MATH 413/513

FINAL

Due December 17^{th}*, 2015 by 12:30 pm*

Your Name: ____

I certify that the work on this final is mine and mine alone (please sign):

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Score:

1.	____________
2a-d.	____________
3.	____________
2e. (Comprehensive)	____________
TCE Extra Credit	__________3
Total	____________/100

1. (32pts) Let V and W be finite-dimensional vector spaces of dimension n and m respectively. Let $T : V \to W$ be a linear transformation. Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ be bases for V and W respectively.

a. Show that if T is one-to-one, then $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is linearly independent.

b. Show that if T is onto, then $T(\beta)$ spans W.

c. Suppose $n \leq m$. Show that there exists a linear map $S : W \to V$ such that $ST = I_V$ if and only if T is one-to-one.

d. If $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ are inner products on V and W respectively, show that there exists a linear transformation $T^* : W \to V$ such that

$$\langle T(x), y \rangle_W = \langle x, T^*(y) \rangle_V$$

for all $x \in V$ and $y \in W$.

2. (32pts) Let F be a field (you can let it be $\mathbb{R}$ if this is confusing you). Let $X \in F^{n \times n}$ be a matrix of unknowns and define the map $T_v : F^{n \times n} \to F^n$ by $T_v(X) = Xv$ where v is a fixed (known) vector.

a. Show that T_v is a linear transformation.

For parts b, c, and d, let $n = 2$ and $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b. Find a basis β for the vector space of 2×2 real-valued matrices (you may use one we have already shown is a basis without proof) and write $[T_v]_\beta^{E_2}$ where E_2 is the standard basis for $\mathbb{R}^2$.

c. Find the nullspace of T_v.

d. For which vectors $w \in \mathbb{R}^2$ is there a solution to $T_v(X) = w$? What is the dimension of the space of solutions when there is a solution?

e. (Comprehensive option only, worth +10pts). If $v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$, write $[T_v]_\beta^{E_n}$ for some basis β of $\mathbb{R}^{n \times n}$.

3. (40pts) Consider the space of continuous functions $\mathcal{F}_c([-\pi, \pi], \mathbb{R})$. Let

$$\beta = \{1, \cos x, \sin x, \cos 2x, \sin 2x\}$$

and let $S = \operatorname{span} \beta$. The vector space $\mathcal{F}_c([-\pi, \pi], \mathbb{R})$ has an inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

and β is an orthogonal set with respect to that inner product (you can check this).

a. Show that $\dim S = 5$.

b. Show that $\cos^2 x \in S$.

c. Show that the derivative $\frac{d^2}{dx^2}$ is a linear operator on S.

d. Show that $\frac{d^2}{dx^2}$ is self-adjoint on S. You may use the orthogonality of β (and the integral facts it implies, such as $\int_{-\pi}^{\pi} \sin x \cos x\ dx = 0$) without proof.

e. Write $\left[\frac{d^2}{dx^2}\right]_\beta$ and give the eigenvalues and eigenvectors of $\frac{d^2}{dx^2}$.

f. Give the characteristic polynomial of $\frac{d^2}{dx^2}$ considered as a linear operator $S \to S$.