Topic 5

Basics of Probability

Equally Likely Outcomes and the Axioms of Probability

Outline

Introduction

A probability model has two essential pieces of its description.

- Ω, the sample space, the set of possible outcomes.
    - An event is a collection of outcomes. We can define an event by explicitly giving its outcomes,

      $$A = \{\omega_1, \omega_2, \cdots, \omega_n\}$$

      or with a description

      $$A = \{\omega; \omega \text{ has property } \mathcal{P}\}.$$

      In either case, A is subset of the sample space, $A \subset \Omega$.
- P, the probability assigns a number to each event.
  Thus, a probability is a function.
    - The domain is the collection of all events.
    - The range is a number.

We will see soon which numbers we will accept as probabilities of events.

Equally Likely Outcomes

If Ω is a finite sample space, then if each outcome is equally likely, we define the probability of A as the fraction of outcomes that are in A. This leads to the formula

$$P(A) = \frac{\#(A)}{\#(\Omega)}.$$

Exercise. Find the probabilities under equal likely outcomes.

(a) Toss a coin.

$$P\{\text{heads}\} = \frac{\#(A)}{\#(\Omega)} = \underline{\hspace{2cm}}.$$

(b) Toss a coin three times.

$$P\{\text{toss at least two heads in a row}\} = \frac{\#(A)}{\#(\Omega)} = \underline{\hspace{2cm}}$$

(c) Roll two dice.

$$P\{\text{sum is } 9\} = \frac{\#(A)}{\#(\Omega)} = \underline{\hspace{2cm}}$$

Equally Likely Outcomes

Because we always have $0 \leq \#(A) \leq \#(\Omega)$, we always have

$$P(A) \geq 0, \qquad P(\Omega) = 1 \tag{1\&2}$$

This gives us 2 of the three axioms.

Toss a coin 4 times. $\#(\Omega) = 16$

$A = \{\text{exactly 3 heads}\} = \{\text{HHHT, HHTH, HTHH, THHH}\}$ $\#(A) = 4$

$$P(A) = 4/16 = 1/4$$

$B = \{\text{exactly 4 heads}\} = \{\text{HHHH}\}$ $\#(B) = 1$

$$P(B) = 1/16$$

Now let's define the set $C = \{\text{at least three heads}\}$. If you are asked the supply the probability of C, your intuition is likely to give you an immediate answer.

$$P(C) = 5/16.$$

Axioms of Probability

The events A and B have *no* outcomes in common. We say that the two events are disjoint or mutually exclusive and write $A \cap B = \emptyset$. In this situation, we have an addition principle

$$\#(A \cup B) = \#(A) + \#(B).$$

Divide by $\#(\Omega)$, then we obtain the following identity: If $A \cap B = \emptyset$, then

$$\frac{\#(A \cup B)}{\#(\Omega)} = \frac{\#(A)}{\#(\Omega)} + \frac{\#(B)}{\#(\Omega)}.$$

$$\text{or} \quad P(A \cup B) = P(A) + P(B). \tag{3}$$

Using this property, we see that

$$P\{\text{at least 3 heads}\} = P\{\text{exactly 3 heads}\} + P\{\text{exactly 4 heads}\} = \frac{4}{16} + \frac{1}{16} = \frac{5}{16}.$$

Any function P that accepts events as its domain, returns numbers as its range and satisfies Axioms 1, 2, and 3 as defined in (1), (2), and (3) can be called a probability.

Axioms of Probability

By iterating the procedure in Axiom 3, we can also state that if the events, $A_1, A_2, \cdots, A_n$, are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n). \tag{3'}$$

This is a sufficient definition for a probability if the sample space Ω is finite.

Consider a rare event - a lightning strike at a given location, winning the lottery, finding a planet with life - and look for this event repeatedly, we can write

$$A_j = \{\text{the first occurrence appears on the } j\text{-th observation}\}.$$

Then, each of the A_j are mutually exclusive and $\{\text{event occurs eventually}\}$

$$= A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{j=1}^{\infty} A_j = \{\omega; \omega \in A_j \text{ for some } j\}.$$

Axioms of Probability

We would like to say that

$$\begin{aligned} P\{\text{event occurs eventually}\} &= P(A_1) + P(A_2) + \cdots + P(A_n) + \cdots \\ &= \sum_{j=1}^{\infty} P(A_j) = \lim_{n\to\infty} \sum_{j=1}^{n} P(A_j). \end{aligned}$$

This would call for an extension of Axiom 3 to an infinite number of mutually exclusive events. This is the general version of Axiom 3 we use when we want to use *calculus*:

For mutually exclusive events, $\{A_j; j \geq 1\}$, then

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j) \qquad (3'')$$

Thus, statements (1), (2), and (3") give us the complete axioms of probability.

Consequences of the Axioms

- Complement Rule. Because A and its complement $A^c = \{\omega; \omega \notin A\}$ are mutually exclusive,

$$P(A) + P(A^c) = P(A \cup A^c) = P(\Omega) = 1$$

or

$$P(A^c) = 1 - P(A).$$

For example, if we toss a *biased* coin. We may want to say that $P\{\text{heads}\} = p$ where p is not necessarily equal to $1/2$. By necessity,

$$P\{\text{tails}\} = 1 - p.$$

Toss a fair coin 4 times.

$$P\{\text{fewer than 3 heads}\} = 1 - P\{\text{at least 3 heads}\} = 1 - \frac{5}{16} = \frac{11}{16}.$$

Consequences of the Axioms

- Difference Rule. Let $B \setminus A$ denote the outcomes in B but *not* in A. If $A \subset B$, then

$$P(B \setminus A) = P(B) - P(A).$$

- Monotonicity Rule. If $A \subset B$, then $P(B \setminus A) \geq 0$ and

$$P(A) \leq P(B).$$

We already know that for any event A, $P(A) \geq 0$.

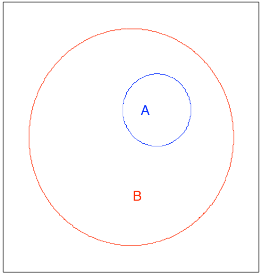

Figure: Monotonicity Rule

By the monotonicity rule,

$$P(A) \leq P(\Omega) = 1.$$

The range of a probability is a subset of $[0, 1]$.

Consequences of the Axioms

- Inclusion-Exclusion Rule. For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

- Bonferroni Inequality. For any two events A and B,

$$P(A \cup B) \leq P(A) + P(B).$$

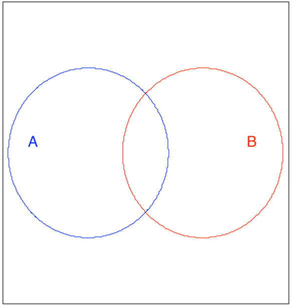

Figure: Inclusion-Exclusion Rule

Consequences of the Axioms

- Continuity Property. If events satisfy

$$B_1 \subset B_2 \subset \cdots \text{ and } B = \bigcup_{i=1}^{\infty} B_i$$

Then, $P(B_i)$ is increasing. In addition,

$$P(B) = \lim_{i\to\infty} P(B_i).$$

Similarly, use the symbol $\supset$ to denote contains. If events satisfy

$$C_1 \supset C_2 \supset \cdots \text{ and } C = \bigcap_{i=1}^{\infty} C_i$$

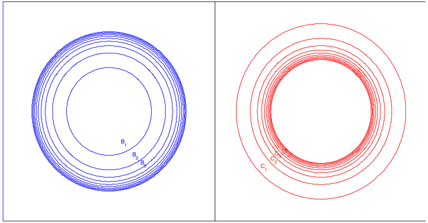

Figure: Continuity Property. (left) B_i increasing to an event B. (right) C_i decreasing to an event C.

Then $P(C_i)$ is decreasing and

$$P(C) = \lim_{i\to\infty} P(C_i).$$

Consequences of the Axioms

Exercise. The statement of $a : b$ odds for an event A indicates that

$$\frac{P(A)}{P(A^c)} = \frac{a}{b}.$$

Show that

$$P(A) = \frac{a}{a+b}.$$

So, for example, $1 : 2$ odds means $P(A) = 1/3$ and $5 : 3$ odds means $P(A) = 5/8$.