Math 443/543 Graph Theory Notes 7: Laplacians on Graphs

David Glickenstein

November 28, 2012

1 Approximating the Laplacian on a lattice

Recall that the Laplacian is the operator

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

acting on functions $f(x, y, z)$, with analogues in other dimension. Let's first consider a way to approximate the one-dimensional Laplacian. Suppose $f(x)$ is a function and I want to approximate the second derivative $\frac{d^2 f}{dx^2}(x)$. We can take a centered difference approximation to the get this as

$$\begin{aligned}
\frac{d^2 f}{dx^2}(x) &\approx \frac{f'\left(x + \frac{1}{2}\Delta x\right) - f'\left(x - \frac{1}{2}\Delta x\right)}{\Delta x} \\
&\approx \frac{1}{\Delta x}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x - \Delta x)}{\Delta x}\right] \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x) - f(x) + (f(x-\Delta x) - f(x))\right] \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x) + f(x-\Delta x) - 2f(x)\right]
\end{aligned}$$

Note that if we take $\Delta x = 1$, then this only depends on the value of the function at the integer points.

Now consider the graph consisting of vertices on the integers of the real line and edges between consecutive integers. Give an function f on the vertices, we can compute the Laplacian as

$$\triangle f(v_i) = f(v_{i+1}) + f(v_{i-1}) - 2f(v_i)$$

for any vertex v_i. Notice that the Laplacian is an infinite matrix of the form

$$\triangle f = \begin{pmatrix} \cdots & \cdots & & & & & \\ \cdots & -2 & 1 & 0 & & & \\ & 1 & -2 & 1 & 0 & & \\ & 0 & 1 & -2 & 1 & 0 & \\ & & 0 & 1 & -2 & 1 & 0 \\ & & & 0 & 1 & -2 & \cdots \\ & & & & & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \cdots \\ f(v_2) \\ f(v_1) \\ f(v_0) \\ f(v_{-1}) \\ f(v_{-2}) \\ \cdots \end{pmatrix}.$$

Also note that that matrix is exactly equal to the adjacency matrix minus twice the identity. The number 2 is the degree of each vertex, so we can write the matrix, which is called the Laplacian matrix, as

$$L = A - D$$

where A is the adjacency matrix and D is the diagonal matrix consisting of degrees (called the degree matrix).

Note that something similar can be done for a two-dimensional grid. We see that

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) &\approx \frac{f_x\left(x + \frac{1}{2}\Delta x, y\right) - f_x\left(x - \frac{1}{2}\Delta x, y\right)}{\Delta x} + \frac{f_y\left(x, y + \frac{1}{2}\Delta y\right) - f_y\left(x, y - \frac{1}{2}\Delta y\right)}{\Delta y} \\
&\approx \frac{1}{\Delta x}\frac{f(x+\Delta x, y) - f(x,y) - [f(x,y) - f(x-\Delta x, y)]}{\Delta x} \\
&+ \frac{1}{\Delta y}\frac{f(x, y+\Delta y) - f(x,y) - [f(x,y) - f(x, y-\Delta y)]}{\Delta y} \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x, y) - f(x,y) + (f(x-\Delta x, y) - f(x,y))\right] \\
&+ \frac{1}{(\Delta y)^2}\left[f(x, y+\Delta y) - f(x,y) + (f(x, y-\Delta y) - f(x,y))\right] \\
&= \frac{1}{(\Delta x)^2}\left[f(x+\Delta x, y) + (f(x-\Delta x, y) - 2f(x,y))\right] \\
&+ \frac{1}{(\Delta y)^2}\left[f(x, y+\Delta y) + (f(x, y-\Delta y) - 2f(x,y))\right].
\end{aligned}$$

If we let $\Delta x = \Delta y = 1$, then we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) \approx f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y).$$

Note that on the integer grid, this translates to the sum of the value of f for the four vertices neighboring the vertex, minus 4 times the value at the vertex. This is precisely the same as the last time, and we see that this operator can again be written as

$$L = A - D.$$

In general we will call this matrix the Laplacian matrix. It can be thought of as a linear operator on functions on the vertices. Sometimes the Laplacian will denote the negative of this operator (which gives positive eigenvalues instead of negative ones), and sometimes a slight variation is used in the graph theory literature.

2 Electrical networks

One finds applications of the Laplacian in the theory of electrical networks. Recall that the current through a circuit is proportional to the change in voltage, and that constant of proportionality is called the conductance (or resistance, depending on where the constant is placed. Thus, for a wire with conductance C between points v and w with voltages $f(v)$ and $f(w)$ respectively, the current from v to w is $C(f(v) - f(w))$. Kirchoff's law says that if we have a network of wires, each with conductance C, the total current through any given point is zero. Thus, we get that

$$C \sum_{v \text{ adjacent to } w} (f(w) - f(v)) = 0$$

which is the same as $\triangle f = 0$. Note that if the conductances are different, then we get an equation like

$$\sum_{v \text{ adjacent to } w} c_{vw} (f(w) - f(v)) = 0,$$

which is quite similar to the Laplacian. Hence we can use the Laplacian to understand graphs by attaching voltages to some of the vertices and seeing what happens at the other vertices. This is very much like solving a boundary value problem for a partial differential equation!

3 Spectrum

This matrix is symmetric, and thus it has a complete set of eigenvalues. The set of these eigenvalues is called the spectrum of the Laplacian. Notice the following.

Proposition 1 *Let G be a finite graph. The eigenvalues of the matrix L are all nonpositive. Moreover, the constant vector $\vec{1} = (1, 1, 1, \ldots, 1)$ is an eigenvector with eigenvalue zero.*

Proof. It is clear that $\vec{1}$ is an eigenvector with eigenvalue 0 since the sum of the entries in each row must be zero. Now, notice that we can write

$$\begin{aligned} v^T L v &= \sum v_i (Lv)_i \\ &= \sum_i v_i \sum_j L_{ij} v_j \\ &= \sum_{v_i v_j \in E} v_i (v_j - v_i) \\ &= \frac{1}{2} \left[\sum_{v_i v_j \in E} v_i (v_j - v_i) + \sum_{v_i v_j \in E} v_j (v_i - v_j) \right] \\ &= -\frac{1}{2} \sum_{v_i v_j \in E} (v_i - v_j)^2 \leq 0. \end{aligned}$$

(The sums over i are over all vertices, but the sums over $v_i v_j \in E$ is the sum over the edges.) Now note that if v is an eigenvector of L with eigenvalue λ, then $Lv = \lambda v$, and

$$v^T L v = \lambda v^T v = \lambda \sum_i v_i^2.$$

Thus we have that

$$\lambda = \frac{-\frac{1}{2} \sum_{v_i v_j \in E} (v_i - v_j)^2}{\sum_i v_i^2} \leq 0.$$

■

Definition 2 *The eigenvalues of* $-L$ *can be arranged* $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{p-1}$, *where* p *is the order of the graph. The collection* $(\lambda_0, \lambda_1, \ldots, \lambda_p)$ *is called the* spectrum *of the Laplacian.*

Remark 3 *Sometimes the Laplacian is taken to be* $D^{-1/2} L D^{-1/2}$. *If there are no isolated vertices, these are essentially equivalent.*

Remark 4 *Note that the Laplacian matrix, much like the adjacency matrix, depends on the ordering of the vertices and must be considered up to conjugation by permutation matrices. Since eigenvalues are independent of conjugation by permutation matrices, the spectrum is an isomorphism invariant of a graph.*

The following is an easy fact about the spectrum:

Proposition 5 *For a graph* G *of order* p,

$$\sum_{i=0}^{p-1} \lambda_i = 2q.$$

Proof. The sum of the eigenvalues is equal to the trace, which is the sum of the degrees. ■

We will be able to use the eigenvalues to determine some geometric properties of a graph.

4 Connectivity and spanning trees

Recall that $\lambda_0 = 0$, which means that the matrix L is singular and its determinant is zero. Recall the definition of the adjugate of a matrix.

Definition 6 *If M is a matrix, the adjugate is the matrix $M^\dagger = \left[M_{ij}^\dagger\right]$ where $M_{ij}^\dagger$ is equal to $(-1)^{i+j} \det\left(\hat{M}_{ij}\right)$, where $\hat{M}_{ij}$ is the matrix with the ith row and jth column removed.*

The adjugate has the property that

$$M\left(M^\dagger\right)^T = (\det M)\, I,$$

where I is the identity matrix. Applying this to L (which is symmetric) gives that

$$LL^\dagger = 0.$$

Now, the $p \times p$ matrix L has rank less than p. If it is less than or equal to $p-2$, then all determinants of $(p-1)\times(p-1)$ submatrices are zero, and hence $L^\dagger = 0$. If L has rank $p-1$, then it has only one zero eigenvalue, which must be $(1,1,\ldots,1)^T$. Since $LL^\dagger = 0$, all columns of $L^\dagger$ must be a multiple of $(1,1,\ldots,1)^T$. But L is symmetric, so that means that $L^\dagger$ must be a multiple of the matrix of all ones. This motivates the following definition.

Definition 7 *We define $t(G)$ by*

$$t(G) = (-1)^{i+j} \det\left(-\hat{L}_{ij}\right)$$

for any i and j (it does not matter since all are the same).

Remark 8 *It follows that $t(G)$ is an integer.*

Proposition 9 $t(G) = \frac{\lambda_1\lambda_2\cdots\lambda_{p-1}}{p}$.

Proof. In general for a matrix A with eigenvalues $\lambda_0,\ldots,\lambda_{p-1}$ we have that

$$\sum_{k=0}^{p-1} \frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{p-1}}{\lambda_k} = \sum_{i=0}^{p-1} \det \hat{A}_{ii}.$$

In our case, $\lambda_0 = 0$ and the right sum is the sum of p of the same entries, so the result follows. ■

Remark 10 *It follows that $t(G) \geq 0$.*

Recall that a spanning tree of G is a subgraph containing all of the vertices of G and is a tree.

Theorem 11 *The number $t(G)$ is equal to the number of spanning trees of G.*

Proof. Omitted, for now. ■

We can apply this, however, as follows.

Example 1, consider the graph K_3. Clearly this has Laplacian matrix

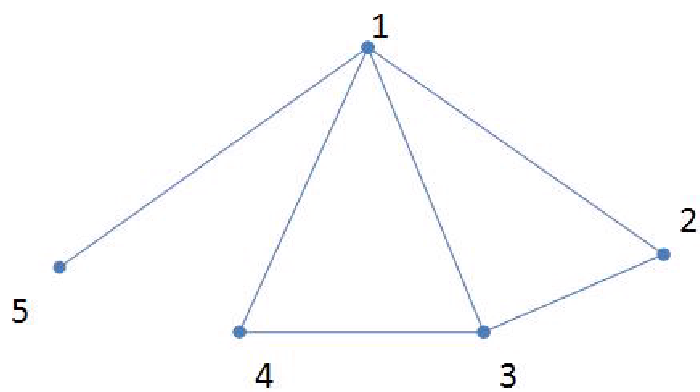
$$L(G) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The number of spanning trees are equal to

$$\det \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = 3.$$

It is clear that each spanning tree is given by omitting one edge, so it is clear there are 3.

Example 2: Consider the following graph.



Its Laplacian matrix is

$$L(G) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The number of spanning trees are equal to

$$\begin{aligned} t(G) &= \det \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \\ &= 2(6-1) + (-2) = 8 \end{aligned}$$

One can check directly that it has eight spanning trees.

Corollary 12 $\lambda_1 \neq 0$ *if and only if* G *is connected.*

Proof. $\lambda_1 = 0$ if and only if $t(G) = 0$ since $t(G)$ is the product of the eigenvalues $\lambda_1 \lambda_2 \cdots \lambda_{p-1}/p$ and λ_1 is the minimal eigenvalue after λ_0. But $t(G) = 0$ means that there are no spanning trees, so G is not connected. ■

Now we can consider the different components.

Definition 13 *The disjoint union of two graphs* $G = G_1 \sqcup G_2$ *is the graph gotten by taking* $V(G) = V(G_1) \sqcup V(G_2)$ *and* $E(G) = E(G_1) \sqcup E(G_2)$ *where* $\sqcup$ *is the disjoint union of sets.*

It is not hard to see that if we number the vertices in G by first numbering the vertices of G_1 and then numbering the vertices of G_2, that the Laplacian matrix takes the form

$$L(G) = \begin{pmatrix} L(G_1) & 0 \\ 0 & L(G_2) \end{pmatrix}.$$

This means that the eigenvalues of $L(G)$ are the union of the eigenvalues of $L(G_1)$ and $L(G_2)$. This implies the following.

Corollary 14 *If* $\lambda_n = 0$, *then there are at least* $n+1$ *connected components of* G.

Proof. Induct on n. We already know this is true for $n = 1$. Suppose $\lambda_n = 0$. We know there must be at least n components, since $\lambda_n = 0$ implies $\lambda_{n-1} = 0$. We can then write the matrix $L(G)$ in the block diagonal form with $L(G_i)$ along the diagonal for some graphs G_i. Since $\lambda_n = 0$, one of these graphs must have $\lambda_1(G_i) = 0$. But that means that there is another connected component, completing the induction. ■

In order to prove the Matrix Tree Theorem, we need another characterization of the Laplacian.

Definition 15 *Let* G *be a directed* (p,q)*-graph. The oriented vertex-edge incidence graph is a* $p \times q$ *matrix* $Q = [q_{ir}]$, *such that* $q_{ir} = 1$ *if* $e_r = (v_i, v_j)$ *for some* j *and* $q_{ir} = -1$ *if* $e_r = (v_j, v_i)$ *for some* j.

Proposition 16 *The Laplacian matrix L satisfies*

$$-L = QQ^T$$

for any oriented vertex-edge incidence graph (so, given an undirected graph, we can take any orientation), i.e.,

$$L_{ij} = -\sum_{r=1}^{q} q_{ir} q_{jr}.$$

Proof. If $i = j$, then we see that $L_{ii} = -\deg v_i$. Notice that q_{ir} is nonzero if v_i is in edge e_r. Thus

$$\sum_{r=1}^{q} (q_{ir})^2 = \deg v_i.$$

Now for $i \neq j$, we have that $q_{ir} q_{jr} = -1$ if $e_r = v_i v_j$, giving the result. ■

Remark 17 *This observation can be used to give a different proof that L_{ij} has all nonpositive eigenvalues.*

The proof of the matrix tree theorem proceeds as follows. We need only consider the case

$$\det\left(-\hat{L}_{11}\right).$$

A property of determinants allows us to use the fact that $-L = QQ^T$ to compute this determinant in terms of determinants of submatrices of Q.

Proposition 18 (Binet-Cauchy Formula) *Let A be an $n \times m$ matrix and B be an $n \times m$ matrix (usually we think $n < m$). We can compute*

$$\det(AB) = \sum_S \det A_S \det B_S^T$$

where A_S is matrix consisting of the columns of A specified by S, and S ranges over all choices of n columns.

We will not prove this theorem, but it is quite geometric in the case that $B = A^T$, which is our case. If we consider the rows of A, then $\det AA^T$ is equal to the square of the volume of the parallelopiped determines by these rows (since we can rotate A to be in $\mathbb{R}^n$). The formula says that the volume squared is equal to the sum of the squares of the volumes when projected onto the coordinate planes. These are generalizations of the pythagorean theorem!

Thus, we have that

$$-L = QQ^T$$

for a choice of directions. To compute $t(G)$, we have that

$$\begin{aligned} t(G) &= \det\left(-\hat{L}_{11}\right) = \det\left(\hat{Q}\hat{Q}^T\right) \\ &= \sum_S \left(\det \hat{Q}_S\right)^2 \end{aligned}$$

where $\hat{Q}$ is Q with the first row removed and S ranges over collections of $p-1$ edges in G. We need to understand what $\det \hat{Q}_S$ represents. The claim is that $\det Q_S = \pm 1$ if the edges represented by S form a spanning tree for G and zero otherwise. If S forms a disconnected graph, then $\hat{Q}_S$ looks like

$$\begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix}$$

and since M' consists of columns with exactly one 1 and one -1, we see that $(0, 0, \ldots, 0, 1, 1, 1, \ldots 1)^T$ (respecting the form above) is a zero eigenvector, and hence the determinant is equal to 0. Now, if S contains a cycle, then since it has $p-1$ edges, the corresponding graph must be disconnected, and the previous comment follows.The problem of showing that $\det \hat{Q}_S = \pm 1$ if the edges represented by S form a spanning tree will be part of your next homework assignment.

It follows that

$$t(G) = \sum_S \left(\det \hat{Q}_S \right)^2$$

is equal to the number of spanning trees.