Topic 7

Random Variables and Distribution Functions

Distribution Functions

Outline

Introduction

statistics	probability
universe of information ⇓ ask a question and collect data ⇓ organize into the empirical cumulative distribution function ⇓ compute sample means and variances	sample space - Ω and probability - P ⇓ define a random variable X ⇓ organize into the cumulative distribution function ⇓ compute distributional means and variances

Definition of a Random Variable

A random variable is a real valued function from the probability space.

$$X : \Omega \to \mathbb{R}.$$

Typically, we shall use capital letters near the end of the alphabet, e.g., X, Y, Z for random variables. The range of a random variable is called the state space.

Exercise. Give some random variables on the following probability spaces, Ω.

1. Roll a die 3 times and consider the sample space $\Omega = \{(i,j,k); i,j,k = 1,2,3,4,5,6\}$.
2. Flip a coin 10 times and consider the sample space Ω, the set of 10-tuples of heads and tails.

We can create new random variables via composition of functions:

$$\omega \mapsto X(\omega) \mapsto g(X(\omega))$$

Thus, if X is a random variable, then so are

$$X^2, \quad \exp \alpha X, \quad \sqrt{X^2+1}, \quad \tan^2 X, \quad \lfloor X \rfloor$$

Distribution Functions

A (cumulative) distribution function of a random variable X is defined by

$$F_X(x) = P\{\omega \in \Omega; X(\omega) \leq x\} = P\{X \leq x\}.$$

For the complement of $\{X \leq x\}$, we have the survival function

$$\overline{F}_X(x) = P\{X > x\} = 1 - P\{X \leq x\} = 1 - F_X(x).$$

Choose $a < b$, then the event $\{X \leq a\} \subset \{X \leq b\}$. Their set theoretic difference

$$\{X \leq b\} \setminus \{X \leq a\} = \{a < X \leq b\}.$$

Consequently, by the difference rule for probabilities,

$$P\{a < X \leq b\} = P(\{X \leq b\} \setminus \{X \leq a\}) = P\{X \leq b\} - P\{X \leq a\} = F_X(b) - F_X(a).$$

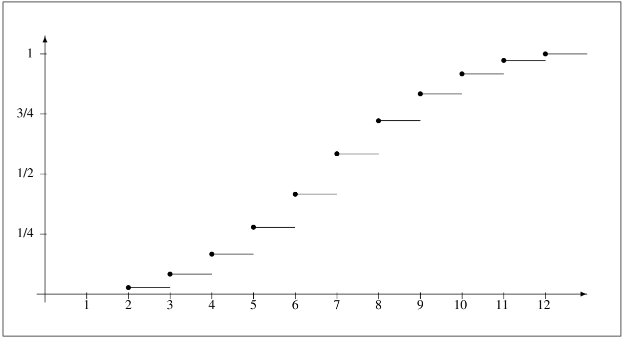
In particular, F_X is non-decreasing.

Distribution Functions

Let X be the sum of the values on two fair dice,

x	2	3	4	5	6	7	8	9	10	11	12
$P\{X = x\}$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Here is F_X the cumulative distribution function.



Distribution Functions

Notice that the distribution function

- is constant in between the possible values for X,
- has a jump size at x is equal to $P\{X = x\}$, and
- is right continuous.

Call X a discrete random variable if its distribution function F_X has these properties.

Examples.

$$\frac{3}{36} = P\{X = 4\} = F_X(4) - F_X(4-) = \frac{6}{36} - \frac{3}{36}$$

$$P\{4 < X \leq 7\} = F_X(7) - F_X(4) = \frac{21}{36} - \frac{6}{36} = \frac{15}{36} = \frac{5}{12}.$$

$$P\{4 \leq X \leq 7\} = F_X(7) - F_X(4-) = \frac{21}{36} - \frac{3}{36} = \frac{18}{36} = \frac{1}{2}.$$

Distribution Functions

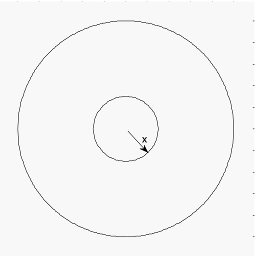
Exercise.

1. Flip a fair coins 3 times. Let X be the number of heads. Under equally likely outcomes, find
$$P\{X = x\} \quad \text{for} \quad x = 0, 1, 2, \text{ and } 3.$$
and use this to sketch a graph of the distribution function F_X.
2. Deal 5 cards out of a deck of 52. Let X be the number of $\diamondsuit$. Under equally likely outcomes, use the `choose` function in R to determine
$$P\{X = x\} \quad \text{for} \quad x = 0, 1, 2, 3, 4, \text{ and } 5.$$
and use this to sketch a graph of the distribution function F_X.

Distribution Functions

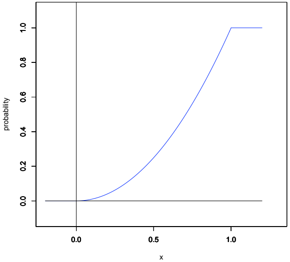
For a dart board with radius 1, assume that the dart lands randomly uniformly. Let X be the distance from the center. For $x \in [0, 1]$,

$$F_X(x) = P\{X \le x\} = \frac{\text{area inside circle of radius } x}{\text{area of circle}} = \frac{\pi x^2}{\pi 1^2} = x^2.$$



Thus, we have the distribution function

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x^2 & \text{if } 0 < x \le 1, \\ 1 & \text{if } x > 1. \end{cases}$$



Distribution Functions

Exercise.

1. Find the probability that the dart no more than $1/2$ unit from the center.
2. Find the probability that the dart lands further $1/3$ unit but no more than $2/3$ unit from the center.
3. Find the median, $x_{1/2}$ so that $P\{X \le x_{1/2}\} = 1/2$.

Definition. X is continuous random variable if it has a cumulative distribution function F_X that is differentiable.

Properties of Distribution Functions

A distribution function F_X has the property that it is right continuous, starts at 0, ends at 1, and does not decrease with increasing values of x.

In mathematical terms,

- For every a, $\lim_{x \to a+} F_X(x) = F_X(a)$.
- $\lim_{x \to -\infty} F_X(x) = 0$.
- $\lim_{x \to \infty} F_X(x) = 1$.
- For every a, b satisfying $a < b$,

$$F_X(a) \leq F_x(b).$$