Topic 16

Interval Estimation

Building Confidence Intervals

Outline

Pivotal Quantities

A random quantity $Q(\mathbf{X}, \theta)$ is called a pivotal quality or a pivot if

- is a function of $\mathbf{X}$ and θ where θ is the only unknown parameter.
- The distribution of $Q(\mathbf{X}, \theta)$ does not depend on θ.

Examples.

For $\mathbf{X} = (X_1, \ldots, X_n)$, independent and identically distributed.

- If the X_i arise from a location family with density $f(x - \theta)$, then

$$Q(\mathbf{X}, \theta) = \sum_{i=1}^{n} a_i(X_i - \theta)$$

is a pivotal quantity. In particular, $\bar{X} - \theta$ is a pivot.

Pivotal Quantities

- If the X_i arise from a scale family with density $\frac{1}{\beta} f\left(\frac{x}{\beta}\right)$, then

$$Q(\mathbf{X}, \theta) = \frac{\sum_{i=1}^n a_i X_i}{\beta}$$

  is a pivotal quantity. In particular, $\sum_{i=1}^n X_I/\beta$ is a pivot.

- If the X_i arise from a location-scale family with density $\frac{1}{\beta} f\left(\frac{x-\theta}{\beta}\right)$

$$Q(\mathbf{X}, \theta) = \frac{\bar{X} - \theta}{S}$$

  is a pivotal quantity.

Note. Even though $Q(\mathbf{X}, \theta)$ does not depend on θ, it is not an ancillary statistic. It is not even a statistic.

Constructing Confidence Intervals

To construct a γ-level confidence interval,

- Find ℓ and u so that
$$P_\theta\{\ell \le Q(\mathbf{x},\theta) \le u\} = \gamma.$$
- Define
$$C(\mathbf{x}) = \{\mathbf{x}; \ell \le Q(\mathbf{X},\theta) \le u\}.$$

Then
$$P_\theta\{\theta \in C(\mathbf{X})\} = \gamma.$$

If, for each $\mathbf{x}$, $Q(\mathbf{x},\theta)$ is monotone increasing in θ, then let

- $\hat{\theta}_\ell(\mathbf{x})$ be the solution to $Q(\mathbf{x},\theta) = \ell$ and
- $\hat{\theta}_u(\mathbf{x})$ be the solution to $Q(\mathbf{x},\theta) = u$.

Then.
$$C(\mathbf{x}) = \{\mathbf{x}; \hat{\theta}_\ell(\mathbf{x}) \le \theta \le \hat{\theta}_u(\mathbf{x})\}.$$

Constructing Confidence Intervals

For $Q(\mathbf{x}, \theta)$ is monotone decreasing in θ, then reverse the roll of $\hat{\theta}_\ell(\mathbf{x})$ and $\hat{\theta}_u(\mathbf{x})$.

Example. For $\mathbf{X} = (X_1, \ldots, X_n)$ from a location family with density $f(x - \theta)$, then take $Q(\mathbf{X}, \theta) = \bar{X} - \theta$ as the pivot. Then choose ℓ and u so that

$$\begin{aligned}\gamma &= P_\theta\{\mathbf{X}; \ell \le Q(\mathbf{X}, \theta) \le u\} = P_\theta\{\mathbf{X}; \ell \le \bar{X} - \theta \le u\} \\ &= P_\theta\{\mathbf{X} : \ell - \bar{X} \le -\theta \le u - \bar{X}\} = P_\theta\{\mathbf{X}; \bar{X} - \ell \ge \theta \ge \bar{X} - u\}\end{aligned}$$

For $\mathbf{X} = (X_1, \ldots, X_n) \sim N(\theta, \sigma_0^2)$, then take

$$\ell = -\frac{z_{(1-\gamma)/2}\sigma_0}{\sqrt{n}} \quad \text{and} \quad u = \frac{z_{(1-\gamma)/2}\sigma_0}{\sqrt{n}}.$$

Constructing Confidence Intervals

Example. For $\mathbf{X} = (X_1, \ldots, X_n)$ from a scale family with density $\frac{1}{\beta} f\left(\frac{x}{\beta}\right)$, take $Q(\mathbf{X}, \beta) = \sum_{i=1}^n x_i/\beta = T(\mathbf{x})/\beta$ as the pivot. Then choose ℓ and u so that

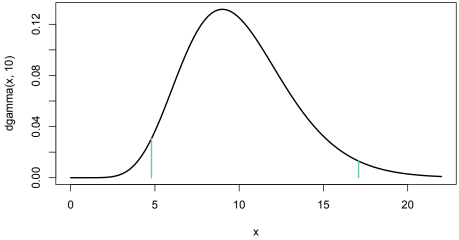
$$
\begin{aligned}
\gamma &= P_\beta\{\mathbf{X}; \ell \leq Q(\mathbf{X}, \beta) \leq u\} \\
&= P_\beta\{\mathbf{X}; \ell \leq T(\mathbf{X})/\beta \leq u\} \\
&= P_\beta\{\mathbf{X}; 1/\ell \geq \beta/T(\mathbf{X}) \geq 1/u\} \\
&= P_\beta\{\mathbf{X}; T(\mathbf{X})/\ell \geq \beta \geq T(\mathbf{X})/u\}
\end{aligned}
$$

For $Exp(\beta)$ random variable, $T(\mathbf{X})/\beta \sim \Gamma(n, 1)$ is a pivot. For an equal tailed 95% confidence interval,

Constructing Confidence Intervals

```
> gamma<-0.95
> alpha<-(1-gamma)/2
> qgamma(c(alpha,1-alpha),10)

[1]  4.795389 17.084803
```



Thus,

$$\begin{aligned} 0.95 &= P_\beta\{\mathbf{X}; 4.795 \leq T(\mathbf{X})/\beta \leq 17.085\} \\ &= P_\beta\{\mathbf{X}; T(\mathbf{X})/4.795 \geq \beta \geq T(\mathbf{X})/17.085\} \end{aligned}$$

Constructing Confidence Intervals

Example. For $\mathbf{X} = (X_1, \ldots, X_n) \sim U(0, \theta)$, take

$$Q(\mathbf{x}, \beta) = \max_i x_i/\theta = T(\mathbf{x})/\theta$$

as the pivot. Then $M = T(\mathbf{x})/\theta$ has distribution function $F_M(t) = t^n$ in the interval $[0, 1]$. For an equal tailed γ-confidence interval, set

$$F_M(u) = \frac{1+\gamma}{2} \quad \text{and} \quad F_M(\ell) = \frac{1-\gamma}{2}.$$

$$u = \left(\frac{1+\gamma}{2}\right)^{1/n} \quad \text{and} \quad \ell = \left(\frac{1-\gamma}{2}\right)^{1/n}.$$

$$\gamma = P_\beta\left\{\mathbf{X};\, T(\mathbf{X})\left(\frac{1-\gamma}{2}\right)^{-1/n} \geq \beta \geq T(\mathbf{X})\left(\frac{1+\gamma}{2}\right)^{-1/n}\right\}.$$

Pivoting Using the Probability Transform

Theorem. Let T be a continuous (test) statistic with distribution function $F_T(t|\theta),\ \theta \in \Theta \subset \mathbb{R}$. Let $\alpha_1 + \alpha_2 = 1 - \gamma$ be the size of the two tails outside the confidence interval.

For $F_T(t|\theta)$ a decreasing function of θ, define $\hat{\theta}_u(t)$ and $\hat{\theta}_\ell(t)$ by

$$F_T(t|\hat{\theta}_u(t)) = \alpha_1 \quad \text{and} \quad F_T(t|\hat{\theta}_\ell(t)) = 1 - \alpha_2,$$

respectively. Then, the random interval

$$[\hat{\theta}_\ell(T), \hat{\theta}_u(T)]$$

is a γ-level confidence interval.

Pivoting Using the Probability Transform

Proof. Because, for any given value of t, $F_T(t|\theta)$ a decreasing function of θ. Thus,

$$1-\alpha_2 > \alpha_1 \quad \text{implies} \quad \hat{\theta}_\ell(t) < \hat{\theta}_u(t).$$

Moreover,

$$\begin{aligned} F_T(t|\theta) < \alpha_1 \quad &\Leftrightarrow \quad \theta > \hat{\theta}_u(t) \\ F_T(t|\theta) > 1-\alpha_2 \quad &\Leftrightarrow \quad \theta < \hat{\theta}_\ell(t). \end{aligned}$$

Consequently,

$$\{\theta; \alpha_1 \leq F_T(T|\theta) \leq 1-\alpha_2\} = \{\theta; \theta_\ell(T) \leq \theta \leq \theta_u(T)\}$$

Remark. The proof can be modified to handle the case of $F_T(t|\theta)$ an increasing function of $\theta$and for one-sided confidence intervals.

Pivoting Using the Probability Transform

Example. For independent $\mathbf{X} = (X_1, \ldots, X_n) \sim U(-\theta, \theta)$, we will use $T(\mathbf{X}) = X_{(n)} - X_{(1)}$ to determine the confidence interval. Set

$$U_i = \frac{X_i + \theta}{2\theta}.$$

Then the independent random variables $\mathbf{U} = (U_1, \ldots, U_n) \sim U(0, 1)$. The

$$T(\mathbf{X}) = X_{(n)} - X_{(1)} = 2\theta((U_{(n)} - \theta) - (U_{(1)} - \theta)) = 2\theta(U_{(n)} - U_{(1)}) = 2\theta D.$$

The distribution

$$F_T(t|\theta) = P_\theta\{T \le t\} = P_\theta\{2\theta D \le t\} = P\{D \le t/(2\theta)\} = F_D(t/(2\theta)),$$

which is a decreasing function of θ.

Pivoting Using the Probability Transform

We have seen that the distribution of $D = U_{(n)} - U_{(1)} \sim Beta(n-1, 2)$.
Let Q_D be the quantile function for D,

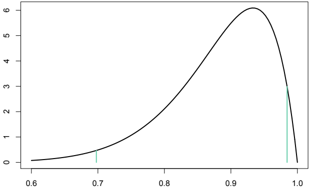
$$\left\{Q_D(\alpha_1) \leq \frac{T(\mathbf{X})}{2\theta} \leq Q_D(1-\alpha_2)\right\} = \left\{\frac{1}{Q_D(\alpha_1)} \geq \frac{2\theta}{T(\mathbf{X})} \geq \frac{1}{Q_D(1-\alpha_2)}\right\}$$
$$= \left\{\frac{T(\mathbf{X})}{2Q_D(\alpha_1)} \geq \theta \geq \frac{T(\mathbf{X})}{2Q_D(1-\alpha_2)}\right\}$$

For $n = 16$ and $\gamma = 0.95$, then the confidence interval with equal probability tails,

```
> gamma<-0.95; alpha<-(1-gamma)/2

> qbeta(c(alpha,1-alpha),15,2)

 [1] 0.6976793 0.9844864
```



Optimizing the Length of the Confidence Interval

As you see the choice of endpoints has the only constraint that the sum of the tail probabilities equals $1-\gamma$. One goal is to have the shortest expected length. This leads to the following constrained optimization problem.

$$\min\{E[\hat{\theta}_u(T)-\hat{\theta}_\ell(T)]; P\{\hat{\theta}_u(T)-\hat{\theta}_\ell(T)\} \geq \gamma\}\}$$

For the case where the density of T is unimodal, to maximize the probability over an interval of length c, we differentiate.

$$\frac{d}{da}(F_T(a+c)-F_T(a)) = f_T(a+c)-f_T(a) = 0.$$

If the mode t_m of the density, then $a < t_m < a+c$. Also,

$$f_T'(a) > 0 \quad \text{and} \quad f_T'(a+c) < 0$$

showing that the probability is maximized. Notice that if the distribution of T is symmetric about its mode, then the equal tailed probabilities are also the shortest.

Optimizing the Length of the Confidence Interval

Returning to the previous example, we want to find the shortest interval for D that has probability γ.

```
> gamma<-0.95
> diff<-function(a) qbeta(gamma+a,15,2)-qbeta(a,15,2)
>(astar<-optimize(diff,interval=c(0.001,0.05),maximum=FALSE))
$minimum
[1] 0.04840239
$objective
[1] 0.2621124
> (u<-qbeta(gamma+astar$minimum,15,2))
[1] 0.9962875
> (l<-qbeta(astar$minimum,15,2))
[1] 0.7341751
> dbeta(c(l,u),15,2)
[1] 0.8433389 0.8457852
```

Optimizing the Length of the Confidence Interval

- Our first confidence interval had equal tails $\alpha_1 = \alpha_2 = 0.025$ and width 0.2868071.
- The density at the left endpoint is lower than on the right.
- This suggests that we can shorten the confidence interval by moving the interval to the right.