

# Math 413/513 Chapter 1 (from Friedberg, Insel, & Spence)

David Glickenstein

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## 1 Vector space definition

**Definition 1** A vector space  $V$  over a field  $F$  is a set together with two operations:

$$\begin{aligned} + : V \times V &\rightarrow V \\ \cdot : F \times V &\rightarrow V \end{aligned}$$

called addition and scalar multiplication. with the results denoted  $x + y$  and  $ax$  respectively, that satisfy the following axioms:

VS 1 (Commutativity of addition) For all  $x, y \in V$ , we have  $x + y = y + x$ .

VS 2 (Associativity of addition) For all  $x, y, z \in V$ , we have  $(x + y) + z = x + (y + z)$ .

VS 3 (Additive identity) There exists an element  $e$  in  $V$  such that  $e + x = x$  for all  $x \in V$ . We usually denote  $e$  as  $0$  or  $\vec{0}$ .

VS 4 (Additive inverse) For each  $x \in V$  there exists a  $y \in V$  such that  $x + y = 0$ . We usually denote  $y$  as  $-x$ .

VS 5 (Multiplication by 1) For each  $x \in V$ ,  $1x = x$ .

VS 6 (Associativity of scalar multiplication) For each  $a, b \in F$  and  $x \in V$ , we have  $a(bx) = (ab)x$ .

VS 7 (Distributivity) For each  $a \in F$  and  $x, y \in V$ , we have  $a(x + y) = ax + ay$ .

VS 8 (Distributivity) For each  $a, b \in F$  and  $x \in V$ , we have  $(a + b)x = ax + bx$ .

We sometimes call  $x + y$  the sum and  $ax$  the product. Elements of the field are called scalars and elements of the vector space are called vectors. We also have the difference  $x - y$  defined to be  $x + (-y)$ .

**Proposition 2** We can denote the finite sum  $x_1+x_2+\cdots+x_n$  without confusion due to VS 2.

**Proposition 3** We can denote the finite product  $a_1a_2\cdots a_kx$  without confusion due to VS 6.

**Proposition 4** In any vector space  $a\vec{0} = \vec{0}$  for any  $a \in F$ .

**Proof.** We know that by VS 7 that for any  $a \in F$ ,

$$a\vec{0} + a\vec{0} = a(\vec{0} + \vec{0}) = a\vec{0}.$$

Adding the additive inverse  $-(a\vec{0})$  to both sides yields that  $a\vec{0} = \vec{0}$ . ■

**Proposition 5** In any vector space,  $0x = \vec{0}$  for each  $x \in V$ .

**Proof.** We know that

$$0x + x = (0 + 1)x = 1x = x$$

by using VS 5 and VS 8. It follows that

$$0x = 0x + x + (-x) = x + (-x) = \vec{0}.$$

■

**Proposition 6** In any vector space,  $(-a)x = -(ax) = a(-x)$  for each  $a \in F$  and  $x \in V$ . In particular,  $(-1)x = -x$ .

**Proof.** We will simply compute:

$$(-a)x + ax = (-a + a)x = 0x = \vec{0}$$

using VS 7, the properties of the field, and Proposition 5. Similarly,

$$a(-x) + ax = a(-x + x) = a0 = \vec{0},$$

where we have used VS 8, the definition of  $-x$ , and the last follows from Proposition 4. ■

## 2 Examples of vector spaces

### 2.1 n-tuples

We are familiar with the vector space structure on  $\mathbb{R}^2$ , where  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$  and  $a(x, y) = (ax, ay)$ . One can check the axioms, and  $\vec{0} = (0, 0)$ .

Similarly, one can take n-tuples of the field, denoted  $F^n$ .

## 2.2 Matrices

Matrices are basically doubly indexed n-tuples, and written  $F^{n \times m}$ , for instance. The  $\vec{0}$  is the matrix of all zeroes.

## 2.3 Polynomials

Polynomials of a fixed degree:  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ , where  $x$  is a variable and  $a_i \in F$ . One can also take polynomials in several variables. There is a correspondence between polynomials of one variable of degree  $n$  and  $(n+1)$ -tuples. One could also take homogeneous polynomials.

## 2.4 Functions

In fact, functions can be added together, as  $(f + g)(x) = f(x) + g(x)$  and  $(af)(x) = a f(x)$ . Note that n-tuples can be considered functions  $f : \{1, \dots, n\} \rightarrow F$  and the addition and scalar multiplication are described analogously. In this way, one can consider sequences to be functions  $f : \mathbb{N} \rightarrow F$  and this gives them a vector space property. The  $\vec{0}$  is the function that takes every element of the domain to  $0 \in F$ .

## 3 Problems

FIS Section 1.2, problems 9, 10, 13, 20, 21.

## 4 Subspaces

**Definition 7** If  $V$  is a vector space over  $F$ , a subset  $W \subseteq V$  is a subspace if  $W$  is a vector space over  $F$  when using the operations of addition and scalar multiplication from  $V$ .

Note that  $V$  and  $\{\vec{0}\}$  are subspaces.  $\{\vec{0}\}$  is called the *trivial* subspace and  $V$  is called an *improper* subspace, so a proper subspace is not  $V$  and a nontrivial subspace is not  $\{\vec{0}\}$ .

The following theorem is a useful way of checking to see if a subset is a subspace.

**Theorem 8** A subset  $W \subseteq V$  is a subspace if and only if the following are all true:

1.  $0 \in W$
2. (closed under addition)  $x, y \in W$  implies  $x + y \in W$
3. (closed under scalar multiplication)  $x \in W$  implies that for all  $a \in F$ ,  $ax \in W$ .

**Proof.** If it is a subspace, it clearly satisfies these properties. If it satisfies these properties, we need to check the axioms to see if  $W$  is a vector space. VS 1, VS 2, VS 5 follow from the fact that the addition is defined in  $V$  and  $V$  is a vector space. VS 3 is assumed. Since additive inverse is the same as multiplication by  $-1$ , the third assumption proves VS 4. VS 6, 7, 8 follow from the fact that each of these is in  $W$  and the corresponding facts in  $V$ . ■

**Proposition 9** *If  $W$  and  $W'$  are subspaces of  $V$ , then  $W \cap W'$  is a subspace of  $V$ .  $W \cup W'$  may not be.*

**Proof.** We can use the theorem. Notice that if  $x, y \in W \cap W'$ , then since  $x, y \in W$ , we have  $x + y \in W$  and similarly for  $W'$ . The argument for scalar multiplication is similar. Notice that the vector spaces  $\mathbb{R} \times \{0\}$  and  $\{0\} \times \mathbb{R}$  are subspaces of  $\mathbb{R}^2$ , but their union is not. ■

## 5 Examples of subspaces

### 5.1 Subspaces of n-tuples

The set of all points in  $\mathbb{R}^3$  satisfying  $x + y - z = 0$  is a subspace. Note that the set of point satisfying  $x + y - z = 2$  is not a subspace (why?).

### 5.2 Subspaces of matrices

The set of diagonal matrices is a subspace, as are upper triangular matrices and lower triangular matrices. The trace of an  $n \times n$  matrix  $M$  is defined as

$$\text{tr}(M) = \sum_{i=1}^n M_{ii}.$$

The set of matrices whose trace is zero is a vector space since the trace satisfied

$$\begin{aligned} \text{tr}(M + M') &= \text{tr}(M) + \text{tr}(M') \\ \text{tr}(aM) &= a \text{tr}(M). \end{aligned}$$

(Check this!)

The matrices over  $\mathbb{R}$  with only positive entries is not a subspace, nor is the set of upper triangular matrices with 1's on the diagonal (unit upper triangular matrices).

### 5.3 Subspaces of functions

The set of continuous functions  $\mathbb{R} \rightarrow \mathbb{R}$  is a subspace of the set of all functions  $\mathbb{R} \rightarrow \mathbb{R}$ . As are the function  $f : X \rightarrow F$  such that a particular point has value 0, e.g., the functions  $\mathbb{R} \rightarrow \mathbb{R}$  such that  $f(1) = 0$ . Note that the functions such that  $f(1) = 1$  are not a subspace.

The set of functions satisfying a linear homogeneous differential equation form a subspace, for instance functions satisfying  $f'' + f = 0$  or functions satisfying  $\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0$  form a subspace. If the differential equation is not homogeneous or nonlinear, the solution space is not a subspace.

## 6 Problems

FIS Section 1.3, problems 13, 17, 18, 19, 20, 23, 28, 30.

Comprehensive/Graduate Option: FIS Section 1.3 problem 31.

## 7 Linear combinations and spans

Notice that given a vector space  $V$  and a vector  $v \in V$ , the subset  $\{tv : t \in F\}$  is a subspace. In fact, this is the smallest subset containing  $v$  that is a subspace. Similarly, if  $w \in V$  also, then we can consider the space  $\{tv + sw : t, s \in F\}$  and this is also a subspace. This subspace is called the *span* of  $\{v, w\}$ , denoted  $\text{span}(\{v, w\})$ , or the subspace *generated by*  $v$  and  $w$ . For a given  $t, s \in F$  and  $v, w \in V$ , we say that  $tv + sw$  is a *linear combination* of  $v$  and  $w$ . We can generalize this in the following way.

**Definition 10** *Let  $V$  be a vector space over a field  $F$  and let  $S \subseteq V$  be nonempty (we usually consider this set to be small, even finite). A vector  $v \in V$  is a linear combination of vectors of  $S$  if there exist vectors  $u_1, \dots, u_n \in S$  and scalars  $a_1, \dots, a_n \in F$  such that*

$$v = a_1u_1 + \dots + a_nu_n.$$

*We say that  $v$  is a linear combination of the vectors  $u_1, \dots, u_n$  and  $a_1, \dots, a_n$  are the coefficients of the linear combination.*

Note that it is a finite sum since that is all that makes sense, even if  $S$  is not finite.

How would one check to see if a vector is a linear combination of a bunch of others? Is  $(1, 2, 3)$  a linear combination of the vectors  $(1, 0, 1)$  and  $(0, 1, 1)$ ? The problem boils down to whether we can find appropriate coefficients:

$$a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

or, written as a matrix problem

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Can this equation be solved? (No! So  $(1, 2, 3)$  is not a linear combination of  $(1, 0, 1)$  and  $(0, 1, 1)$ ) If we could solve it, then we would have found the appropriate coefficients.

**Definition 11** Given a nonempty subset  $S \subseteq V$ , the span of  $S$ , denoted  $\text{span}(S)$ , is the set of all linear combinations of vectors in  $S$ . We also define  $\text{span}(\emptyset) = \{0\}$ .

As in the example above, we see that the span gives a subspace.

**Theorem 12** Given any nonempty subset  $S \subseteq V$ , the span of  $S$  is a subspace of  $V$ . Moreover, any subspace of  $V$  that contains  $S$  must contain  $\text{span}(S)$ .

**Proof.** Straightforward by showing that it contains  $\vec{0}$ , is closed under addition (hardest part – think about it), and closed under scalar multiplication. For the second part, we need to know that linear combinations must be in the subspace, but this follows by induction and the fact that linear combinations of two vectors must be in the subspace (from Theorem 8). ■

Determining vector spaces as spans is quite easy and natural. However, some vector spaces are described in a different way: as the solution to some equations. The process of going from describing a vector space as a set of equations to describing it as a span is usually called “finding the general solution.” For instance, the set of functions satisfying

$$f'' + f = 0$$

can be shown to be a vector space. But that vector space is generated by  $\sin t$  and  $\cos t$  and so the vector space can be described as

$$\{a_1 \sin t + a_2 \cos t : a_1, a_2 \in \mathbb{R}\}.$$

## 8 Problems

FIS Section 1.4, problems 7, 11, 12, 13, 14, 15, 16.

## 9 Linear dependence and independence

**Definition 13** A subset  $S \subseteq V$  of a vector space  $V$  over  $F$  is called linearly dependent if there exists a finite number of distinct vectors  $u_1, \dots, u_n \in S$  and scalars  $a_1, \dots, a_n \in F$  not all zero such that

$$a_1 u_1 + a_2 u_2 + \dots + a_n u_n = 0.$$

We may also say the vectors in  $S$  are linearly dependent. If  $S$  is not linearly dependent, it is said to be linearly independent, or also that its vectors are linearly independent.

The importance and elegance of this definition cannot be overstated. Note that if a finite set of vectors is linearly dependent, that means that some vector can be written as a linear combination of the others. But we do not have to

specify which one, and it is often the case that not every vector can be written as a linear combination of others. For instance, consider the vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Notice that the vector  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  cannot be written as a linear combination of the

others, but  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  can. We can see the set is linearly dependent, however, but seeing that

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

How does one check to see if vectors are linearly dependent? Try to solve

$$a_1 v_1 + \cdots + a_n v_n = 0$$

for a solution of  $a_1, \dots, a_n$  that is not all zero (notice we can always take all the  $a_i$ 's to be zero; that is called the trivial solution). For vectors in  $F^n$ , this can be translated into a matrix equation

$$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

where you are trying to find the coefficients. This is the kind of linear algebra you learned in Math 215 or Math 313 (for instance, row reduce to row echelon form).

**Theorem 14** *Let  $V$  be a vector space, and let  $S_1 \subseteq S_2 \subseteq V$ . If  $S_1$  is linearly dependent then  $S_2$  is linearly dependent. If  $S_2$  is linearly independent then  $S_1$  is linearly independent.*

**Theorem 15** *Let  $S$  be a linearly independent subset of  $V$  and let  $v \in V \setminus S$ . Then  $S \cup \{v\}$  is linearly dependent if and only if  $v \in \text{span}(S)$ .*

**Proof.** Suppose  $S \cup \{v\}$  is linearly dependent. Then we know that there exist distinct  $u_1, \dots, u_n \in S$  and  $a_1, \dots, a_n, b \in F$ , not all 0, such that

$$a_1 u_1 + \cdots + a_n u_n + b v = \vec{0}.$$

Since  $S$  is linearly independent,  $b \neq 0$ . Thus

$$v = -b^{-1} a_1 u_1 - \cdots - b^{-1} a_n u_n$$

and so  $v$  is a linear combination of vectors in  $S$  and hence in the span. Conversely, suppose  $v \in \text{span}(S)$ . Then we can find scalars (not all zero) and vectors in  $S$  such that

$$v = a_1u_1 + \cdots + a_ku_k$$

so

$$a_1u_1 + \cdots + a_ku_k - v = \vec{0}$$

and so  $S \cup \{v\}$  is linearly dependent. ■

## 10 Problems

FIS Section 1.5, problems 4, 6, 9, 10, 12, 14.

## 11 Basis and dimension

**Definition 16** A basis  $\beta$  for a vector space  $V$  is a linearly independent subset of  $V$  that generates  $V$ , i.e.,  $V = \text{span}(\beta)$ . If  $\beta$  is a basis for  $V$ , we also say that the vectors of  $\beta$  form a basis for  $V$ .

We often have a natural basis that is called the standard basis:

- The standard basis for  $F^n$  is  $e_1 = (1, 0, \dots, 0)$ ,  $e_2 = (0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $e_n = (0, 0, \dots, 0, 1)$ .
- The standard basis for the polynomials of degree  $n$  with coefficients in  $F$ , denoted  $P_n(F)$ , is  $\{1, x, x^2, \dots, x^n\}$ . The standard basis for all polynomials  $P(F)$  is  $\{1, x, x^2, x^3, \dots\}$ .

Note that a basis need not be finite. It is a curious fact that every vector space has a basis (the proof is hard and uses Zorn's Lemma), though not always a useful basis.

**Theorem 17** Let  $V$  be a vector space and  $\beta = \{u_1, u_2, \dots, u_n\}$  be a subset of  $V$ . The set  $\beta$  is a basis for  $V$  if and only if each  $v \in V$  can be uniquely expressed as a linear combination of vectors of  $\beta$ , i.e., for each  $v \in V$  there exist unique scalars  $a_1, \dots, a_n$  such that

$$v = a_1u_1 + \cdots + a_nu_n.$$

**Proof.** First suppose  $\beta$  is a basis. Since  $\beta$  spans, there must be scalars such that  $v = a_1u_1 + \cdots + a_nu_n$ . Since  $\beta$  is linearly independent, if we take  $v = \vec{0}$ , then since  $\beta$  is linearly independent, the coefficients must be all equal to 0. We now need to show that the representation of  $v$  is unique. Suppose  $v = b_1u_1 + \cdots + b_nu_n$  as well. Then we find

$$\begin{aligned} \vec{0} &= v - v = a_1u_1 + \cdots + a_nu_n - (b_1u_1 + \cdots + b_nu_n) \\ &= (a_1 - b_1)u_1 + \cdots + (a_n - b_n)u_n. \end{aligned}$$



It follows that  $a_i - b_i = 0$  for all  $i$ , or  $a_i = b_i$  and hence the coefficients are unique.

Conversely, we know that both  $\beta$  spans  $V$  and also that  $\vec{0}$  has a unique representation as a linear combination, and since that must be  $a_1 = a_2 = \dots = a_n = 0$ ,  $\beta$  must be linearly independent. ■

**Theorem 18** *If a vector space  $V$  is generated by a finite set  $S$ , then some subset of  $S$  is a basis for  $V$ , and hence  $V$  has a finite basis.*

**Proof.** Suppose  $V$  is generated by a finite set  $S$ . Let  $S' = \{u'_1, \dots, u'_k\}$  be the subset of  $S$  with the largest cardinality such that  $S'$  is linearly independent (since a set containing one vector is linearly independent,  $S'$  must contain at least one element). It follows that for any  $w \in S \setminus S'$ , we can write  $w$  as a linear combination of vectors in  $S'$  (since  $aw + a_1u'_1 + \dots + a_ku'_k = \vec{0}$  must have a nontrivial solution since otherwise  $S' \cup \{w\}$  would be a larger linearly independent subset). So that means that  $\text{span } S = \text{span } S'$ . Since  $\text{span } S = V$ , the result follows. ■

**Theorem 19 (Replacement Theorem)** *Let  $V$  be a vector space that is generated by a set  $G$  containing exactly  $n$  vectors and let  $L$  be a linearly independent subset of  $V$  containing exactly  $m$  vectors. Then  $m \leq n$  and there exists a set  $H \subseteq G$  containing exactly  $n - m$  vectors such that  $L \cup H$  generates  $V$ .*

**Proof.** See worksheet. ■

**Corollary 20** *Let  $V$  be a vector space having a finite basis. Then any basis for  $V$  contains the same number of vectors.*

**Proof.** Let  $\beta$  and  $\beta'$  be bases with  $n$  and  $m$  elements, respectively. Using the replacement theorem with  $G = \beta$  and  $L = \beta'$  yields  $m \leq n$ , while using it with  $G = \beta'$  and  $L = \beta$  yields  $n \leq m$ , hence  $n = m$ . ■

**Definition 21** *A vector space called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for  $V$  is called the dimension of  $V$  and is denoted  $\dim V$ . A vector space that is not finite-dimensional is called infinite-dimensional.*

**Corollary 22** *Let  $V$  be vector space of dimension  $n$ .*

1. *Any finite generating set for  $V$  contains at least  $n$  vectors, and a generating set for  $V$  that contains exactly  $n$  vectors is a basis for  $V$ .*
2. *Any linearly independent subset of  $V$  that contains exactly  $n$  vectors is a basis for  $V$ . Hence any linearly independent set has at most  $n$  vectors.*
3. *Every linearly independent subset of  $V$  can be extended to a basis for  $V$ .*

**Proof.** 1. Given any generating set, it has a subset that is a basis by Theorem 18. That basis must have  $n$  elements and so the set must have at least  $n$  elements. If the generating set has  $n$  elements, then the subset must be the set itself.

2. This is a special case of the Replacement Theorem with  $G = \beta$ , and in this case  $m = n$ , and so the subset must already span. Once we have  $n$  vectors, since the subset must span, any additional vector added to the set cannot be linearly independent.

3. We can use the Replacement Theorem with  $G = \beta$ . Given a linearly independent set  $L$ , we can extend it to a generating set with  $n$  elements taking  $H \subseteq \beta$ . By the first part of this corollary, it must be a basis. ■

**Theorem 23** *Let  $W$  be a subspace of a finite-dimensional vector space  $V$ . The  $W$  is finite dimensional and  $\dim W \leq \dim V$ . Moreover, if  $\dim W = \dim V$ , then  $W = V$ .*

**Proof.** If  $W = \{\vec{0}\}$ , then  $\dim W = 0 \leq \dim V$ . If  $W \neq \{\vec{0}\}$ , then  $W$  contains a nonzero vector  $x_1$ . The set  $\{x_1\}$  is a linearly independent set. We can continue to choose vectors  $x_2, \dots, x_k \in W$  such that  $\{x_1, \dots, x_k\}$  are linearly independent until we cannot do this any more. We know that there can be no linearly independent set in  $W$  with more than  $\dim V$  elements, so  $k \leq \dim V$ . This set must generate  $W$ , otherwise we could extend it, so  $\{x_1, \dots, x_k\}$  forms a basis. If  $k = \dim V$ , then by the corollary above we must have that the basis for  $W$  spans  $V$ , and hence  $V = W$ . ■

**Corollary 24** *If  $W$  is a subspace of a finite-dimensional vector space  $V$ , then any basis for  $W$  can be extended to a basis for  $V$ .*

**Proof.** We can produce the basis for  $V$  by starting with a basis for  $W$  and adding in points one by one as in the proof of the previous theorem. Since any linearly independent set has fewer than  $\dim V$  elements, this will eventually produce a basis for  $V$  that extends the basis for  $W$ . You could also use the Replacement Theorem. ■

## 12 Problems

FIS Section 1.6, problems 4, 11, 12, 13, 16, 17, 20, 23, 33, 34.

Comprehensive/Graduate Option: FIS Section 1.6 problem 35.