Math 322 Section 3 Written Homework 3

1) The delta function $\delta(x)$ is defined to be the a function such that for any function $f(x)$,

$$\int_a^b \delta(x) f(x)\, dx = \begin{cases} f(0) & \text{if } a < 0 < b \\ 0 & \text{if } a \geq 0 \text{ or } b \leq 0 \end{cases}$$

Actually, it can be shown that no such function exists, but it still has a Fourier series since, for instance,

$$\int_{-\pi}^{\pi} \delta(x) \cos 3x dx = \cos(0) = 1.$$

1) Calculate $\int_a^b \delta\left(x - \frac{\pi}{2}\right) f(x)\, dx$ for any function $f(x)$.
Answer:

$$\int_a^b \delta\left(x - \frac{\pi}{2}\right) f(x)\, dx = \begin{cases} f\left(\frac{\pi}{2}\right) & \text{if } a < \frac{\pi}{2} < b \\ 0 & \text{otherwise} \end{cases}$$

The easiest way to see this is to substitute $y = x - \frac{\pi}{2}$.

2) Write the Fourier series for the delta function $\delta\left(x - \frac{\pi}{2}\right)$ gotten as an odd periodic half range expansion on the interval $[0, \pi]$.

Answer: For the odd periodic half range expansion, we have

$$b_n = \frac{2}{\pi} \int_0^{\pi} \delta\left(x - \frac{\pi}{2}\right) \sin nx\ dx = \frac{2}{\pi} \sin \frac{n\pi}{2}.$$

Note that if n is even, this is zero so the expansion only has odd n. We get the Fourier series

$$\frac{2}{\pi} \sin x - \frac{2}{\pi} \sin 3x + \frac{2}{\pi} \sin 5x - \frac{2}{\pi} \sin 7x + \cdots$$

or

$$\sum_{n=1}^{\infty} \frac{2}{\pi} \sin \frac{n\pi}{2} \sin nx$$

or even better

$$\sum_{n=0}^{\infty} \frac{2}{\pi} (-1)^n \sin(2n+1) x$$

3) Show that the series does not converge for $x = \frac{\pi}{2}$.

Answer: for $x = \frac{\pi}{2}$ we get the series

$$\frac{\pi}{2} (1 + 1 + 1 + 1 + 1 + \cdots)$$

which clearly does not converge.

4) Use separation of variables to find the solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ satisfies $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = \delta\left(x - \frac{\pi}{2}\right)$.

Answer: We start with solutions of the form $u(x,t) = X(x)T(t)$ and find that

$$XT' = X''T$$

and so

$$\frac{T'}{T} = \frac{X''}{X}$$

and since the left is a funciton of t only and the right is a funciton of x only, we must have that

$$\begin{aligned} T &= kT \\ X'' &= kX \end{aligned}$$

for some constant k. The boundary conditions lead to

$$X(0) = X(\pi) = 0$$

and hence we must have that

$$X_n(x) = B_n \sin nx$$

are solutions which satisfy the boundary condition, with

$$k = -n^2.$$

Thus we have

$$T_n(t) = e^{-n^2 t}.$$

We use superposition to get a general solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx.$$

Using the initial condition and the solution to question 3, we have that

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{2}{\pi} \sin \frac{n\pi}{2} e^{-n^2 t} \sin nx \\ &= \sum_{n=0}^{\infty} \frac{2}{\pi} (-1)^n e^{-(2n+1)^2 t} \sin(2n+1)x \\ &= \frac{2}{\pi} e^{-t} \sin x - \frac{2}{\pi} e^{-9t} \sin 3x + \frac{2}{\pi} e^{-25t} \sin 5x - \frac{2}{\pi} e^{-49t} \sin 7x + \cdots \end{aligned}$$

5) Explain why we say that the heat equation shrinks high frequency modes faster than low frequency modes.

Answer: The amplitude of the mode corresponding to function $\sin nx$ is at most $\frac{2}{\pi}e^{-n^2t}$, and so as t gets bigger, this shrinks to zero. For larger n, this shrinks to zero faster.