Part One. Linear Algebra

Column expansion

1. $A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix}$

$$\det A = -(1)\cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + (-1) \det \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$= -1 \cdot (-1) + (-1)\cdot 6 = 1-6 = -5$$

2. $A = \begin{pmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{pmatrix} \xrightarrow{\text{Row 2} - \text{Row 1}\left(\frac{b}{a}\right)} \begin{pmatrix} a & b & 0 & 0 \\ 0 & a-\frac{b^2}{a} & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{pmatrix}$

$\xrightarrow{\text{Row 4} - \text{Row 3}\cdot\left(\frac{d}{c}\right)} \begin{pmatrix} a & b & 0 & 0 \\ 0 & a-\frac{b^2}{a} & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & c-\frac{d^2}{c} \end{pmatrix}$

$$\det(A) = a\cdot\left(a-\frac{b^2}{a}\right)\cdot c\cdot\left(c-\frac{d^2}{c}\right)$$

$$= (a^2-b^2)\cdot(c^2-d^2)$$

3. $A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix}$

$$\det A = 1 \det \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} + 3 \det \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

$$= 1 \cdot 0 + 3 \cdot 2 = 6$$

$$A^{-1} = \frac{\begin{pmatrix} +(0) & -(1) & +(2) \\ -(-6) & +(4) & -(2) \\ +(0) & -(-3) & +(0) \end{pmatrix}^T}{\det A}$$

$$= \frac{\begin{pmatrix} 0 & -1 & 2 \\ 6 & 4 & -2 \\ 0 & 3 & 0 \end{pmatrix}^T}{6} = \frac{\begin{pmatrix} 0 & 6 & 0 \\ -1 & 4 & 3 \\ 2 & -2 & 0 \end{pmatrix}}{6} = \boxed{\begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{6} & \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}}$$

A^{-1}

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4. $A = \begin{pmatrix} -1 & 1 & 3 \\ 2 & 8 & 9 \\ 3 & 7 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 3 \\ 0 & 10 & 15 \\ 0 & 10 & 15 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 3 \\ 0 & 10 & 15 \\ 0 & 0 & 0 \end{pmatrix}$

$\det(A) = (-1) \cdot (10) \cdot (0) = 0.$

Since "det A" needs to be divided through when ~~[illegible]~~ we try to find A^{-1},

when $\det A = 0$, the inverse doesn't exist !!

Problem 5

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = -\lambda^3-\lambda^2+21\lambda+45 = -(\lambda-5)(\lambda+3)^2 = 0$$

$\Rightarrow \lambda_1 = 5, \quad \lambda_2 = \lambda_3 = -3$

(1) $\lambda_1 = 5$

$$A-\lambda_1 I = \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{pmatrix}$$

Choose $x_3 = -1$, then from $-\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0 \Rightarrow x_2 = 2$

from $-7x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = 1$

$\Rightarrow$ for $\lambda_1 = 5$, $\quad \vec{x_1} = (1 \quad 2 \quad -1)^T$

(2) $\lambda_2 = \lambda_3 = -3$

$$A-\lambda_2 I = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Choose $x_2 = 1$, $x_3 = 0$, from $x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = -2$

Choose $x_2 = 0$, $x_3 = 1$, $\cdots\cdots \Rightarrow x_1 = 3$

$\Rightarrow$ for $\lambda_2 = \lambda_3 = -3 \qquad \vec{x_2} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x_3} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Problem 6

$$|A-\lambda I| = \begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix} = \lambda^2 - 2\cos\theta\,\lambda + 1$$

$$\Rightarrow \lambda_1 = \cos\theta + \sin\theta\, i \quad , \quad \lambda_2 = \cos\theta - \sin\theta\, i$$

(1) $\lambda_1 = \cos\theta + \sin\theta\, i$

$$A-\lambda I = \begin{pmatrix} -\sin\theta\, i & -\sin\theta \\ \sin\theta & -\sin\theta\, i \end{pmatrix} \longrightarrow \begin{pmatrix} -\sin\theta\, i & -\sin\theta \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

(2) $\lambda_2 = \cos\theta - \sin\theta\, i$

$$A-\lambda I = \begin{pmatrix} \sin\theta\, i & -\sin\theta \\ \sin\theta & \sin\theta\, i \end{pmatrix} \longrightarrow \begin{pmatrix} \sin\theta\cdot i & -\sin\theta \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \vec{x}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Part II

Problem 1

$\left.\begin{array}{l} y_1 = \cos wx \\ y_2 = \sin wx \end{array}\right\} \Rightarrow \pm wi$ are the roots of the characteristic equation

$\Rightarrow$ The characteristic equation is $\lambda^2 + w^2 = 0$

$\Rightarrow$ The ODE is $y'' + w^2 y = 0$

$$W(\cos wx, \sin wx) = \begin{vmatrix} \cos wx & \sin wx \\ -w\sin wx & w\cos wx \end{vmatrix} = w\cos^2 wx + w\sin^2 wx = w$$

For $w \neq 0$, y_1 and y_2 are independent

Problem 2

$\left.\begin{array}{l} y_1 = e^x \\ y_2 = xe^x \end{array}\right\} \Rightarrow 1$ is a double root of the characteristic equation

$\Rightarrow$ The characteristic equation is $(\lambda-1)^2 = \lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow$ The ODE is $y'' - 2y' + y = 0$

$$w(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x}$$

$$= e^{2x} \neq 0$$

$\Rightarrow y_1$ and y_2 are independent

Part Two ODEs

3. $$\frac{d^3y}{dx^3} - \frac{dy}{dx} = 0$$

characteristic eqn : $\lambda^3 - \lambda = 0$

$$\lambda(\lambda^2 - 1) = 0$$

$$\lambda(\lambda+1)(\lambda-1) = 0$$

$$\lambda = 0, \quad \lambda = -1, \quad \lambda = 1$$

The general form of solution is

$$y = c_1 e^{0x} + c_2 e^{-x} + c_3 e^{x}$$

$$\Rightarrow \quad y = c_1 + c_2 e^{-x} + c_3 e^{x}$$

where c_1, c_2, c_3 are arbitrary constants

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4. $$\frac{d^2y}{dx^2} - 4y = 0$$

Note : Don't put "4λ" here !!

(b) characteristic eqn is $\lambda^2 - 4 = 0$

$$(\lambda + 2)(\lambda - 2) = 0$$

$$\lambda = -2 \qquad \lambda = 2$$

The general form of solution is $y = c_1 e^{-2x} + c_2 e^{2x}$

$y(0) = c_1 + c_2$ has to match the initial cond $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$

$y'(0) = -2c_1 + 2c_2$

$$\Rightarrow \begin{cases} c_1 + c_2 = 1 \\ -2c_1 + 2c_2 = 0 \end{cases} \Rightarrow \quad c_1 = c_2 = \frac{1}{2}$$

So, the solution to the initial value problem is

$$y = \frac{1}{2} e^{-2x} + \frac{1}{2} e^{2x}$$

(continued on next page)

A comment on 4(a):

Apparently, the eqn has a solution $y = \frac{1}{2}e^{-2x} + \frac{1}{2}e^{2x}$

(We found it!) and it is unique because all the coefficients are uniquely determined. (in other words, there's no ambiguity any where)

But, actually 4(a) can be answered even without solving 4(b). since the theory says that for a linear ODE, if the coefficients are continuous, then the solution exist and is unique.

5. $$\frac{d^2y}{dx^2} + 5y = \cos x$$

characteristic eqn : $\lambda^2 + 5 = 0 \Rightarrow \lambda = \pm\sqrt{5}\,i$

So, the homogeneous solution is $y_h = C_1\cos(\sqrt{5}x) + C_2\sin(\sqrt{5}x)$

To find a particular solution y_p, we guess the form of y_p first, and then find out the undetermined coefficient.

We guess $y_p = A\cos x + B\sin x$ because 1 is not among $\pm\sqrt{5}$

Plug it into the eqn:

$$y_p' = -A\sin x + B\cos x, \quad y_p'' = -A\cos x - B\sin x$$

$$y_p'' + 5y_p = (-A\cos x - B\sin x) + 5(-A\cos x - B\sin x)$$

has to match the Right Hand side $\cos x$.

$$\text{So, } \begin{cases} -A - 5A = 1 \\ -B - 5B = 0 \end{cases} \Rightarrow \begin{array}{l} A = -\frac{1}{6} \\ B = 0 \end{array}$$

So, $y_p = -\frac{1}{6}\cos x$

So, the general form of solution is $y = y_h + y_p$

$$\Rightarrow \quad y = C_1\cos(\sqrt{5}x) + C_2\sin(\sqrt{5}x) - \frac{1}{6}\cos x$$

where C_1, C_2 are arbitrary constants

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6. $\frac{d^2y}{dx^2} - 4y = xe^{2x}$

characteristic eqn : $\lambda^2 - 4 = 0 \quad \Rightarrow (\lambda+2)(\lambda-2) = 0 \Rightarrow \lambda = -2, \lambda = 2$

So. $y_h = C_1 e^{-2x} + C_2 e^{2x}$ (homogeneous part)

Note the Right hand side is xe^{2x} and the number "2" in the exponent is one of the solutions to the characteriste eqn then we would guess a particular solution should look like

$$y_p = Ax^2e^{2x} + Bxe^{2x}$$

plug in the eqn.

$$y_p' = 2Axe^{2x} + 2Ax^2e^{2x} + Be^{2x} + 2Bxe^{2x}$$

$$y_p'' = 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2e^{2x} + 2Be^{2x} + 2Be^{2x} + 4Bxe^{2x}$$

$$= (4B+2A)e^{2x} + (4B+8A)xe^{2x} + 4Ax^2e^{2x}$$

$$y_p'' - 4y = (4B+2A)e^{2x} + (4B+8A)xe^{2x} + \cancel{4Ax^2e^{2x}}$$

$$- \cancel{4Ax^2e^{2x}} - 4Bxe^{2x}$$

$$= (4B+2A)e^{2x} + 8Axe^{2x}$$

has to match the Right Hand side xe^{2x}

$$\begin{cases} 4B + 2A = 0 \\ 8A = 1 \end{cases} \Rightarrow \begin{array}{l} A = \frac{1}{8} \\ B = -\frac{A}{2} = -\frac{1}{16} \end{array}$$

So, $Y_p = \frac{1}{8} x^2 e^{2x} - \frac{1}{16} x e^{2x}$

The general form of solution is $y = Y_h + Y_p$

$$\Rightarrow y = C_1 e^{-2x} + C_2 e^{2x} + \frac{1}{8} x^2 e^{2x} - \frac{1}{16} x e^{2x}$$

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7. $\frac{d^2y}{dx^2} + y = 2\sin x$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm 1$

$\Rightarrow Y_h = C_1 \cos(x) + C_2 \sin(x)$

$Y_p = Ax\sin x + Bx\cos x$

$Y_p' = A\sin x + Ax\cos x + B\cos x - Bx\sin x$

$Y_p'' = A\cos x + A\cos x - Ax\sin x - B\sin x - B\sin x - Bx\cos x$

$Y_p'' + Y_p = (2A\cos x - B\cancel{x}\cos x - 2B\sin x - \cancel{Ax\sin x}) + (A\cancel{x\sin x} + Bx\cancel{\cos x})$

has to match the right hand side. $2\sin x$

$$\begin{cases} 2A = 0 \\ -2B = 2 \end{cases} \Rightarrow \begin{array}{l} A = 0 \\ B = -1 \end{array} \Rightarrow Y_p = -x\cos x$$

The general form of solution is $y = Y_h + Y_p$

$$\Rightarrow y = C_1\cos x + C_2\sin x - x\cos x$$

Now, use the initial condition to determine C_1 and C_2

$$y(0) = C_1 + 0 - 0 = C_1$$

$$y'(x) = -C_1 \sin x + C_2 \cos x - \cos x + x \sin x$$

$$y'(0) = 0 + C_2 - 1 + 0 = C_2 - 1$$

they have to match $y(0) = 1$, $y'(0) = 0$

So $\begin{cases} C_1 = 1 \\ C_2 - 1 = 0 \end{cases} \Rightarrow \begin{matrix} C_1 = 1 \\ C_2 = 1 \end{matrix}$

So, the solution that satisfy the initial ~~value~~ condition

is $y = \cos x + \sin x - x \cos x$

8.
$$\begin{cases} \dfrac{dy_1}{dx} = y_1 + 2y_2 & ① \\ \dfrac{dy_2}{dx} = 5y_1 - 2y_2 & ② \\ y_1(0) = & ③ \\ y_2(0) = & ④ \end{cases}$$

Let $\vec{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 2 \\ 5 & -2 \end{pmatrix}$

Eqn ① & ② is then equivalent to

$$\frac{d\vec{Y}}{dx} = A\vec{Y}$$

First find the eigenvalues and eigenvectors of A

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 5 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) - 10$$

$$= \lambda^2 + \lambda - 2 - 10 \quad = \lambda^2 + \lambda - 12 = 0$$

$$(\lambda + 4)(\lambda - 3) = 0$$

$$\lambda = -4, \quad \lambda = 3$$

For $\lambda = -4$: $\begin{pmatrix} 1+4 & 2 \\ 5 & -2+4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} 5x_1 + 2x_2 = 0 \\ 5x_1 + 2x_2 = 0 \end{matrix}$

$\Rightarrow x_1 = -\frac{2x_2}{5} \Rightarrow \begin{pmatrix} -\frac{2}{5}x_2 \\ x_2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -2/5 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -4$

For $\lambda = 3$:
$$\begin{pmatrix} 1-3 & 2 \\ 5 & -2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{cases} -2x_1 + 2x_2 = 0 \\ 5x_1 - 5x_2 = 0 \end{cases}$$

$$\Rightarrow \quad x_1 = x_2 \quad \Rightarrow \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ is an eigenvector corresponding to $\lambda = 3$

So, the general form of solution is
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Y = c_1 e^{-4x}\begin{pmatrix} -\frac{2}{5} \\ 1 \end{pmatrix} + c_2 e^{3x}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e. $$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-4x}\left(-\frac{2}{5}\right) + c_2 e^{3x} \\ c_1 e^{-4x} + c_2 e^{3x} \end{pmatrix}$$

i.e. $y_1 = -\frac{2}{5}c_1 e^{-4x} + c_2 e^{3x}$

$y_2 = c_1 e^{-4x} + c_2 e^{3x}$

where c_1, c_2 are arbitrary constants

If the initial condition is given, c_1, c_2 can be determined

$y_1(0) = -\frac{2}{5}c_1 + c_2$

$y_2(0) = c_1 + c_2$

They have to match $y_1(0) = 2$, $y_2(0) = 9$

$$\begin{cases} -\frac{2}{5}c_1 + c_2 = 2 \\ c_1 + c_2 = 9 \end{cases} \Rightarrow c_2 = 9 - c_1 \Rightarrow -\frac{2}{5}c_1 + 9 - c_1 = 2$$

$$\Rightarrow -\frac{7}{5}c_1 = -7 \Rightarrow c_1 = 5 \Rightarrow c_2 = 4$$

So, the solution that satisfy the initial condition is

$y_1 = -2e^{-4x} + 4e^{3x}$, $\quad y_2 = 5e^{-4x} + 4e^{3x}$ unique solution.

9. $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & -2 & 3 \\ 0 & 5 & -4 \end{pmatrix}$. $\vec{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. Solve $\frac{d}{dt}\vec{Y} = A\vec{Y}$

Again, we need to find the eigenvalues and eigenvectors of A

$$\det\begin{pmatrix} 3-\lambda & 0 & 2 \\ 0 & -2-\lambda & 3 \\ 0 & 5 & -4-\lambda \end{pmatrix} = (3-\lambda)\det\begin{pmatrix} -2-\lambda & 3 \\ 5 & -4-\lambda \end{pmatrix}$$

$$=(3-\lambda)\left[(-2-\lambda)(-4-\lambda)-15\right] = (3-\lambda)(\lambda^2+6\lambda+8-15)$$

$$=(3-\lambda)(\lambda^2+6\lambda-7) = (3-\lambda)(\lambda+7)(\lambda-1) = 0$$

$\lambda = 3$. $\lambda = -7$. $\lambda = 1$

For $\lambda = 3$. $\begin{pmatrix} 3-3 & 0 & 2 \\ 0 & -2-3 & 3 \\ 0 & 5 & -4-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{matrix} 2x_3 = 0 \\ -5x_2+3x_3=0 \\ 5x_2-7x_3=0 \end{matrix} \Rightarrow \begin{matrix} x_3=0 \\ x_2=0 \\ x_1=\text{anything} \end{matrix}$

So, $\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$, and in particular, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector for $\lambda=3$

For $\lambda = -7$ $\begin{pmatrix} 3+7 & 0 & 2 \\ 0 & -2+7 & 3 \\ 0 & 5 & -4+7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{cases} 10x_1+2x_3=0 \\ 5x_2+3x_3=0 \\ 5x_2+3x_3=0 \end{cases} \Rightarrow \begin{matrix} x_1 = -\frac{1}{5}x_3 \\ x_2 = \frac{-3}{5}x_3 \end{matrix}$

So, $\begin{pmatrix} -\frac{1}{5}x_3 \\ -\frac{3}{5}x_3 \\ x_3 \end{pmatrix}$, and in particular $\begin{pmatrix} -1/5 \\ -3/5 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda=-7$

For $\lambda = 1$. $\begin{pmatrix} 3-1 & 0 & 2 \\ 0 & -2-1 & 3 \\ 0 & 5 & -4-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{cases} 2x_1+2x_3=0 \\ -3x_2+3x_3=0 \\ 5x_2-5x_3=0 \end{cases} \Rightarrow \begin{matrix} x_1=-x_3 \\ x_2=x_3 \end{matrix}$

So, $\begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix}$, and in particular $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda=1$

So, $Y = c_1e^{3t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2e^{-7t}\begin{pmatrix} -1/5 \\ -3/5 \\ 1 \end{pmatrix} + c_3e^{t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is the general form of solution

Convert the single second-order ODE

$$\frac{d^2y}{dx^2} - 4y = 0$$

into a first-order ODE system :

Let $\frac{dy}{dx} = z$ ①

then $\frac{dz}{dx} = \frac{d^2y}{dx^2} = 4y$ ② because $\frac{d^2y}{dx^2} = 4y$

$$\Rightarrow \begin{cases} \frac{dy}{dx} = z \\ \frac{dz}{dx} = 4y \end{cases} \Leftrightarrow \frac{d}{dz}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}$$

Find the eigenvalues and eigenvectors of A :

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 0-\lambda & 1 \\ 4 & 0-\lambda \end{pmatrix} = (-\lambda)(-\lambda) - 4 = \lambda^2 - 4 = 0$$

$$(\lambda+2)(\lambda-2) = 0$$

$$\lambda = -2 \quad \lambda = 2$$

For $\lambda = -2$. $\begin{pmatrix} 0-(-2) & 1 \\ 4 & 0-(-2) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} 2x_1 + x_2 = 0 \\ 4x_1 + 2x_2 = 0 \end{matrix}$

$$\Rightarrow x_2 = -2x_1 \Rightarrow \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ is an}$$

eigenvector for $\lambda = -2$

For $\lambda = 2$. $\begin{pmatrix} 0-2 & 1 \\ 4 & 0-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} -2x_1 + x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{matrix}$

$\Rightarrow x_2 = 2x_1 \Rightarrow \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector for $\lambda = 2$

so, $\begin{pmatrix} y \\ z \end{pmatrix} = c_1 e^{-2x}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{2x}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$y = c_1 e^{-2x} + c_2 e^{2x}$$

$$z = -2c_1 e^{-2x} + 2c_2 e^{2x}$$

But we are only interested in solving for y. (z is only auxillary in this case)

So, $y = c_1 e^{-2x} + c_2 e^{2x}$.

and we found it the same as the result we obtained in problem 2 by directly solving the eqn.

Problem 11:

$a_m = m!$

$\Rightarrow \frac{a_{m+1}}{a_m} = \frac{(m+1)!}{m!} = m+1 \longrightarrow \infty$ as $m \to \infty$

$\Rightarrow R = 1 / \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right| = 0$

this series converges only at $x=0$

Problem 12:

Let $t = x^3 \longrightarrow \sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} t^m$

$a_m = \frac{(-1)^m}{8^m}$

$\Rightarrow \frac{a_{m+1}}{a_m} = (-1) \cdot \frac{1}{8}$

$\Rightarrow R = 1 / \lim_{m\to\infty} \left| \frac{a_{m+1}}{a_m} \right| = 8$

$\Rightarrow |t| = |x^3| < 8 \Rightarrow |x| < 2 \Rightarrow R = 2$

Problem 13

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y' = 2xy$$

$$\Rightarrow 1 \cdot a_1 x^0 + \sum_{m=2}^{\infty} m a_m x^{m-1} = 2x \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} 2 a_m x^{m+1}$$

$$\Rightarrow a_1 + \sum_{s=0}^{\infty} (s+2) a_{s+2} x^{s+1} = \sum_{s=0}^{\infty} 2 a_s x^{s+1}$$

$$\Rightarrow a_1 = 0, \quad (s+2) a_{s+2} = 2 a_s$$

$$\Rightarrow a_{2n+1} = 0, \quad a_2 = a_0, \quad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \quad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \quad \cdots$$

$$\Rightarrow y = a_0 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) = a_0 e^{x^2}$$

Problem 14

$$y = \sum_{m=0}^{\infty} a_m x^m, \qquad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$y'' + y = 0,$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{s=0}^{\infty} (s+2)(s+1) a_{s+2} x^s = -\sum_{s=0}^{\infty} a_s x^s$$

$$\Rightarrow (s+2)(s+1) a_{s+2} = -a_s$$

$$\Rightarrow a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{a_0}{2!}, \quad a_3 = -\frac{a_1}{3!}, \quad a_4 = \frac{a_0}{4!}, \quad a_5 = \frac{a_1}{5!}$$

$$\Rightarrow y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots$$

$$\Rightarrow y = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= a_0 \cos x + a_1 \sin x$$

Problem 15

$$y=\sum_{m=0}^{\infty} a_m x^m, \quad y'=\sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y' = y + x$$

$$\Rightarrow \sum_{m=1}^{\infty} m a_m x^{m-1} = \sum_{m=0}^{\infty} a_m x^m + x$$

$$\Rightarrow \sum_{s=0}^{\infty}(s+1)a_{s+1} x^s = a_0 + (a_1+1)x + \sum_{s=2}^{\infty} a_s x^s$$

$$\Rightarrow a_1 = a_0, \quad 2a_2 = a_1+1 = a_0+1, \quad a_3 = \frac{a_2}{3} = \frac{a_0+1}{3!}, \quad \cdots$$

$$\Rightarrow y = a_0 + a_0 x + \frac{1}{2}(a_0+1)x^2 + \frac{1}{3!}(a_0+1)x^3 + \cdots$$
$$= (a_0+1)e^x - x - 1$$

Problem 16

For positive $\lambda = \nu^2$

a general solution for $y'' + \lambda y = 0$ is

$$y(x) = A\cos\nu x + B\sin\nu x$$

$y(0) = 0 \Rightarrow A = 0$

$y(5) = B\sin 5\nu = 0 \Rightarrow 5\nu = n\pi \Rightarrow \nu = \frac{n\pi}{5}, \quad n = \pm 1, \pm 2, \cdots$

$\Rightarrow \lambda = \nu^2 = \left(\frac{n\pi}{5}\right)^2 \quad (n=1,2,\cdots)$ (here $n=-1,-2\cdots$ give the same λ as $n=1,2,\cdots$)

take $B=1$, $y_n(x) = \sin\frac{n\pi}{5}x \quad (n=1,2,\cdots)$

Problem 17

For positive $\lambda = \nu^2$

a general solution for $y'' + \lambda y = 0$ is

$$y(x) = A\cos \nu x + B \sin \nu x$$

$y(0) = 0 \Rightarrow A = 0$

$y'(L) = B\nu \cos \nu \cdot L = 0 \Rightarrow \nu L = \frac{2n+1}{2}\pi$

$\Rightarrow \nu = \frac{2n+1}{2L}\cdot\pi \Rightarrow \lambda = \nu^2 = \left(\frac{2n+1}{2L}\pi\right)^2 \quad (n = 0, 1, 2, \cdots)$

take $B = 1 \Rightarrow y_n(x) = \sin\left(\frac{2n+1}{2L}\pi\, x\right) \quad (n = 0, 1, 2, \cdots)$

Problem 18

For positive $\lambda = \nu^2$

a general solution for $y'' + \lambda y = 0$ is

$$y(x) = A\cos \nu x + B \sin \nu x$$

$y(0) = y(2\pi) \Rightarrow A = A\cos 2\pi\nu + B\sin 2\pi\nu \quad (1)$

$y'(0) = y'(2\pi) \Rightarrow \nu B = -\nu A \sin 2\pi\nu + \nu B \cos 2\pi\nu \quad (2)$

$(1) \times A\nu + (2) \times B$

$\Rightarrow (A^2+B^2)\nu = (A^2+B^2)\nu \cdot \cos 2\pi\nu$

$\Rightarrow \cos 2\pi\nu = 1 \Rightarrow 2\pi\nu = 2\pi \cdot n \Rightarrow \nu = n \quad (n = 0, 1, 2, \cdots)$

$\Rightarrow \lambda = \nu^2 = n^2$

$\Rightarrow y_0 = 1,\ y_n = \cos nx,\ \sin nx \quad . \ n = 1, 2, \cdots$

(y_n are eigenfunctions)