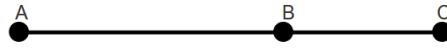


Problem 1 - Finding the Golden Ratio Consider the line segment below and assume it satisfies $\frac{AB}{BC} = \frac{AC}{AB}$.



(a) What is this ratio? (Hint: Set $AC = 1$, $AB = x$, $BC = ?$ and solve for x . Why can we set $AC = 1$?). Can you determine the ratio without assuming that $AC = 1$?

(b) What is the quadratic equation that you solved in part (a)? Can you find a similar quadratic equation whose solution is the reciprocal of x ?

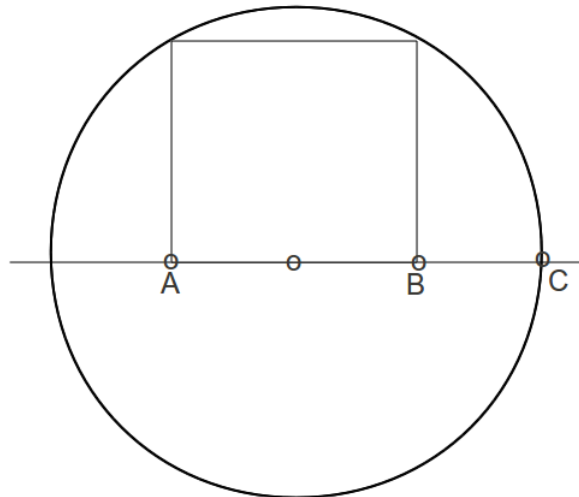
From now on we will denote the above ratio as φ . It is called the **golden ratio**.

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.61803\dots$$

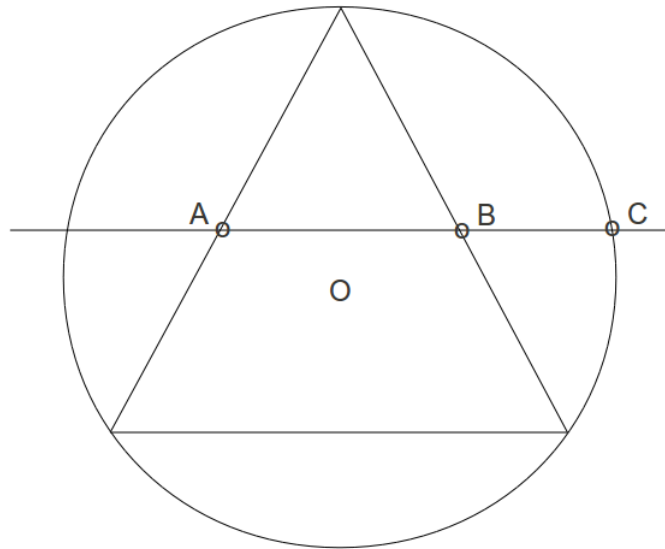
$$\frac{1}{\varphi} = \frac{-1 + \sqrt{5}}{2} = 0.61803\dots$$

Problem 2 - Appearances in Geometry The Golden Ratio appears in certain simple geometric constructions.

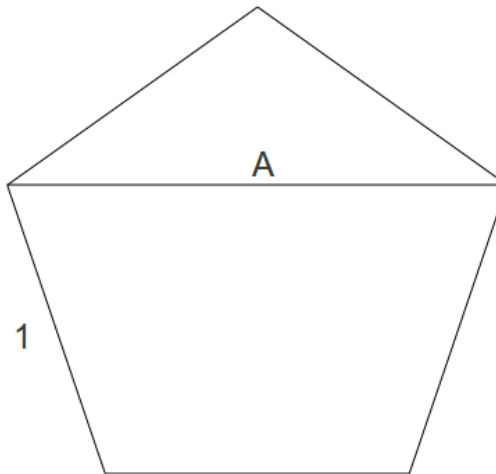
(a) A square is inscribed in a semicircle. The ratio $\frac{AB}{BC}$ is the golden ratio. Prove it.



(b) An equilateral triangle is inscribed in a circle. We draw a line through the midpoints of two sides of the triangle. The ratio $\frac{AB}{BC}$ is the golden Ratio. Prove it.



(c) A regular pentagon has 5 sides of equal length with interior angles equal to 108° . The line segment A is equal to the golden ratio. Prove it.



Problem 3 - Immortal rabbits A pair of adult rabbits produces a pair of baby rabbits once each month. Each pair of baby rabbits requires one month to grow to be adults and subsequently produces one pair of baby rabbits each month thereafter. Determine the number of pairs of adult and baby rabbits after some number of months. The rabbits are immortal.

Problem 4 - Golden Ratio and Fibonacci Sequence Let $a_0 = 1$ and $a_1 = 1$. The Fibonacci sequence is defined recursively by $a_{n+2} = a_{n+1} + a_n$. The first few numbers of the sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

(a) Recall that the golden ratio φ is a solution to $x^2 - x - 1 = 0$. Form a relation between φ^n and φ where n is a positive integer. In other words, express φ^n linearly in φ . (Hint: So we know $\varphi^2 = \varphi + 1$. Hence $\varphi^3 = \varphi^2 + \varphi$. Substituting for φ^2 gives $\varphi^3 = 2\varphi + 1$. Now generalize to φ^n . Your answer will involve the Fibonacci sequence.)

(b) Denote $\frac{1}{\varphi}$ by ρ . Recall that $\rho = \frac{-1 + \sqrt{5}}{2}$ and also that it satisfies $(-\rho)^2 = (-\rho) + 1$. Is it clear that $-\rho$ satisfies the same relation as φ is part (a)?

(c) Use parts (a) and (b) to simplify the expression $\varphi^n - (-\rho)^n$. You should get an expression involving a_n . Solve for a_n .

(d) Using this expression for a_n show that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \varphi$.