Math 443/543 additional problems sheet 1

November 28, 2012

1) Let A be a link matrix for a web with no dangling nodes. Show that if we are able to compute rankings using only the link matrix (that is, we take $m = 0$), then adding a new page that links only into the previous web does not change the rankings, and the new page is ranked last.

2) Suppose $x = (1, 0, 0, \ldots, 0)^T$. Let A be the link matrix for a web and $M = (1 - m) A + mS$ the matrix used for PageRank. Describe an interpretation of $A^n x$ and $M^n x$ in terms of the interpretation of A and M as a matrix of probabilities (this is similar to the interpretation of powers of the adjacency matrix).

3) Using the probabalistic interpretation of M, discuss why you would want a small value of m instead of a large one. What happens if $m = 1$?

4) Find the eigenvalues of $-L$ for the graph $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1 v_2, v_3 v_4\}$. Note: two eigenvalues should be easy to find using our theorems. You will still need to find two more eigenvalues either by guessing eigenvectors (which is not too hard) or by using the fact that the eigenvalues are the solution to $\det (A - \lambda I) = 0$. The fact that the sum of the eigenvalues is equal to the trace may also be helpful.

5) Use the Matrix Tree Theorem to compute the number of spanning trees of $K_{1,m}$ (the answer should be fairly obvious, but I want to see that this number actually equals $t(G) = \det\left(-\hat{L}_{11}\right)$).

6) Compute the number of spanning trees of $K_{3,3}$ and compare to the result using the Matrix Tree Theorem (use matlab or wolfram alpha or some other tool).

7) Let $Q \in \mathbb{R}^{p \times q}$ be the directed edge-vertex adjacency matrix, $\hat{Q}$ be the matrix Q with the first row removed, and S be a subset of the columns representing edges forming a spanning tree in the graph as in the proof of the Matrix Tree Theorem, show that $\det \hat{Q}_S = \pm 1$. Hint: induct on p using the fact that a tree must have at least two vertices of degree one.

Alternative formulation. Let $Q \in \mathbb{R}^{p \times (p-1)}$ be a directed edge-vertex adjacency matrix corresponding to a tree. Let $\hat{Q}$ be the $\mathbb{R}^{(p-1)\times(p-1)}$ matrix gotten by removing the first row from Q. Show that $\det \hat{Q} = \pm 1$. Hint: induct on p using the fact that a tree must have at least two vertices of degree one.

8) Suppose we have a network (G, Φ) where $\Phi : E \to \mathbb{R}_+$ give positive weights to the edges. The weighted Laplacian matrix is given by $L_\Phi = A - D$ where A is the weighted adjacency matrix $A_{ij} = \Phi(v_i v_j)$ if $v_i v_j \in E$ and 0 otherwise, and D is the diagonal matrix given by $D_{ii} = \sum_{j=1}^{p} A_{ij}$.

A) Show that for a function f on the vertices,

$$(L_\Phi f)_i = \sum_{\substack{j \text{ such that} \\ v_i v_j \in E}} \Phi(v_i v_j) \quad (f_j - f_i)$$

B) Show that $\det(-L_\Phi) = 0$ and that if λ is an eigenvalue of $-L_\Phi$ then $\lambda \geq 0$.

9) Give an interpretation of the weighted Laplacian in terms of electrical networks and Kirchoff's law. What do the weights represent?