CHAPTER 1: SMOOTH MANIFOLDS

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1. Introduction

This semester we will focus primarily on the basics of smooth manifold theory. We will spend some time on what a manifold is and what its properties are. This involves a discussion of the differential theory of manifolds, including vector fields and tensor fields. We will then discuss a bit about integration on manifolds, specifically the integration of differential forms. Finally, the relationship between the two is given by Stokes' Theorem.

Next semester we will tackle algebraic topology and its relation to differential forms and the work from this semester.

Reminder: the qualifying exam will also cover the basics of complex analysis. This will not be covered in this class, though I hope to give some questions about it throughout the semester.

You are expected to have a good grounding in multivariable calculus and point-set topology already. You may want to review a bit.

2. Topological manifolds

We define a topological manifold as follows:

Definition 1. *A topological space M is a* manifold of dimension n *if*

1. *M is Hausdorff, and*
2. *M is second countable, and*
3. *M is locally Euclidean of dimension n.*

This definition depends on the following definitions:

Definition 2. *A topological space X is* Hausdorff *if for all $x, y \in X$ such that $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Definition 3. *A topological space X is* second countable *if it has a countable basis for the topology, i.e., there exists a countable collection of open sets $\{U_\alpha\}_{\alpha \in \mathbb{N}}$ such that for any open set $U \subset X$ containing a point x, there exists a $\beta \in \mathbb{N}$ such that $x \in U_\beta \subseteq U$.*

Definition 4. *A topological space X is* locally Euclidean of dimension n *if for each $x \in X$, there exists an open set $U \subset X$ containing X and a map $\phi : U \to \mathbb{R}^n$ such that $\phi : U \to \phi(U)$ is a homeomorphism (in particular, $\phi(U)$ is an open subset of $\mathbb{R}^n$).*

The first two conditions are essentially technical conditions, with the third condition giving the main condition on being a manifold.

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Remark 1. *We sometimes use the terminology M^n is a manifold to mean that M is a manifold of dimension n. We also say M is an n-dimensional manifold.*

The set and map in the definition of locally Euclidean is called a coordinate chart.

Definition 5. *A* coordinate chart *on M is a pair (U,ϕ) where $U \subseteq M$ is open and $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism. The set U is called a* coordinate domain *or* coordinate neighborhood *or* coordinate patch. *If $\phi(U)$ is a ball in $\mathbb{R}^n$, U is called a* coordinate ball. *A coordinate chart (U,ϕ) is* centered at p *if $\phi(p) = 0$.*

Theorem 6. *If M is a manifold, every point $x \in M$ is contained in a coordinate ball centered at x.*

Proof. Since M is locally Euclidean, x must be contained in a coordinate chart (U,ϕ). Since $\phi(U)$ is an open set containing $\phi(x)$, by the topology of $\mathbb{R}^n$ there must be an open ball B containing $\phi(x)$ and contained in $\phi(U)$. The appropriate coordinate ball is $\left(\phi^{-1}(B), \phi|_{\phi^{-1}(B)}\right)$. If we compose ϕ with a translation taking $\phi(x)$ to 0, we have completed the proof. □

Example 1. *The graph of a continuous function $f : U \to \mathbb{R}^k$, where $U \subseteq \mathbb{R}^n$, is a manifold:*

$$\Gamma(f) = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k : x \in U \text{ and } y = f(x)\right\}.$$

$\Gamma(f)$ is given the subspace topology (of $\mathbb{R}^{n+k}$), so it is automatically Hausdorff and second countable. It has a single coordinate chart given by $(\Gamma(f), \pi_1)$ where π_1 is projection onto the first coordinate. The inverse of π_1 is the map $x \to (x, f(x))$. This map is continuous if and only if f is continuous.

Example 2. *Spheres are manifolds. An n-sphere is defined as*

$$\mathbb{S}^n = \left\{x \in \mathbb{R}^{n+1} : |x|^2 = 1\right\}.$$

Since it is a subspace of $\mathbb{R}^n$, it is Hausdorff and second countable. We can define coordinate charts as follows

$$\begin{aligned} U_k^+ &= \{x \in \mathbb{S}^n : x_k > 0\}, \\ \phi_k^+ &= \left(x^1, x^2, \ldots, x^{k-1}, \widehat{x^k}, x^{k+1}, \ldots, x^{n+1}\right), \end{aligned}$$

where $\widehat{x^k}$ denotes that x^k is not there (so $\phi_k : U_k \to \mathbb{R}^n$). The inverse of ϕ_k is

$$\psi_k\left(y^1, \ldots, y^n\right) = \left(y^1, y^2, \ldots, y^{k-1}, \sqrt{1 - \sum_{j=1}^n \left(y^j\right)^2}, y^k, \ldots, y^n\right)$$

(since $x^k > 0$). To get charts that cover the sphere, we also need the corresponding $\left(U_k^-, \phi_k^-\right)$ charts.

Remark 2. *There are other important coordinate charts for spheres, notably stereographic projection. See problems.*

Example 3. *Real projective space $\mathbb{RP}^n$ is a manifold. We define real projective space $\mathbb{RP}^n$ to be the set of lines in $\mathbb{R}^{n+1}$. We can represent it more formally, by* *writing a line in* $\mathbb{R}^{n+1}$ *as an equivalence class of points* $v \in \mathbb{R}^{n+1} \setminus \{0\}$ *such that* $v \sim v'$ *if and only if* $v = tv'$ *for some* $t' \in \mathbb{R} \setminus \{0\}$*. Then* $\mathbb{RP}^n$ *is the quotient*

$$\left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim .$$

We can then give $\mathbb{RP}^n$ *the quotient topology (i.e., a set is open if and only if its pre-image under the quotient map is open). We write* $\left(x^0 : x^1 : \cdots : x^n\right)$ *for an element in the quotient. We can define coordinate charts*

$$\phi_i\left(x^0 : x^1 : \cdots : x^n\right) = \left(\frac{x^0}{x^i}, \frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^n}{x^i}\right),$$

where the domain is

$$U_i = \left\{\left(x^0 : x^1 : \cdots : x^n\right) : x^i \neq 0\right\}.$$

We can see that this map is continuous by seeing that the quotient map composed with this map is continuous. Notice that the image $\phi_i\left(U_i\right) = \mathbb{R}^n$ *and its inverse is*

$$\phi_i^{-1}\left(y^1, \ldots, y^n\right) = \left(y^1 : \cdots : y^{i-1} : 1 : y^{i+1} : \cdots : y^n\right).$$

(Check this: it is not totally obvious!) This map is also continuous. One can check that the space is Hausdorff and second countable.

Remark 3. *It is not too hard to show that* $\mathbb{RP}^n$ *is compact. You can rewrite* $\mathbb{RP}^n$ *as a quotient of* $\mathbb{S}^n$*, showing that* $\mathbb{RP}^n$ *is the continuous image of a compact space, and hence compact.*

Example 4. *Products of manifolds are manifolds. If we have manifolds* M^m *and* N^n*, then we can give* $M \times N$ *the product topology. If we have coordinate charts* (U, ϕ) *for* M *and* (V, ψ) *for* N*, then there is a coordinate chart* $(U \times V, \phi \times \psi)$ *for* $M \times N$*. Given charts which cover* M *and* N*, we can construct charts which cover* $M \times N$*. By induction we can show that any finite product of manifolds is a manifold as well.*

Example 5. *The torus* $\mathbb{S}^1 \times \mathbb{S}^1$ *is a manifold. So are other tori* $\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$*.*

Finally, we give some topological properties of manifolds that may come in useful.

Theorem 7. *Every manifold has a countable basis of coordinate balls.*

Corollary 8. *Every manifold is locally compact (i.e., every point has a neighborhood contained in a compact set).*

Remark 4. *We will use neighborhood of a point to mean an open set containing that point. Some authors use neighborhood to mean any set which contains an open set containing that point. Thus, we will always assume that a neighborhood is open.*

Definition 9. *A topological space* X *is* connected *if there do not exist two disjoint, nonempty sets whose union is* X*. The space* X *is* path connected *if every two points are connected by a path (i.e., for any* $x, y \in X$*, there exists a continuous map* $\gamma : [0,1] \to X$ *such that* $\gamma(0) = x$ *and* $\gamma(1) = y$*). A topological space is* locally path connected *if* X *has a basis of path connected sets (i.e., every point has a path connected neighborhood).*

Note that connected need not imply path connected, though the reverse implication is true. In a manifold, however, they are equivalent.

Theorem 10. *A connected manifold is connected if and only if it is path connected. Furthermore, the components of a manifold are the same as its path components.*

Proof. Since path connected implies connected, we need only prove that connected implies path connected. Consider a manifold M^n and let $x \in M$. Let S be the set of all points y in M for which there exists a path from x to y. We will show that S is open and closed. Suppose $y \in S$. Then there is a coordinate chart (U, ϕ) around y (i.e., U is a neighborhood of y) and so $\phi(U) \subseteq \mathbb{R}^n$. Since $\phi(U)$ is open, there exists a path γ from $\phi(y)$ to z for every z in a sufficiently small open ball B around $\phi(y)$. B is an open set, so $\phi^{-1}(B)$ is open in M, and contained in S (since we can extend the continuous path from x to y by $\phi^{-1} \circ \gamma$). Thus S is open. Now suppose y is a limit point of S. Then for every open neighborhood of y there is a point $x' \in S$. Take a coordinate ball (U, ϕ) centered at y. Then there exists $x' \in S \cap U$. Since $x' \in S$, there is a path γ from x to x'. Furthermore, since $\phi(U)$ is a ball, there is a path α from $\phi(x')$ to 0, so by juxtaposing γ with $\phi^{-1} \circ \alpha$, there is a path from x to y, thus $y \in S$ and S is closed. Since S is open and closed and M is connected, we must have that S is everything and M is path connected. The second part follows easily. □

Furthermore, we have the following two theorems.

Theorem 11. *Every topological manifold is locally path connected.*

Proof. Every point is contained in a coordinate ball, so the result follows. □

Theorem 12. *A topological manifold has at most countably many components, each of which is a topological manifold.*

Proof. Suppose the manifold had uncountably many components. Then it is impossible to have a countable basis for the topology, since one can take each of the components separately and each must contain a different coordinate neighborhood. Each component is clearly a manifold. □

3. Smooth manifolds

First we review smooth maps between subsets of $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets.

Definition 13. *A map $f : U \to V$ is* smooth *(or C^∞) if each of its component functions has continuous partial derivatives of all orders at every point. If f is bijective with smooth inverse, it is called a* diffeomorphism.

Since a smooth map is continuous, we have that a diffeomorphism is a homeomorphism.

Remark 5. *In this field, usually functions refer to maps whose codomain is $\mathbb{R}$, whereas maps can be between any two manifolds. However, sometimes the term function is used in place of map.*

Since each point in a manifold is contained in a coordinate patch, one can consider smoothness when pre- and post-composing with the coordinate maps. I.e., we want a map $f : M \to N$ to be smooth if $\phi \circ f \circ \psi^{-1} : U \to V$, is smooth where $\left(\tilde{U}, \psi\right)$ is a coordinate patch for M, $\left(\tilde{V}, \phi\right)$ is a coordinate patch for N, and $U \subseteq \psi\left(\tilde{U}\right) \subseteq \mathbb{R}^n$ and $V \subseteq \phi\left(\tilde{V}\right) \subseteq \mathbb{R}^m$. In particular, the identity map should be smooth.

Definition 14. *Let* (U,ϕ) *and* (V,ψ) *be coordinate patches such that* $U\cap V\neq\varnothing$. *A map of the form* $\psi\circ\phi^{-1}:\phi(U\cap V)\to\psi(U\cap V)$ *is called a* transition map *(note that it is a homeomorphism). The two charts are* smoothly compatible *if the transition map is a diffeomorphism. (Note if* $U\cap V=\varnothing$, *we still say the transition map is smooth.)*

Definition 15. *An* atlas *for* M *is a collection of charts whose domains cover* M. *A* smooth atlas *is an atlas such that any two charts are smoothly compatible.*

Definition 16. *A function* $f:M\to\mathbb{R}$ *on a manifold with an atlas* $\mathcal{A}$ *is smooth if for every* $(U,\phi)\in\mathcal{A}$, *the function* $f\circ\phi^{-1}$ *is smooth.*

It may be the case that different atlases give the same collection of smooth functions.

Proposition 17. *If* $\mathcal{A}$ *and* $\mathcal{A}'$ *are smooth atlases on* M, *then they determine the same set of smooth functions if and only if* $\mathcal{A}\cup\mathcal{A}'$ *is a smooth atlas.*

Proof. Suppose they determine the same set of smooth functions. Then one can decompose the set of transition maps into components functions. The fact that the functions are smooth shows that transitions between patches in each atlas must be smooth. Conversely, suppose there is a function $f:M\to\mathbb{R}$ that is smooth in $\mathcal{A}$ but not in $\mathcal{A}'$. Then there exists $(U',\phi')\in\mathcal{A}$' such that $f\circ(\phi')^{-1}$ is not smooth, say at a point $\phi'(p)$. Let (U,ϕ) be a coordinate neighborhood of p. We claim that $\phi\circ(\phi')^{-1}$ is not smooth, since if it were smooth, then

$$f\circ(\phi')^{-1}=\left[f\circ\phi^{-1}\right]\circ\left[\phi\circ(\phi')^{-1}\right]$$

is a composition of smooth maps and hence smooth. Thus $\mathcal{A}\cup\mathcal{A}'$ is not a smooth atlas. □

Let's give some examples of atlases:

Example 6. *On the manifold* $\mathbb{R}$, *the set* $\{(\mathbb{R},Id)\}$ *is a smooth atlas.*

Example 7. *We can also consider the manifold* $\mathbb{R}$ *together with the set* $\{(\mathbb{R},\phi)\}$, *where* $\phi:\mathbb{R}\to\mathbb{R}$ *is defined by* $\phi(x)=x^{1/3}$. *This is an atlas. Note, however, that with this atlas,* $f:\mathbb{R}\to\mathbb{R}$ *given by* $f(x)=x^{1/3}$ *is a smooth function, since* $f\circ\phi^{-1}(x)=x$ *is smooth. However, in the previous example,* f *is not a smooth function!*

Example 8. *The charts we gave for the sphere and* $\mathbb{RP}^n$ *also make atlases.*

There are two ways to describe smooth structure: (1) as an equivalence class of manifolds with smooth atlases and (2) as a maximal smooth atlas. They amount to the same thing.

Definition 18. *Two smooth atlases* $\mathcal{A}$ *and* $\mathcal{A}'$ *on a manifold* M *are equivalent if* $\mathcal{A}\cup\mathcal{A}'$ *is a smooth atlas. This is an equivalence relation.*

Note there is a partial ordering of atlases determined by the property of being a subset. Since unions are atlases, there is a maximal atlas determined as the union of all equivalent smooth atlases.

Definition 19. *An atlas is* maximal *if it is not contained in another atlas. A* smooth structure *on a manifold* M *is a maximal smooth atlas. A* smooth manifold *is a manifold together with a maximal smooth atlas.*

Lemma 20. *Every smooth atlas for M is contained in a unique maximal smooth atlas.*

Proof. Let $\mathcal{A}$ be a smooth atlas for M. Let $\overline{\mathcal{A}}$ denote the set of all charts smoothly compatible with every chart in $\mathcal{A}$. We need to show that if (U,ϕ) and (V,ψ) are charts in $\overline{\mathcal{A}}$, then they are smoothly compatible, which would imply that $\overline{\mathcal{A}}$ is a smooth atlas. We need only show that the transition map $\psi\circ\phi^{-1}:\phi(U\cap V)\to\psi(U\cap V)$ is smooth, since it must already be a bijection and the continuity of the inverse follows from the fact that its inverse is $\phi\circ\psi^{-1}$ (which would also be smooth since ϕ and ψ are arbitrary). To show smoothness, it suffices to show it at a point $p\in\phi(U\cap V)$. Since $\mathcal{A}$ is an atlas, there is a chart $(W,\eta)\in\mathcal{A}$ such that $\phi^{-1}(p)\in W$. Since (U,ϕ) and (V,ψ) are smoothly compatible with (W,η), we have that $\psi\circ\eta^{-1}$ and $\eta\circ\phi^{-1}$ are smooth in a neighborhood of $\eta\circ\phi^{-1}(p)$ and p respectively. But since the composition of smooth maps is smooth, we must have that $\psi\circ\phi^{-1}$ is smooth in a neighborhood of $\phi(p)$ and we have shown that $\overline{\mathcal{A}}$ is an atlas.

We now need to show that $\overline{\mathcal{A}}$ is maximal and unique. Suppose $\overline{\mathcal{A}}$ is contained in another atlas, $\mathcal{A}'$. Then clearly $\mathcal{A}$ is contained in $\mathcal{A}'$, and so all charts in $\mathcal{A}'$ are compatible with all charts in $\mathcal{A}$. Thus $\mathcal{A}'$ is contained in $\overline{\mathcal{A}}$. It follows that $\overline{\mathcal{A}}=\mathcal{A}'$. Now suppose that there is another maximal atlas $\mathcal{A}''$ containing $\mathcal{A}$. Since every chart in $\mathcal{A}''$ is compatible with every chart in $\mathcal{A}$, we must have that $\mathcal{A}''\subseteq\overline{\mathcal{A}}$. Since $\mathcal{A}''$ is maximal, it follows that $\mathcal{A}''=\overline{\mathcal{A}}$. □

This says that we do not need to specify the maximal atlas to determine the smooth structure, only some atlas. Many of the examples of manifolds we have given so far have standard smooth structures, such as the spheres, projective spaces, and $\mathbb{R}$.

Remark 6. *An interesting question is whether there are many smooth structures on simple objects like the sphere. The answer is quite remarkable, and we will talk about this a bit later in the course.*

Remark 7. *By changing the compatibility requirement to C^k or analytic or something else, one creates other types of manifolds.*

Now, we have the notion of smooth coordinate chart (which must be in the atlas), and all of the corresponding terminology as in the topological manifold case. In the smooth setting, coordinate charts are often described in the following way. Suppose (U,ϕ) is a smooth coordinate chart. We can then identify U and $\tilde{U}=\phi(U)\subseteq\mathbb{R}^n$. For any $p\in U$, we often write

$$\phi(p)=\left(x^1(p),x^2(p),\ldots,x^n(p)\right)$$

as the local coordinates at p. Note that if $p\in V$, another coordinate chart, then there is an identification with $y(p)=\phi(p)$ given by a (smooth) transition map.

Example 9. *The standard smooth structure on $\mathbb{R}^n$ is generated by the identity map when one chooses the standard basis. If one has an abstract vector space V of dimension n, one can identify it with $\mathbb{R}^n$ by choosing a basis $E_1,\ldots,E_n$, and the identification is*

$$\phi\left(x^1E_1+\cdots+x^nE_n\right)=\left(x^1,\ldots,x^n\right).$$

One gets additional charts in the smooth structure by considering a change of basis (invertible) matrix A_i^j, *so that there is another basis*

$$F_k = \sum_{i=1}^{n} A_k^i E_i.$$

In general, we do not write the sum, since we are summing on one up and one down index that is repeated, so we write

$$F_k = A_k^i E_i.$$

We call this Einstein summation notation*. Note that in these coordinates, the point looks different. The transition map sends* $(x^1, \ldots, x^n)$ *to* $\left(\left(A^{-1}\right)_k^1 x^k, \ldots, \left(A^{-1}\right)_k^n x^k\right)$ *since*

$$x^k E_k = x^j \left(A^{-1}\right)_j^k A_k^i E_i = x^j \left(A^{-1}\right)_j^k F_k.$$

Remark 8. *We will discuss Einstein summation notation and up/down indices in more detail later in the course. However, it is important to note that coordinate functions always have upper indices since basis vectors have lower indices. Linear transformations always have one up and one down index.*

Example 10. *The following are also all examples of smooth manifolds:*

- *Open subsets of smooth manifolds.*
- *The set of all matrices.*
- *The set of invertible matrices (an open subset of the set of all matrices)*
- *Products of smooth manifolds.*

There are two important ways of constructing manifolds: as zero sets of functions (using implicit function theorems) and by gluing together open sets. The main result in this section is to characterize how to glue together charts to make a manifold.

Lemma 21 (Smooth manifold construction lemma)**.** *Let* M *be a set and suppose we are given a collection* $\{U_\alpha\}$ *of subsets of* M *together with injective maps* $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ *for each* α *satisfying:*

1. *For each* α, *the set* $\phi_\alpha(U_\alpha)$ *is an open subset of* $\mathbb{R}^n$.
2. *For each* α *and* β, *the set* $\phi_\alpha(U_\alpha \cap U_\beta)$ *is an open subset of* $\mathbb{R}^n$.
3. *Whenever* $U_\alpha \cap U_\beta \neq \varnothing$, *the map* $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ *is a diffeomorphism.*
4. *There is a countable subset of* $\{U_\alpha\}$ *that covers* M.
5. *Whenever* p *and* q *are distinct points in* M, *either there is an* α *such that* $p, q \in U_\alpha$ *or there are disjoint sets* U_α *and* U_β *such that* $p \in U_\alpha$ *and* $q \in U_\beta$.

Then M *has a unique smooth manifold structure such that* (U_α, ϕ_α) *are smooth coordinate charts for each* α.

Proof. We claim that the sets $\phi_\alpha^{-1}(V)$, where V is open in $\mathbb{R}^n$, form a basis for a topology, and we take this topology on M. To check that these form a basis, we need that if $\phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(V') \neq \varnothing$, then for any $p \in \phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(V')$ there exists a neighborhood $\phi_\gamma(V'')$ of p (such that V'' is open in $\mathbb{R}^n$). In fact, we can show that $\phi_\alpha^{-1}(V) \cap \phi_\beta^{-1}(V') = \phi_\alpha^{-1}(V'')$. Let $U = \phi_\alpha^{-1}(V)$ and $U' = \phi_\beta^{-1}(V')$. We simply let

$$V'' = V \cap \phi_\alpha \circ \phi_\beta^{-1}(V').$$

The set V'' is open since the range of $\phi_\alpha \circ \phi_\beta^{-1}$, namely $\phi_\beta\left(U_\alpha \cap U_\beta\right)$, is open by (2) and since $\phi_\alpha \circ \phi_\beta^{-1}$ is a diffeomorphism (hence homeomorphism). One can then check that $\phi_\alpha^{-1}\left(V''\right) = \phi_\alpha^{-1}\left(V\right) \cap \phi_\beta^{-1}\left(V'\right)$ since ϕ_α is injective.

With this topology, the coordinate maps are homeomorphisms, and give that the manifold is locally Euclidean. We can use (5) to show that M is Hausdorff (one can separate points either within a coordinate chart using balls in $\mathbb{R}^n$ or by separating them in disjoint coordinate neighborhoods. Finally, if one takes the countable subset from (4), each one can be given a countable basis by pulling back one from $\mathbb{R}^n$. The union of these is a countable basis for M.

By (2) and (3), the U_α form a smooth atlas, determining a unique smooth structure. □

4. Manifolds with boundary

I just want to introduce the notion of manifold with boundary, but not do much with it at this time. A manifold is locally Euclidean, meaning that it is like the model space $\mathbb{R}^n$ near every point. Manifolds with boundary can sometimes be like a half space.

Definition 22. *The upper half space $\mathbb{H}^n \subset \mathbb{R}^n$ is the set*

$$\mathbb{H}^n = \left\{ \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n : x^n \geq 0 \right\}.$$

Definition 23. *An n-dimensional (topological) manifold with boundary is a second-countable, Hausdorff space in which every point is homeomorphic to an open subset of $\mathbb{H}^n$.*

Definition 24. *A smooth atlas for an n-dimensional manifold with boundary is a collection of charts so that transition maps can be extended to smooth maps between open subsets of $\mathbb{R}^n$.*