Math 413/513 Chapter 2 (from Friedberg, Insel, & Spence)

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1 Linear Transformations

Definition 1 *Let V and W be vector spaces over F. We say a function $T : V \to W$ is a* linear transformation *from V to W if for all $x, y \in V$ and $c \in F$, we have*

1. $T(x + y) = T(x) + T(y)$ *and*
2. $T(cx) = cT(x)$.

Sometimes we will just call T *linear*.

Proposition 2 *The following are properties of linear transformations:*

1. $T\left(\vec{0}\right) = \vec{0}$
2. *T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.*
3. *$T(x - y) = T(x) - T(y)$ for all $x, y \in V$*
4. *T is linear if and only if for $x_1, \ldots, x_n \in V$ and $a_1, \ldots, a_n \in F$, we have*

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$

We have the identity transformation $I_V : V \to V$ given by $I_V(x) = x$ and the zero transformation $T_0 : V \to W$ given by $T_0(x) = \vec{0}$ for all $x \in V$.

Definition 3 *Let V, W be vector spaces and let $T : V \to W$ be a linear transformation. The null space (or kernel) $N(T)$ of T is the set of vectors $x \in V$ such that $T(x) = 0$, i.e.,*

$$N(T) = \{x \in V : T(x) = 0\}.$$

The range (or image) of $R(T)$ (or $T(V)$) of T is the subset of W consisting of all images of vectors in V under T, i.e.,

$$R(T) = \{T(x) : x \in V\} = \{y \in W : \exists x \in V \text{ s.t. } y = T(x)\}.$$

Theorem 4 *$N(T)$ and $R(T)$ are subspaces of V and W, respectively.*

Proof. Since $T\left(\vec{0}\right) = \vec{0}$, we must have $\vec{0} \in N(T)$ and $\vec{0} \in R(T)$. If $x, y \in N(T)$ and $a \in F$ then $T(ax + y) = aT(x) + T(y) = \vec{0} + \vec{0} = \vec{0}$ so $ax + y \in N(T)$. Similarly, if $v, w \in R(T)$ and $c \in F$ then there exist x and y in V such that $v = T(x)$ and $w = T(y)$. Then $cv + w = cT(x) + T(y) = T(cx + y)$ and so $cv + w \in R(T)$. ■

Theorem 5 *If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V, then*

$$R(T) = \operatorname{span} T(\beta) = \operatorname{span}(\{T(v_1), \ldots, T(v_n)\}).$$

Proof. Since every element of V is a linear combination of elements of β, we have that if $v \in R(T)$ then $v = T(x) = T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i T(v_i)$ so $v \in \operatorname{span} T(\beta)$. The other inclusion is trivial. ■

Definition 6 *If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T, denoted* $\operatorname{nullity}(T)$, *and the rank of T, denoted* $\operatorname{rank}(T)$, *as the dimensions of $N(T)$ and $R(T)$ respectively.*

Theorem 7 (Dimension Theorem) *Let V, W be vector spaces and $T : V \to W$ a linear transformation. If V is finite-dimensional, then*

$$\operatorname{nullity} T + \operatorname{rank} T = \dim V.$$

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis for $N(T)$ and extend it to a basis $\{v_1, \ldots, v_n\}$ of V. We claim that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis of $R(T)$, which would complete the proof. First we prove that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is linearly independent. Suppose $\sum_{i=1}^{k-n} a_i T(v_{k+i}) = \vec{0}$, then $T\left(\sum_{i=1}^{k-n} a_i v_{k+i}\right) = \vec{0}$, which means that $\sum_{i=1}^{k-n} a_i v_{k+i} \in N(T)$. This is only possible if $\sum_{i=1}^{k-n} a_i v_{k+i} = \vec{0}$ (why?) and this is only possible if a_i are all equal to zero, showing that $\{T(v_{k+1}), \ldots, T(v_n)\}$ are linearly independent. Now suppose $v \in R(T)$, then $v = T(x)$ for some $x \in V$. We can express $x = \sum_{i=1}^{n} a_i v_i$. Since $v = T(x) = \sum_{i=1}^{n} a_i T(v_i) = \vec{0} + \sum_{i=k+1}^{n} a_i T(v_i)$ we see that $\{T(v_{k+1}), \ldots, T(v_n)\}$ spans. ■

Theorem 8 *If $T : V \to W$ is linear then T is one-to-one if and only if $N(T) = \left\{\vec{0}\right\}$.*

Proof. Suppose T is one-to-one. Then if $x \in N(T)$, then $T(x) = \vec{0}$. However, this means that $x = \vec{0}$ since $T\left(\vec{0}\right) = \vec{0}$ for any linear transformation and one-to-one means this is the only one. Hence $N(T) = \left\{\vec{0}\right\}$. Now suppose $N(T) =$

$\left\{\vec{0}\right\}$. If $T(u) = T(v)$ then $T(u-v) = \vec{0}$. But this implies that $u - v \in N(T)$ and hence $u - v = \vec{0}$, so $u = v$ and T is one-to-one. ■

Theorem 9 *Let V and W be vector spaces of equal (finite) dimension and let $T : V \to W$ be linear. Then the following are equivalent:*

1. *T is one-to-one.*
2. *T is onto.*
3. $\operatorname{rank} T = \dim V$.

Proof. If $\operatorname{rank} T = \dim V$ then the Dimension Theorem says that $\operatorname{nullity} T = 0$ and thus $N(T) = \left\{\vec{0}\right\}$ and so T is one-to-one.

If T is onto then $T(V) = W$ so $\operatorname{rank} T = \dim W = \dim V$.

If T is one-to-one then $\operatorname{nullity} T = 0$ so $\operatorname{rank} T = \dim V = \dim W$. Since $R(T)$ is a subspace of W with the same dimension, $R(T) = W$. ■

Theorem 10 *Let V and W be vector spaces over F and suppose $\{v_1, \ldots, v_n\}$ is a basis for V. Given $w_1, \ldots, w_n \in W$, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, \ldots, n$.*

Proof. Clearly we can define a linear transformation, since $\{v_1, \ldots, v_n\}$ is a basis, by

$$T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i w_i$$

since each vector can be *uniquely* written as a linear combination of vectors in $\{v_1, \ldots, v_n\}$. If we have two transformation T and T' that agree on $T(v_i)$ for $i = 1, \ldots, n$ then

$$T\left(\sum_{i=1}^{n} a_i v_i\right) - T'\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i \left(T(v_i) - T'(v_i)\right) = \vec{0}.$$

■

Corollary 11 *Let V, W be vector spaces and suppose $\{v_1, \ldots, v_n\}$ is a basis for V. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, \ldots, n$, then $U = T$.*

2 Problems

FIS Section 2.1, exercises 2,3,4,5,7,9,11,14,20,21,26,28,30,33,38

Comprehensive/Graduate option: 40.

3 Matrix representation of a linear transformation

From the end of last section, we see that linear transformations on finite dimensional vector spaces are determined entirely by what they do on a basis of the vector space. For this reason, if the vector spaces are finite dimensional, we can represent a linear transformation by a matrix.

Definition 12 *Let V be a finite-dimensional vector space. An* ordered basis *for V is a basis for V endowed with a specific order.*

This is not too unusual. We usually consider the ordered basis of F^n to be $\{e_1, e_2, \ldots, e_n\}$ where e_i is 1 in the ith slot and 0 elsewhere. This is called the *standard ordered basis* for F^n. Polynomials of degree at most n also have a standard ordered basis: $\left\{1, x, x^2, \ldots, x^n\right\}$.

Definition 13 *Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for a finite-dimensional vector space V over the field F. For $x \in V$, let $a_1, a_2, \ldots, a_n$ be the unique set of scalars such that*

$$x = \sum_{i=1}^{n} a_i u_i$$

(since β is a basis, such scalars are unique!) We define the coordinate vector *of x relative to β, denoted $[x]_\beta \in F^n$, by*

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Proposition 14 *The map $T_\beta : V \to F^n$ given by $T_\beta(x) = [x]_\beta$ is linear bijection.*

Definition 15 *Let V and W be finite dimensional vector spaces with ordered bases $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ (note that n and m can be different numbers). Then we can write any linear transformation $T : V \to W$ in terms of the basis elements of β as*

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

for each $j = 1, \ldots, n$. The $m \times n$ matrix $A = [a_{ij}]$ is called the matrix representation of T in the ordered bases β and γ *and is written $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, we write $A = [T]_\beta$.*

Remark 16 *This choice of placement of the decorations is not standard and there are better ways; we choose to stick with the convention in the book but I'm not particularly happy with it.*

Proposition 17 *If β and γ are ordered basis of V and W and $T : V \to W$ is a linear transformation, the the following is true*

$$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta .$$

Proof. In fact, if $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_n\}$ it is sufficient to take $x = v_i$ for all $i = 1, \ldots, n$ (why?) By definition of the matrix, we see that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

so

$$[T(v_j)]_\gamma = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

when $[a_{ij}] = [T]_\beta^\gamma$. We see that if $x = \sum_{i=1}^n c_i v_i$ then

$$\begin{aligned} T(x) &= \sum_{j=1}^{n} c_j T(v_j) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} c_j a_{ij} w_i \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_j a_{ij} \right) w_i \end{aligned}$$

so

$$[T(x)]_\gamma = \begin{pmatrix} \sum_{j=1}^n c_j a_{1j} \\ \sum_{j=1}^n c_j a_{2j} \\ \vdots \\ \sum_{j=1}^n c_j a_{mj} \end{pmatrix} = Ac$$

if $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = [x]_\beta$. Hence we can re-write

$$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta .$$

■

Linear transformations actually form a vector space once we define some operations.

Definition 18 *Let $T, U : V \to W$ be functions, where V, W are vector spaces over F. Let $a \in F$. Then we can define $T + U : V \to W$ and $aT : V \to W$ as*

$$\begin{aligned}(T+U)(x) &= T(x) + U(x) \\ (aT)(x) &= aT(x).\end{aligned}$$

Theorem 19 *Let $T, U : V \to W$ be linear, where V, W are vector spaces over F.*

1. *For all $a \in F$, $aT + U$ is linear.*
2. *Using these operations, the collection of linear transformations from V to W forms a vector space over F.*

Proof. The first is a direct computation:

$$\begin{aligned}(aT+U)(cx+y) &= aT(cx+y) + U(cx+y) \\ &= acT(x) + aT(y) + cU(x) + U(y) \\ &= acT(x) + cU(x) + aT(y) + U(y) \\ &= c(aT+U)(x) + (aT+U)(y).\end{aligned}$$

(Note the second equality follows from linearity of T and U.) If we let T_0, the transformation that takes all of V to $0 \in W$, we easily see that T_0 satisfies the properties of the additive identity. The additive inverse is defined as $(-T)(x) = -T(x)$, and the other properties of a vector space are easily verified. ■

Definition 20 *We denote the vector space of all linear transformations from V to W by $\mathcal{L}(V, W)$. If $V = W$, we often write $\mathcal{L}(V)$ instead of $\mathcal{L}(V, V)$.*

Theorem 21 *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $T, U \in \mathcal{L}(V, W)$. Then*

1. $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
2. $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ *for all scalars a.*

It follows that the map $\mathcal{L}(V, W) \to F^{m \times n}$ given by $T \to [T]_\beta^\gamma$ is a linear transformation.

Proof. This proof is a very good exercise in understanding $[T]_\beta^\gamma$. It is left as an exercise. ■

The previous theorem allows us to do the following. We recall that if V and W are finite dimensional vector spaces with corresponding bases $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$, the map

$$\begin{aligned}\mathcal{L}(V, W) &\to F^{m \times n} \\ T &\mapsto [T]_\beta^\gamma\end{aligned}$$

is linear. There is also another map

$$F^{m\times n} \to \mathcal{L}(V,W)$$
$$A \mapsto L_A^{\gamma,\beta}$$

where $L_A^{\gamma,\beta}$ is the linear transformation determined by

$$L_A^{\gamma,\beta}(v_j) = \sum_{i=1}^m A_{ij} w_i$$

(why does this *uniquely* determine a linear transformation?) Notice that

$$L_{A+cB}^{\gamma,\beta}(v_j) = \sum_{i=1}^m (A_{ij} + cB_{ij}) w_i = \sum_{i=1}^m A_{ij} w_i + c \sum_{i=1}^m B_{ij} w_i = L_A^{\gamma,\beta}(v_j) + cL_B^{\gamma,\beta}(v_j),$$

so this is also a linear map (why is it enough to check this on the basis vectors?). We can check the compositions in both directions: Given T, if we write $A = [T]_\beta^\gamma$, then

$$L_A(v_j) = \sum_{i=1}^m A_{ij} w_i = T(v_j)$$
$$[L_A]_\beta^\gamma = A$$

and so these two linear maps are inverses of each other.

The map $L_A^{\gamma,\beta}$ is denoted with the letter L since if the vector space $V = F^n$, $W = F^m$, and the standard bases are used, then the transformation L_A is called *left-multiplication*, given by matrix multiplication:

$$L_A(x) = Ax.$$

We will often omit notation β, γ in L_A in this case.

4 Problems

FIS section 2.2, exercises 3,4,5,7,8,9,11,12,15

5 Composition of linear transformations

Theorem 22 *Let V, W, and Z be vector spaces over the same field F. Let $T : V \to W$ and $U : W \to Z$ be linear. Then $U \circ T = UT : V \to Z$ is linear.*

Proof. We need to verify for vectors $x, y \in V$ and $a \in F$,

$$\begin{aligned} UT(x+ay) &= U(T(x+ay)) \\ &= U(T(x) + aT(y)) \\ &= U(T(x)) + aU(T(y)) \\ &= UT(x) + aUT(y). \end{aligned}$$

■

Theorem 23 *Let V be a vector space. Let $T, U_1, U_2 \in \mathcal{L}(V)$. Recall that I is the linear transformation $I(x) = x$. Then*

1. *(additivity of composition) $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$,*

2. *(associativity of composition) $T(U_1U_2) = (TU_1)U_2$,*

3. *(identity transformation) $TI = IT = T$,*

4. *(composition respects scalar multiplication) $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars a.*

This theorem is true even if the domains and codomains are not the same; this will be an exercise in the book.

For the next part, one needs to understand the definition of matrix multiplication. Recall that if A is a $m \times n$ matrix and B is a $n \times p$ matrix, then the $m \times p$ matrix $C = AB$ is defined by making its entries

$$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj},$$

where $1 \leq i \leq m$ and $1 \leq j \leq p$.

We can now see that for finite linear transformations over finite dimensional vector spaces, the maps that associates $\mathcal{L}(V, W)$ with $F^{m\times n}$ also respects the algebra structure, mapping between composition and matrix multiplication:

Theorem 24 *Let $T : V \to W$ and $U : W \to Z$ be linear transformations and let $\alpha = \{v_1, \ldots, v_n\}, \beta = \{w_1, \ldots, w_m\}, \gamma = \{z_1, \ldots, z_p\}$ be ordered bases of V, W, Z respectively. Then*

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

Also, $L_{BA}^{\gamma,\alpha} = L_B^{\gamma,\beta} L_A^{\beta,\alpha}$ if $A \in F^{m\times n}$ and $B \in F^{p\times m}$.

Proof. Let's let $A = [T]_\alpha^\beta$, $B = [U]_\beta^\gamma$ and $C = [UT]_\alpha^\gamma$. By definition,

$$T(v_j) = \sum_{i=1}^{m} A_{ij}w_i,$$
$$U(w_i) = \sum_{k=1}^{p} B_{ki}z_k,$$
$$UT(v_j) = \sum_{k=1}^{p} C_{kj}z_k.$$

But, also

$$\begin{aligned} UT(v_j) &= U(T(v_j)) \\ &= U\left(\sum_{i=1}^{m} A_{ij} w_i\right) \\ &= \sum_{i=1}^{m} A_{ij} U(w_i) \\ &= \sum_{i=1}^{m} A_{ij} \sum_{k=1}^{p} B_{ki} z_k \\ &= \sum_{k=1}^{p} \sum_{i=1}^{m} B_{ki} A_{ij} z_k \\ &= \sum_{k=1}^{p} (BA)_{kj} z_k. \end{aligned}$$

It follows that $C = BA$.

Similarly,

$$\begin{aligned} L_{BA}^{\gamma,\alpha}(v_j) &= \sum_{k=1}^{p} (BA)_{kj} z_k \\ &= \sum_{k=1}^{p} \sum_{i=1}^{m} B_{ki} A_{ij} z_k \\ &= \sum_{i=1}^{m} A_{ij} \sum_{k=1}^{p} B_{ki} z_k \\ &= \sum_{i=1}^{m} A_{ij} L_B(w_i) \\ &= L_B L_A(v_j). \end{aligned}$$

■

This theorems allows us to compare matrices and linear transformations, and there are corresponding theorems to Theorems 22 and 23 with regard to matrices. See the book for details.

Here is a definition we may need later:

Definition 25 *We define the Kronecker delta δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The $n \times n$ identity matrix I_n is defined by $(I_n)_{ij} = \delta_{ij}$. We often omit the subscript n for the identity matrix.*

6 Problems

Problem 1: Recall the transpose of a matrix, defined by $\left(A^T\right)_{ij} = A_{ji}$. Show that $(AB)^T = B^T A^T$.

FIS section 2.3, exercises 2, 3, 5, 6, 7, 8, 11, 14, 15, 16

7 Invertibility and isomorphism

Definition 26 *Let V and W be vector spaces and $T : V \to W$ be a linear transformation. A function $U : W \to V$ is said to be an* inverse *of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, we say T is* invertible*.*

Recall that a function has an inverse if and only if it is both one-to-one and onto. Also, if a function has an inverse, that inverse is unique; we usually denote the inverse of T by the symbol T^{-1} if T is invertible.

Proposition 27 *For invertible function T and U,*

$$(TU)^{-1} = U^{-1}T^{-1}$$

and

$$\left(T^{-1}\right)^{-1} = T.$$

In particular, the inverse of T is itself invertible.

Remark 28 *We generally do not write $1/T$ instead of T^{-1} since the former seems to refer to division and inverse need not be related to division.*

Proposition 29 *If V, W are finite-dimensional with the same dimension, then a linear transformation $T : V \to W$ is invertible if and only if it is either one-to-one (or $N(T) = \{0\}$) or onto or $\operatorname{rank}(T) = \dim V$.*

Proof. This follows from Theorem 9. ■

An important fact is that if T is invertible, then its inverse function is also linear.

Theorem 30 *If $T : V \to W$ is an invertible invertible linear transformation, then $T^{-1} : W \to V$ is also an invertible linear transformation.*

Proof. Let $y, y' \in W$ and $a \in F$ and consider

$$T^{-1}(y + ay').$$

Since T is invertible, it is onto, so there exist $x, x' \in V$ such that $y = T(x)$, $y' = T(x')$, and so

$$\begin{aligned} T^{-1}(y + ay') &= T^{-1}(T(x) + aT(x')) \\ &= T^{-1}(T(x + ax')) \\ &= x + ax' \\ &= T^{-1}(y) + aT^{-1}(y') \end{aligned}$$

where the second equality is from linearity of T, the third is the definition of inverse, and the fourth follows from $y = T(x)$, so $x = T^{-1}(y)$, and similarly for x' and y'. ■

Definition 31 *If $T : V \to W$ is an invertible linear transformation, we say T is an* isomorphism*. If V and W are two vector spaces, we say they are* isomorphic *if there exists an isomorphism between them (there are usually many such isomorphisms if there is at least one).*

Remark 32 *In most fields of math, we use isomorphism to denote a map between spaces that satisfies the appropriate property and is invertible and its inverse also satisfies that property. For linear algebra, the property is that of being linear. By the previous theorem, we do not need to check to see if the inverse is linear! If we were to use "differentiable" as the property instead of linear, the inverse is not automatically differentiable, since $f(x) = x^3$ is a differentiable map that is one-to-one and onto, but its inverse is not differentiable.*

Lemma 33 *Suppose V and W are isomorphic vector spaces. Then V is finite dimensional if and only if W is finite dimensional. If they are both finite dimensional, then $\dim V = \dim W$.*

Proof. There exists an isomorphism $T : V \to W$. Suppose V is finite dimensional. Then $R(T) = W$ since T is onto, but we proved previously that if β is a basis of V, then $T(\beta)$ is a basis for $R(T)$, so W is finite dimensional with the same dimension. Using T^{-1} we get the other implication. ■

Theorem 34 *If V and W are finite dimensional, then V and W are isomorphic if and only if $\dim V = \dim W$.*

Proof. In the lemma, we proved one direction. Now suppose that $\dim V = \dim W$. Taking a basis $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_n\}$ (note the same size!), we can write down the linear map determined by

$$T(v_i) = w_i.$$

It is easy to see that this map is onto and hence an isomorphism. ■

Note that the isomorphism we produced in the proof depends on the choices of bases, and thus there are many such isomorphisms!

Corollary 35 *If V is a finite dimensional vector space, then V is isomorphic to F^n if and only if $\dim V = n$.*

Proof. If $\dim V = n$, then we have the isomorphism $V \to F^n$ given by $v \to [v]_\beta$ where β is a basis of dimension n (why is it an isomorphism? it is linear and injective and the dimensions of the domain and codomain are the same). Conversely, if V is isomorphic to F^n, the theorem says the dimensions are the same. ■

Theorem 36 *If V and W are finite-dimensional vector spaces with bases β and γ, then the map $\mathcal{L}(V,W) \to F^{m\times n}$ given by $T \to [T]_\beta^\gamma$ is an isomorphism.*

Proof. We have essentially been through all elements of the proof, even defining the inverse map $A \to L_A^{\gamma,\beta}$. It was very important that we showed both maps are (or at least one map is) linear. ■

Corollary 37 *If V and W are finite-dimensional vector spaces with* $\dim V = n$ *and* $\dim W = m$, *then* $\dim \mathcal{L}(V,W) = mn$.

Proof. We can use the isomorphism from the theorem, and then we see that the dimensions of $\mathcal{L}(V,W)$ and $F^{m\times n}$ are the same. The dimension of $F^{m\times n}$ is known to be mn (recall that the basis consists of matrices with a single nonzero entry). ■

The pervious work shows that the vector space of linear transformations finite dimensional vector spaces is isomorphic to a space of matrices. We summarize some facts that are easy to prove that further describes the connections between matrices and linear transformations:

Theorem 38 *Let V and W be finite-dimensional vector spaces with bases β and γ. Let $T: V \to W$ be a linear transformation and let A be a matrix. Then the following are true:*

1. *Let $B = [T]_\beta^\gamma$. T is an invertible linear transformation if and only if B is an invertible matrix. In this case, $B^{-1} = \left[T^{-1}\right]_\beta^\gamma$*
2. *A is invertible if and only if $L_A^{\gamma,\beta}$ is an invertible linear transformation. Moreover, $\left(L_A^{\gamma,\beta}\right)^{-1} = L_{A^{-1}}^{\beta,\gamma}$.*

8 Problems

FIS section 2.4, exercises 4, 5, 6, 9, 10, 12, 15, 17, 20, 21

Comprehensive/Graduate option: FIS section 2.4, exercise 24

9 Change of coordinates matrix

We have a way of associating a finite dimensional vector space V with F^n using the map $v \to [v]_\beta$. However, this is dependent on a choice of basis β, and one may ask what happens if I chose a different basis β'? We know that it will result in another map $V \to F^n$ given by $[v]_{\beta'}$. How are the vectors $[v]_\beta$ and $[v]_{\beta'}$ related? The answer is given by the following theorem. Recall the every vector space V has the identity map $I_V : V \to V$ given by $I_V(v) = v$.

Theorem 39 *Let β, β' be two ordered bases for the finite dimensional vector space V, and let $Q = [I_V]_{\beta'}^\beta$. Then*

1. *Q is invertible and $Q^{-1} = [I_V]_{\beta}^{\beta'}$.*
2. *For any $v \in V$, $[v]_\beta = Q [v]_{\beta'}$.*

Proof. Since I_V is invertible (in fact, $I_V^{-1} = I_V$), Q is invertible. The second statement follows from Theorem 24. Also, we have that

$$[v]_\beta = [I_V (v)]_\beta = [I_V]_{\beta'}^{\beta} [v]_{\beta'} = Q [v]_{\beta'}.$$

■

Definition 40 *The matrix $Q = [I_V]_{\beta'}^{\beta}$ is called the* change of coordinate matrix*. We say Q* changes β'-coordinates into β-coordinates*.*

Remark 41 *The theorem also tells us something about different basis. In fact, given any invertible matrix Q, we can use it to turn a basis $\beta' = \{u_1, \ldots, u_n\}$ into a new basis by looking at*

$$\beta = \left\{ \sum_{i=1}^{n} Q_{i1} u_i, \ldots, \sum_{i=1}^{n} Q_{in} u_i \right\}.$$

It is easy to see that β is also a basis with change of basis matrix Q.

We can do the same thing for linear operators:

Theorem 42 *Let $T \in \mathcal{L}(V)$ for a finite dimensional vector space V with ordered bases β, β'. If Q is the change of coordinate matrix that changes β' coordinates into β coordinates, then $[T]_{\beta'} = Q^{-1} [T]_\beta Q$.*

Proof. We can check:

$$Q [T]_{\beta'} = [I_V]_{\beta'}^{\beta} [T]_{\beta'} = [I_V T]_{\beta'}^{\beta} = [T]_{\beta'}^{\beta} = [T I_V]_{\beta'}^{\beta} = [T]_\beta Q.$$

■

The fact that linear transformations correspond to matrices give the a reason that the following definition is useful.

Definition 43 *Let A and B be $n \times n$ matrices. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1} A Q$.*

10 Problems

FIS 2.5, exercises 2, 3, 4, 8, 9, 13, 14

11 Dual spaces (Comprehensive/Graduate option)

The space $\mathcal{L}(V, F)$ has a special name.

Definition 44 *Let V be a vector space over a field F. The* dual space *is defined to be the space of linear functions $V \to F$ (these are called* linear functionals*), i.e., $V^* = \mathcal{L}(V, F)$.*

Here are some facts about dual spaces:

Theorem 45 *If V is finite dimensional, then V^* is finite dimensional and* $\dim V = \dim V^*$. *In fact, given a basis $\beta = \{x_1, \ldots, x_n\}$ for V there is an associated subset $\beta^* = \{f_1, \ldots, f_n\}$ of V^* such that*

$$f_i(x_j) = \delta_{ij}$$

and β^* is a basis for V (we call β^* the dual basis to *β).*

Theorem 46 *There is a linear map $V \to V^{**}$ that takes $x \in V$ to a linear transformation $T_x : V^* \to F$ such that $T_x(f) = f(x)$. If V is finite-dimensional, then the map $x \to T_x$ is an isomorphism.*

12 Problems (Comprehensive/Graduate option)

FIS 2.6, exercises 2, 3, 5, 9

Read about the proofs of the above theorems and about the transpose of a linear transformation in the chapter.