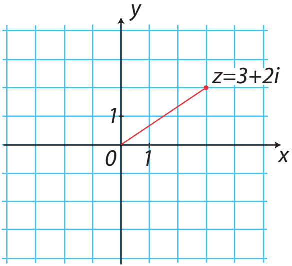
  - The complex number $z = x + iy$ may be represented in the complex plane as the point with cartesian coordinates (x, y).



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- If z_1 and z_2 are two complex numbers, then

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}. \tag{2}$$

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- **Examples:** Evaluate the following
  - $|i|$
  - $|2 - 3i|$

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- Get used to writing a complex number in the form

$$z = (\text{real part}) + i\ (\text{imaginary part}),$$

no matter how complicated this expression might be.

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  - Find $\Im m\left(\frac{1}{\bar{z_1}^3}\right)$.

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  - # 13.2.27: Solve $z^2 - (8-5i)z + 40 - 20i = 0$.

3. Polar coordinates form of complex numbers

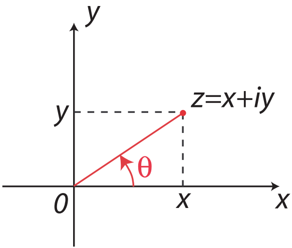
- In polar coordinates,

  $$x = r\cos(\theta), \qquad y = r\sin(\theta),$$

  where

  $$r = \sqrt{x^2 + y^2} = |z|,$$

  is the modulus of z.



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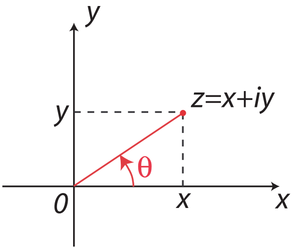

- The angle θ is called the argument of z. It is defined for all $z \neq 0$, and is given by

$$\arg(z) = \theta = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x \geq 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \end{cases} \pm 2n\pi.$$

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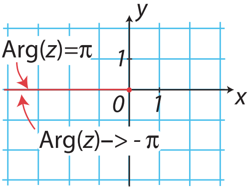
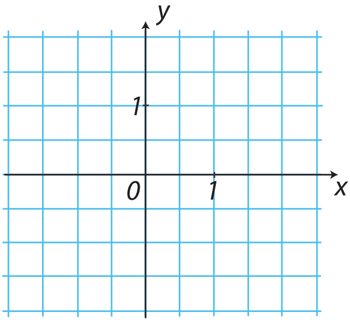
- Note that $\mathrm{Arg}(z)$ jumps by -2π when one crosses the negative real axis from above.



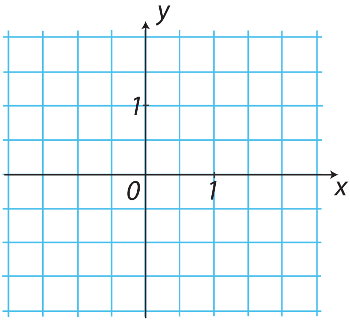
Principal value $\mathrm{Arg}(z)$ (continued)

- **Examples:**
  - Find the principal value of the argument of $z = 1 - i$.



Principal value $\mathrm{Arg}(z)$ (continued)

- **Examples:**
  - Find the principal value of the argument of $z = 1 - i$.
  - Find the principal value of the argument of $z = -10$.



Polar and cartesian forms of a complex number

- You need to be able to go back and forth between the polar and cartesian representations of a complex number.

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  - Convert $\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$ to cartesian coordinates.
  - What is the cartesian form of the complex number such that $|z| = 3$ and $\mathrm{Arg}(z) = \pi/4$?

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- This formula is extremely useful for calculating powers and roots of complex numbers, or for multiplying and dividing complex numbers in polar form.

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$$z^n = |z|^n \exp(in\theta) = |z|^n (\cos(n\theta) + i\sin(n\theta)).$$
3. In particular, if z is on the unit circle ($|z| = 1$), we have
$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$

This is De Moivre's formula.

Integer powers of a complex number (continued)

- **Examples of application:**
  - Trigonometric formulas

$$\begin{cases} \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \\ \sin(2\theta) = 2\sin(\theta)\cos(\theta). \end{cases} \tag{3}$$

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  - Find $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Product of two complex numbers

- The product of $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$ is

$$\begin{aligned} z_1 z_2 &= (r_1 \exp(i\theta_1))\ (r_2 \exp(i\theta_2)) \\ &= r_1 r_2 \exp(i(\theta_1 + \theta_2)). \end{aligned} \tag{4}$$

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- As a consequence,

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- As a consequence,

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- We can use Equation (4) to show that

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \, , \\ \sin(\theta_1 + \theta_2) &= \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \, . \end{aligned} \tag{5}$$

Ratio of two complex numbers

- Similarly, the ratio $\frac{z_1}{z_2}$ is given by

$$\frac{z_1}{z_2} = \frac{r_1 \exp(i\theta_1)}{r_2 \exp(i\theta_2)} = \frac{r_1}{r_2} \exp(i(\theta_1 - \theta_2)).$$

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- **Example:** Assume $z_1 = 2 + 3i$ and $z_2 = -1 - 7i$. Find $\left|\frac{z_1}{z_2}\right|$.

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2. Then take the n-th root (or the $1/n$-th power)
$$\sqrt[n]{z} = z^{1/n} = r^{1/n} \exp\left(i \frac{\theta + 2p\pi}{n}\right) = \sqrt[n]{r} \exp\left(i \frac{\theta + 2p\pi}{n}\right).$$

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3. There are thus n roots of z, given by

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \quad k = 0, \cdots, n-1.$$

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- The n-th roots of unity are given by

$$\sqrt[n]{1} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) = \omega^k, \qquad k = 0, \cdots, n-1$$

where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$.

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  where $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$.
- In particular, if w_1 is any n-th root of $z \neq 0$, then the n-th roots of z are given by

$$w_1,\ w_1\omega,\ w_1\omega^2, \cdots,\ w_1\omega^{n-1}.$$

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  - Find the three cubic roots of 1.
  - Find the four values of $\sqrt[4]{i}$.
  - Give a representation in the complex plane of the principal value of the eighth root of $z = -3 + 4i$.

Triangle inequality

- If z_1 and z_2 are two complex numbers, then

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

This is called the triangle inequality. Geometrically, it says that the length of any side of a triangle cannot be larger than the sum of the lengths of the other two sides.

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- More generally, if $z_1, z_2, \ldots, z_n$ are n complex numbers, then

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$