Math 413/513 Chapter 4 (from Friedberg, Insel, & Spence)

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1 Determinants

1.1 Definitions and main results

Definition 1 *Let* $A = (A_{ij}) \in F^{n\times n}$*. The* determinant $\det(A)$ *is a scalar defined recursively as* $\det(A) = A_{11}$ *if* $n = 1$ *and if* $n \geq 2$,

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det\left(\tilde{A}_{1j}\right)$$

where $\tilde{A}_{ij} \in F^{(n-1)\times(n-1)}$ *is the matrix obtained from* A *by deleting the* i*th row and* j*th column. Sometimes* $\det A$ *is denoted as* $|A|$*. The scalar* $(-1)^{i+j} A_{ij} \det\left(\tilde{A}_{ij}\right)$ *is called the* i, j cofactor*.*

We note that in the 2×2 case, we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Theorem 2 *The determinant is a linear function of each row if the other rows are held fixed. That is, for any* r *between* 1 *and* n*,*

$$\det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k \det \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

for k *a scalar and* a_j, u, v *row vectors.*

Note that this tells us how the determinant is affected by one type of elementary row operation.

Corollary 3 *If A has a row of all zeroes, then* $\det A = 0$

Theorem 4 *The determinant can be defined by a cofactor expansion in any row, i.e., for any i,*

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det\left(\tilde{A}_{ij}\right).$$

Corollary 5 *If A has two identical rows, then* $\det A = 0$.

Theorem 6 *If B is obtained from A by exchanging any two rows,* $\det B = -\det A$.

Theorem 7 *Let B be obtained from A by adding a multiple of one row to another. Then* $\det B = \det A$.

Corollary 8 *If $A \in F^{n\times n}$ has* $\operatorname{rank} A < n$, *then* $\det A = 0$.

Theorem 9 *For $A \in F^{n\times n}$,* $\det\left(A^T\right) = \det A$.

Corollary 10 *The determinant can be defined by a cofactor expansion in any column.*

Theorem 11 *For $A, B \in F^{n\times n}$,* $\det(AB) = (\det A)(\det B)$

Corollary 12 *A matrix $A \in F^{n\times n}$ is invertible if and only if* $\det A \neq 0$. *If A is invertible,* $\det\left(A^{-1}\right) = \frac{1}{\det A}$.

1.2 Proofs of theorems and corollaries

Proof of Theorem 2. The proof is by induction on n. Clearly this is true for $n = 1$. Now suppose it is true for n and let $A, B, C \in F^{(n+1)\times(n+1)}$ be of the form

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}, \quad B = \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}, \quad C = \begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

Notice that by the inductive hypothesis, $\det \tilde{A}_{1j} = \det \tilde{B}_{1j} + k \det \tilde{C}_{1j}$ for each j and also note that if $i \neq r$, $A_{ij} = B_{ij} = C_{ij}$ Then if $r > 1$, then

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det \left(\tilde{A}_{1j} \right) \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \left(\det \tilde{B}_{1j} + k \det \tilde{C}_{1j} \right) \\
&= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det \tilde{B}_{1j} + k \sum_{j=1}^{n} (-1)^{1+j} C_{1j} \det \tilde{C}_{1j} \\
&= \det B + k \det C.
\end{aligned}$$

We leave the case of $r = 1$ as an exercise. ■

Proof of Corollary 3. Exercise. ■

Proof of Theorem 4. We do induction on n. The base case is easy. Notice that since we can expand in any row in $\tilde{A}_{ij}$, we can compute the following if we let $\tilde{A}_{ij,k\ell}$ be the matrix obtained by removing the ith and kth rows and jth and ℓth columns from A.

$$\begin{aligned}
\det A &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det \left(\tilde{A}_{1j} \right) \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \left(\sum_{k=1}^{j-1} (-1)^{k+i-1} A_{ik} \det \left(\tilde{A}_{1j,ik} \right) + \sum_{k=j+1}^{n} (-1)^{k+i} A_{ik} \det \left(\tilde{A}_{1j,ik} \right) \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{j-1} (-1)^{j+k+i} A_{1j} A_{ik} \det \left(\tilde{A}_{1j,ik} \right) + \sum_{j=1}^{n} \sum_{k=j+1}^{n} (-1)^{j+k+i+1} A_{1j} A_{ik} \det \left(\tilde{A}_{1j,ik} \right) \\
&= \sum_{k<j} (-1)^{j+k+i} A_{1j} A_{ik} \det \left(\tilde{A}_{1j,ik} \right) + \sum_{k>j} (-1)^{j+k+i+1} A_{1j} A_{ik} \det \left(\tilde{A}_{1j,ik} \right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=1}^{n} (-1)^{i+k} A_{ik} \det \left(\tilde{A}_{ik} \right) \\
&= \sum_{k=1}^{n} (-1)^{i+k} A_{ik} \left(\sum_{j=1}^{k-1} (-1)^{1+j} A_{1j} \det \left(\tilde{A}_{ik,1j} \right) + \sum_{j=k+1}^{n} (-1)^{1+j-1} A_{1j} \det \left(\tilde{A}_{ik,1j} \right) \right) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{k-1} (-1)^{1+j+k+i} A_{ik} A_{1j} \det \left(\tilde{A}_{ik,1j} \right) + \sum_{k=1}^{n} \sum_{j=k+1}^{n} (-1)^{j+k+i} A_{ik} A_{1j} \det \left(\tilde{A}_{ik,1j} \right) \\
&= \sum_{j<k} (-1)^{1+j+k+i} A_{ik} A_{1j} \det \left(\tilde{A}_{ik,1j} \right) + \sum_{j>k} (-1)^{j+k+i} A_{ik} A_{1j} \det \left(\tilde{A}_{ik,1j} \right).
\end{aligned}$$

■

Proof of Corollary 5. We leave this as an exercise if $n \leq 2$. For $n \geq 3$, if we assume rows 1 and j are the same, then we can expand in a row other than those two rows. We see that each of the determinants in the expansion also has two identical rows, and by induction these determinants are all zero. Hence the determinant is zero. ∎

Proof of Theorem 6. Let the rows of A be labeled $a_1, \ldots, a_n$. We see that by linearity in the rows we get

$$0 = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i + a_j \\ \vdots \\ a_i + a_j \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix} + \det \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix}.$$

The first and last determinants are zero, and so if A' is gotten by exchanging rows i and j, then $\det A + \det A' = 0$. ∎

Proof of Theorem 7. By linear in the rows, we get that

$$\det \begin{pmatrix} a_1 \\ \vdots \\ a_i + k a_j \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} = \det \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} + k \begin{pmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} = \det A + 0$$

since the last matrix has two of the same rows. ∎

Proof of Corollary 8. We now know how row operations affect the calculation of the determinant. If the rank is less than n, we can perform row operations to get a row of all zeros. This matrix will have determinant zero and the row operations will all show that the determinant of A is still zero. ∎

Proof of Theorem 11. We know that if A or B is not rank n, then its determinant is zero. Also, if A or B is not rank n, AB is not rank n (why? Show L_A is not onto implies L_{AB} is not onto, and L_B is not one-to-one implies L_{AB} is not one-to-one). Hence the theorem is true if A or B is not rank n. If both A and B are rank n, so is AB. Using what we know about how row operations affect determinants, we can easily see that for any matrix C and elementary matrix E, $\det(EC) = (\det E)(\det C)$. Since any invertible matrix is a product of elementary matrices, we can show that $\det(AB) = \det(E_k \cdots E_1 B) = \det(E_k \cdots E_1)(\det B) = (\det A)(\det B)$. ∎

Proof of Corollary 12. If A is not invertible, then it has rank less than n and so $\det A = 0$. If A is invertible, then $1 = \det I = \det\left(A^{-1}A\right) = \left(\det A^{-1}\right)(\det A)$ so $\det A \neq 0$ and $\det A^{-1} = \frac{1}{\det A}$. ∎

Proof of Theorem 9. If A is not invertible, then neither is A^T, and so $\det A^T = \det A = 0$. If A is invertible, then $A = E_1 \cdots E_k$ for some elementary matrices. We then have that $A^T = E_k^T \cdots E_1^T$. We now see that $\det A^T = \left(\det E_k^T\right) \cdots \det \left(E_1^T\right)$ and the result follows from checking that $\det E_k^T = \det E_k$ for each type of elementary matrix. ■

1.3 Remarks on determinants as volumes

If $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$, then it turns out that $\det A$, where the rows of A are the vectors $v_1, \ldots, v_n$ is equal to $\pm$ the volume of the parelelopiped determined by the vectors. Notice that this is zero if the vectors form a degenerate paralellopiped (lower dimensional), which is geometrically the same as saying the vectors are linearly dependent. The sign has to do with the ordering of the vectors, and obeys the right hand rule for $n = 2, 3$, and gives a way of defining an analogue of the right hand rule in higher dimensions. This is called a choice of orientation and is important in algebraic topology and differential geometry.

2 Problems

- FIS Section 4.1 exercises 2, 3, 6, 7, 10
- FIS Section 4.2 exercises 3, 5-25,27, 29
- FIS Section 4.3 exercises 9-13, 17, 21, 28

3 Characterization of the determinant (Comprehensive/Graduate option)

Definition 13 *A function $\delta : F^{n\times n} \to F$ is called an n-*linear function *if it is a linear function of each row when the remaining rows are held fixed.*

Definition 14 *An n-linear function $\delta : F^{n\times n} \to F$ is called* alternating *if $\delta(A) = 0$ whenever two rows are identical.*

Note that the determinant satisfies both of these properties. In fact, it is essentially the only such function.

Theorem 15 *Let $\delta : F^{n\times n} \to F$ be an alternating n-linear function such that $\delta(I) = 1$. Then $\delta(A) = \det A$ for all $A \in F^{n\times n}$.*

Proof (sketch). If you carefully look at our proofs from the previous section, all we used is n-linearity and alternating to get the characterization of what row operations do. It then follows that $\delta(A) = 0$ if $\operatorname{rank} A < n$, and that if $\operatorname{rank} A = n$, then we can perform row operations to row reduce A to I, and so $\delta(I)$ determines $\delta(A)$. ■

Corollary 16 *We can write the determinant in the following way*

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1\ \sigma(1)} \cdots a_{n\ \sigma(n)}$$

where S_n *is the set of permutations of* $\{1, \ldots, n\}$ *and* $\operatorname{sgn}(\sigma)$ *is the sign of the permutations (*-1 *to the number of transpositions to form the permutation* σ*).*

Proof. It is easy to see that the function on the right satisfy the properties of Theorem 15. ■

4 Problems

- FIS Section 4.5 exercises 16, 18, 19