Chapter 2

Transformations and Expectations

Expected Values

Outline

Continuous Random Variables

For X a continuous random variable with density function f_X, consider the discrete random variable $\tilde{X}$ obtained from X by rounding down. (Δ has a different meaning here than in the previous section).

Say, for example, we give lengths by rounding down to the nearest millimeter. Thus, $\tilde{X} = 1.655$ meters for any lengths X satisfying

$$1.655 \text{ meters} < X \leq 1.656 \text{ meters}.$$

The random variable $\tilde{X}$ is discrete and has a mass function $f_{\tilde{X}}$. Thus, the expected value

$$Eg(\tilde{X}) = \sum_{\tilde{x}} g(\tilde{x}) f_{\tilde{X}}(\tilde{x}).$$

Continuous Random Variables

Let Δx be the spacing between values for $\tilde{X}$. Then, $\tilde{x}$, an integer multiple of Δx, represents a possible value for $\tilde{X}$,

$$\tilde{X} = \tilde{x} \quad \text{if and only if} \quad \tilde{x} < X \leq \tilde{x} + \Delta x.$$

With this, we can give the mass function

$$f_{\tilde{X}}(\tilde{x}) = P\{\tilde{X} = \tilde{x}\} = P\{\tilde{x} < X \leq \tilde{x} + \Delta x\}.$$

Now, by the property of the density function,

$$P\{\tilde{x} \leq X < \tilde{x} + \Delta x\} \approx f_X(\tilde{x})\Delta x.$$

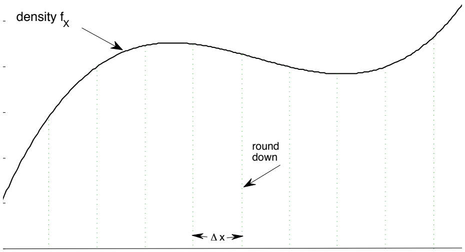


Figure: The value of the mass function $f_{\tilde{X}}(\tilde{x})$ is the area of the rectangular region above and to the right of $\tilde{x}$.

Continuous Random Variables

For this discrete random variable $\tilde{X}$, we can use the approximation of its mass function to approximate the expected value.

$$\begin{aligned} Eg(\tilde{X}) &= \sum_{\tilde{x}} g(\tilde{x}) f_{\tilde{X}}(\tilde{x}) = \sum_{\tilde{x}} g(\tilde{x}) P\{\tilde{x} \leq X < \tilde{x} + \Delta x\} \\ &\approx \sum_{\tilde{x}} g(\tilde{x}) f_X(\tilde{x}) \Delta x. \end{aligned}$$

This last sum is a Riemann sum and so taking limits as $\Delta x \to 0$, we have that $\tilde{X}$ converges to X and the Riemann sum converges to the definite integral. Thus,

$$Eg(X) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

Continuous Random Variables

Exercise. For the dart example, the density $f_X(x) = 2x$ on the interval $[0, 1]$ and 0 otherwise. Determine EX and EX^2.

$$EX = \int_0^1 x f_X(x)\ dx = \int_0^1 2x^2\ dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}.$$

$$EX^2 = \int_0^1 x^2 f_X(x)\ dx = \int_0^1 2x^3\ dx = \frac{1}{2}x^3\Big|_0^1 = \frac{1}{2}.$$

Indeed, for $p > 0$,

$$EX^p = \frac{2}{p+1}$$

Continuous Random Variables

As in the case of discrete random variables, a similar formula to holds for a vector of random variables $X = (X_1, X_2, \ldots, X_n)$, f_X, the joint probability density function and g a real-valued function of the vector $x = (x_1, x_2, \ldots, x_n)$.

The expectation in this case is an n-dimensional Riemann integral. For example, if X_1 and X_2 has joint density $f_{X_1,X_2}(x_1, x_2)$, then

$$Eg(X_1, X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2) f_{X_1,X_2}(x_1, x_2)\ dx_2 dx_1$$

provided that the improper Riemann integral converges.

Survival Function

We learned that the sample mean is equal to the area under the empirical survival function for nonnegative observations. We check to see if an analogous identity holds for continuous random variables.

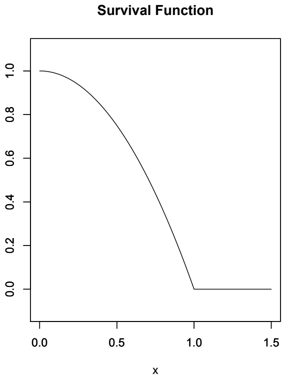
Let X be a nonnegative random variable with distribution function F_X and density f_X.

Then the survival function

$$\overline{F}_X(x) = P\{X > x\} = 1 - F_X(x).$$

The question we are asking is if the following identity holds:

$$EX = \int_0^\infty P\{X > x\}\, dx.$$



Survival Function

We integrate by parts.

$$\begin{aligned}\int_0^\infty (1 - F_X(x))dx &= x(1 - F_X(x))\Big|_0^\infty - \int_0^\infty x(-f_X(x))\ dx \\ &= \int_0^\infty x f_X(x)\ dx = EX.\end{aligned}$$

$$\begin{aligned} u(x) &= 1 - F_X(x) & v(x) &= x \\ u'(x) &= -f_X(x) & v'(x) &= 1 \end{aligned}$$

Survival Function

More generally, for $g(x)$ increasing with $g(0) = 0$ and $Eg(X) < \infty$

$$\begin{aligned}\int_0^\infty g'(x)(1 - F_X(x))dx &= g(x)(1 - F_X(x))\Big|_0^\infty - \int_0^\infty g(x)(-f_X(x))\ dx \\ &= 0 + \int_0^\infty g(x)f_X(x)\ dx = Eg(X).\end{aligned}$$

$$\begin{aligned}u(x) &= 1 - F_X(x) & v(x) &= g(x) \\ u'(x) &= -f_X(x) & v'(x) &= g'(x)\end{aligned}$$

$$0 \le g(b)(1 - F_X(b)) = \int_b^\infty g(b)f_X(x)\,dx \le \int_b^\infty g(x)f_X(x)\,dx \to 0$$

as $b \to \infty$ because $Eg(X) < \infty$.

Exercise. For the identity above, show that it is sufficient to have $|g(x)| < h(x)$ for some increasing h with $Eh(X)$ finite.

Survival Function

Example. For the dart example, the survival function

$$\bar{F}_X(x) = P\{X > x\} = 1 - x^2.$$

Thus,

$$EX = \int_0^1 \bar{F}_X(x)\,dx = \int_0^1 (1 - x^2)dx = x - \frac{1}{3}x^3\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

$$EX^2 = \int_0^1 2x\bar{F}_X(x)\,dx = \int_0^1 (2x - 2x^3)dx = x^2 - \frac{1}{2}x^4\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

Survival Function

Example. Let T be an exponential random variable, then for some λ, $P\{T > t\} = \exp -(\lambda t)$. Then

$$ET = \int_0^\infty P\{T > t\}\,dt = \int_0^\infty \exp -(\lambda t)\,dt = -\frac{1}{\lambda}\exp -(\lambda t)\Big|_0^\infty = 0 - (-\frac{1}{\lambda}) = \frac{1}{\lambda}.$$

$$\begin{aligned} ET^2 &= \int_0^\infty 2tP\{T > t\}\,dt = \int_0^\infty 2t\exp -(\lambda t)\,dt = \frac{2}{\lambda}\int_0^\infty t\ \lambda\exp -(\lambda t)\,dt \\ &= \frac{2}{\lambda}ET = \frac{2}{\lambda^2}. \end{aligned}$$

We can now show by mathematical induction that $EX^n = n!/\lambda^n$.

Normal Random Variables

The most important density function we shall encounter is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}), \quad z \in \mathbb{R}.$$

for Z, the standard normal random variable.

Because the function ϕ has no simple antiderivative, we use a numerical approximation to compute the distribution function, denoted Φ.

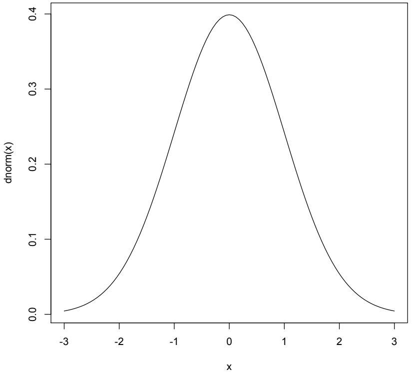


Figure: Normal density, plotted by entering `curve(dnorm(x),-3,3)`

Normal Random Variables

The expectation,

$$EZ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \exp(-\frac{z^2}{2})\, dz = 0$$

because the integrand is an odd function. Next to evaluate

$$EZ^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \exp(-\frac{z^2}{2})\, dz = \frac{1}{\sqrt{2\pi}} \left(-z \exp(-\frac{z^2}{2}) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp(-\frac{z^2}{2}) dz \right) = 1.$$

we integrate by parts.

$$\begin{array}{rclcrcl} u(z) & = & z & & v(z) & = & -\exp(-z^2/2) \\ u'(z) & = & 1 & & v'(z) & = & z \exp(-z^2/2) \end{array}$$

Use l'Hôpital's rule to see that the first term is 0. The fact that the integral of a probability density function is 1 shows that the second term equals 1.

Summary

Using the Riemann-Stielitjes integral we can write the expectation in a unified manner,

$$Eg(X) = \int_{-\infty}^{\infty} g(x)\, dF_X(x).$$

For the Riemann-Stieltjes integral

$$\int_a^b g(x)\, dF_X(x),$$

we begin with a partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n+1} = b\}$and a Riemann-Stieltjes sum

$$R(\mathcal{P}, g, F) = \sum_{i=1}^{n} g(x_i)\Delta F_X(x_i)$$

where $\Delta F_X(x_i) = F_X(x_{i+1}) - F_X(x_i)$. The integral is the limit as mesh $\mathcal{P} \to 0$.

Summary

For discrete random variables, $\Delta F_X(x_i) = 0$ if the interval, $(x_i, x_{i+1}]$ does not contain a possible value for the random variable X. If mesh $\mathcal{P}$ is sufficiently small, the jump points $x \in (x_i, x_{i+1}]$ of size $\Delta F_X(x_i) = f_X(x)$ are isolated in distinct intervals. Thus, the Riemann-Stieltjes sums converges to

$$\sum_x g(x) f_X(x)$$

for X having mass function f_X.

For continuous random variables, $\Delta F_X(x_i) \approx f_X(x_i)\Delta x$. Thus, the Riemann-Stieltjes sum is approximately a Riemann sum for the product $g \cdot f_X$ and converges to

$$\int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

for X having density function f_X.

Summary

	distribution function $F_X(x) = P\{X \leq x\}$	
discrete	random variable	continuous
mass function $f_X(x) = P\{X = x\}$		density function $f_X(x)\Delta x \approx P\{x \leq X < x + \Delta x\}$
$f_X(x) \geq 0$ $\sum_{\text{all } x} f_X(x) = 1$	properties	$f_X(x) \geq 0$ $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
$P\{X \in A\} = \sum_{x \in A} f_X(x)$	probability	$P\{X \in A\} = \int_A f_X(x)\, dx$
$Eg(X) = \sum_{\text{all } x} g(x) f_X(x)$	expectation	$Eg(X) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$