Topic 5

Basics of Probability

Counting

Outline

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Fundamental Principle of Counting

We start with the fundamental principle of counting.

Suppose that two experiments are to be performed.

- Experiment 1 can have n_1 possible outcomes and
- for *each* outcome of experiment 1, experiment 2 has n_2 possible outcomes.

Then together there are $n_1 \times n_2$ possible outcomes.

Example. For a group of n individuals, one is chosen to become the president and a second is chosen to become the treasurer. By the multiplication principle, if these position are held by different individuals, then this task can be accomplished in

$$n \times (n-1)$$

possible ways.

Fundamental Principle of Counting

Exercise. Find the number of ways to draw two cards and the number of ways to draw two aces from a deck of 52 cards.

Exercise. Generalize the fundamental principle of counting to k experiments.

Assume that we have a collection of n objects and we wish to make an ordered arrangement of k of these objects. Using the generalized principle of counting, the number of possible outcomes is

$$n \times (n-1) \times \cdots \times (n-k+1).$$

We will write this as $(n)_k$ and say n falling k.

The Birthday Problem

In a list the birthday of k people, there are

$$365^k$$

possible lists (ignoring leap year day births) and

$$(365)_k$$

possible lists with no date written twice. Thus, the probability, under equally likely outcomes, that no two people on the list have the same birthday is

$$prob(k) = \frac{(365)_k}{365^k} = \frac{365 \cdot 364 \cdots (365 - k + 1)}{365^k}$$

and, by the complement rule,

$$P\{\text{at least one pair of individuals share a birthday}\} = 1 - \frac{(365)_k}{365^k}.$$

The Birthday Problem

Exercise. We can create an iterative process to compute $prob(k)$ by noting that

$$prob(k) = \frac{(365)_k}{365^k} = \frac{(365)_{k-1}}{365^{k-1}} \cdot \frac{(365-k+1)}{365} = prob(k-1) \cdot \frac{(365-k+1)}{365}$$

Compute $prob(k)$ for a range of values for k (say $k = 1$ to $n = 60$ or $n = 90$).

Begin by creating a vector `prob = rep(1,n)` to set a vector of n ones. Note that $prob(1) = 1$. Compute $prob(k)$ using a `for` loop for $k = 2$ to n. For what value of k is $prob(k) \approx 1/2$.

Definition. The number of ordered arrangements of all n objects (also called permutations) is

$$(n)_n = n \times (n-1) \times \cdots \times 1 = n!,$$

n factorial. We take $0! = 1$.

The Birthday Problem

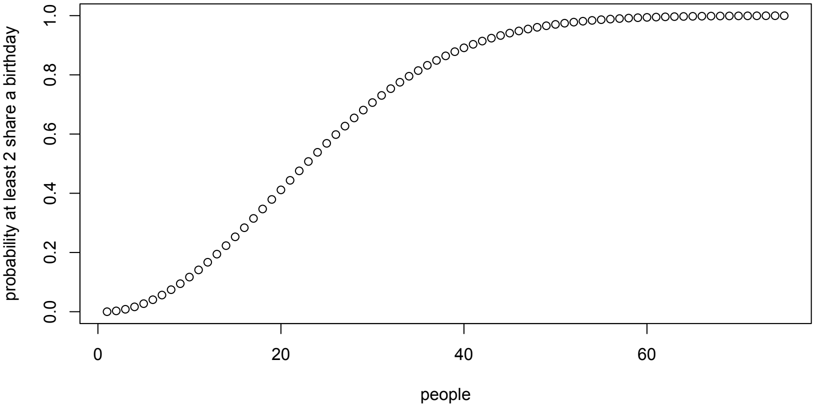

Figure: Plot of $1 - prob(k)$ versus k, the number of people. Notice the value for which $1 - prob(k) \approx 1/2$

Combinations

Write

$$\binom{n}{k}$$

for the number of different groups of k objects that can be chosen from a collection of size n.

We will next find a formula for this number by counting the number of possible outcomes in two different ways. To introduce this with a concrete example.

Suppose 3 cities will be chosen out of 8 under consideration for a vacation. If we think of the vacation as visiting three cities in a particular order, for example,

New York then Boston then Montreal.

Combinations

Then we are counting the number of *ordered arrangements*. For the choice of 3 cities out of 8, we have

$$(8)_3 = 8 \cdot 7 \cdot 6$$

outcomes. If we are just considering the 3 cities we visit, irrespective of order, then these unordered choices are combinations. The number of ways of doing this is written

$$\binom{8}{3}.$$

After we have chosen the 3 cities, we will also have to pick an order to see them, giving

$$\binom{8}{3} \times 3 \cdot 2 \cdot 1 = \binom{8}{3} 3!$$

possible vacations if the order of the cities is included in the choice. These two strategies are counting the same possible outcomes and so must be equal.

$$(8)_3 = \binom{8}{3} 3! \quad \text{or} \quad \binom{8}{3} = \frac{(8)_3}{3!}.$$

Combinations

Then we are counting the number of *ordered arrangements*. For the choice of k cities out of n, we have

$$(n)_k = n \cdots (n-k+1)$$

outcomes. If we are just considering the k cities we visit, irrespective of order, then these unordered choices are combinations. The number of ways of doing this is written

$$\binom{n}{k}.$$

After we have chosen the k cities, we will also have to pick an order to see them, giving

$$\binom{n}{k} \times k \cdots 1 = \binom{n}{k} k!$$

possible vacations if the order of the cities is included in the choice. These two strategies are counting the same possible outcomes and so must be equal.

$$(n)_k = \binom{n}{k} k! \quad \text{or} \quad \binom{n}{k} = \frac{(n)_k}{k!}.$$

Combinations

Exercise. Explain in words the following identities.

1. $\binom{n}{1} = \binom{n}{n-1} = n$
2. $\binom{n}{k} = \binom{n}{n-k}$.
3. The binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

From 2, we set

$$\binom{n}{n} = \binom{n}{0} = 1.$$

The number of combinations is computed in R using `choose`. In the vacation example above, $\binom{8}{3}$ is determined by entering

```
> choose(8,3)
[1] 56
```

Pascal's Triangle

Theorem. From the example on vacations

$$\binom{8}{3} = \binom{7}{2} + \binom{7}{3}.$$

Assume that New York is one of 8 vacation cities. Then of the $\binom{8}{3}$ possible vacations,

- If New York is on the list, then we must choose the remaining 2 cities from the remaining 7.
- If New York in *not* on the list, then all 3 choices must be from the remaining 7.
- Because New York is either on the list or off the list, but never both, the two types of choices have no overlap.

Pascal's Triangle

Theorem. Pascal's triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Assume that New York is one of n vacation cities. Then of the $\binom{n}{k}$ possible vacations,

- If New York is on the list, then we must choose the remaining $k-1$ cities from the remaining $n-1$.
- If New York in *not* on the list, then all k choices must be from the remaining $n-1$.
- Because New York is either on the list or off the list, but never both, the two types of choices have no overlap.

Pascal's Triangle

	$k-1$	k	
$n-1$	$\binom{n-1}{k-1}$	$\binom{n-1}{k}$	$\leftarrow$ the sum of these two numbers
n		$\binom{n}{k}$	$\leftarrow$ equals this number

Exercise. Use the identity above to construct Pascal's triangle up to $n = 8$