Proof of the Replacement Theorem

September 2, 2015

Theorem 1 (Replacement Theorem) *Let V be a vector space that is generated by a set G containing exactly n vectors and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a set $H \subseteq G$ containing exactly $n - m$ vectors such that $L \cup H$ generates V.*

The proof will be by induction on m.

1. Show the base case $m = 0$.

2. Suppose the theorem is true for $m \geq 0$, and we will show that it is true for $m + 1$. Let $L = \{v_1, \ldots, v_{m+1}\}$ be linearly independent. Use the inductive hypothesis to find a set $\{u_1, \ldots, u_{n-m}\}$ such that $\{v_1, \ldots, v_m\} \cup \{u_1, \ldots, u_{n-m}\}$ generate V and also to argue that $m \leq n$.

3. Argue that $n > m$ by showing that if $m = n$, then v_{m+1} is a linear combination from $\{v_1, \ldots, v_m\}$, which it is not by assumption (which assumption?) Conclude $m + 1 \leq n$.

4. Using the fact that $v_m \in \operatorname{span}(\{v_1, \ldots, v_m\} \cup \{u_1, \ldots, u_{n-m}\})$, argue that some u_i, say u_1, can be written as a linear combination of the other vectors. Conclude that $\{v_1, \ldots, v_m, v_{m+1}\} \cup \{u_2, \ldots, u_{n-m}\}$ generates V and the induction is completed if we take $H = \{u_2, \ldots, u_{n-m}\}$.