**MATH 413/513**

# $\mathcal{EXAM}$ 2

*November* $18^{th}$, *2015*

**Your Name:** ______________________________

**Directions:**

a. You may NOT use your book or your notes.
b. Please ask for extra scrap paper if needed.
c. Show all work. Unless otherwise noted, a solution without work is worth nothing.
d. The total possible points are 105, but your score will be counted out of 100.
e. Good Luck!

**Score:**

| | |
|---|---|
| 1. | ________ |
| 2. | ________ |
| 3. | ________ |
| 4. | ________ |
| 5. | ________ |
| Total | ________ /100 |

**1. (20pts)** Suppose $A$ is a $10 \times 10$ matrix with entries in $\mathbb{C}$ and the rank of $A$ is 7. For the following, determine whether the statement is necessarily true, necessarily false, or there is not enough information to determine if it is true or false. Give short justifications.

**a.** $\det A = 0$

**b.** The reduced row echelon form of $A$ contains at least one row of zeroes.

**c.** There exists an invertible matrix $Q$ such that $Q^{-1}AQ$ is diagonal.

**d.** There exist invertible matrices $Q$ and $P$ such that $QAP$ is diagonal.

**e.** $A$ is a product of elementary matrices.

**2. (20pts)** Consider the matrix $A = \begin{pmatrix} 1 & -3 & -7 & 5 & -4 \\ 2 & 2 & 2 & 6 & -8 \\ -1 & 2 & 5 & -2 & -1 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix}$. Its reduced row echelon form is $\begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Answer the following questions about $A$:

**a.** What is the rank of $A$?

**b.** Give a basis for the column space of $A$.

**c.** Give a basis for the nullspace of $A$.

**d.** Suppose for some vector $b \in \mathbb{R}^4$, $x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$ is a solution to $Ax = b$. Find all solutions of $Ax = b$ in $\mathbb{R}^5$.

**3. (20pts)**

**a. (10pts)** Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

**b. (10pts)** Suppose a matrix $A$ is diagonalizable, so there is an invertible matrix $Q$ such that $Q^{-1}AQ$ is diagonal. Show that for each standard basis element $e_i$, the vector $Qe_i$ is an eigenvector for $A$. (Hint: each $e_i$ is an eigenvector for the diagonal matrix.)

**4. (25pts)** Let $T$ be an invertible linear operator on a finite-dimensional vector space $V$.
**a. (15pts)** Prove that the scalar $\lambda$ is an eigenvalue of $T$ if and only if $\lambda^{-1}$ is an eigenvalue of $T^{-1}$. Hint: Notice that $T - \lambda I_V = -\lambda T\left(T^{-1} - \lambda^{-1} I_V\right).$

**b. (10pts)** Prove that the eigenspace of $T$ corresponding to $\lambda$ is the same as the eigenspace of $T^{-1}$ corresponding to $\lambda^{-1}$ (Note: you may use the result of part a even if you cannot prove it.)

**5. (20pts)** Recall that an real inner product space is a vector space $V$ over $\mathbb{R}$ together with a product $\langle \cdot, \cdot \rangle$ such that for all $x, y, z \in V$ and $c \in \mathbb{R}$,

- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ ,
- $\langle cx, z \rangle = c \langle x, z \rangle$ ,
- $\langle x, z \rangle = \langle z, x \rangle$ ,
- $\langle x, x \rangle > 0$ if $x \neq \vec{0}$.

Consider the vector space $\mathbb{R}^{2\times 2}$ of two-by-two matrices with real entries and define a product $\langle A, B \rangle = tr\left(B^T A\right)$ , where $tr$ denotes the trace (recall that $tr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$). Show that this is an inner product space. You may use the following facts without proof:

1. $\mathbb{R}^{2\times 2}$ is a vector space over $\mathbb{R}$.
2. The trace is linear: $tr\,(cA + B) = c\; tr\,(A) + tr\,(B)$ for all $c \in \mathbb{R}$ and $A, B \in \mathbb{R}^{2\times 2}$.
3. The trace is invariant under transpose: $tr\left(A^T\right) = tr\,(A)$ for all $A \in \mathbb{R}^{2\times 2}$.
4. $(AB)^T = B^T A^T$ for all $A, B \in \mathbb{R}^{2\times 2}$.