Chapter 8

Interval Estimation

Bayesian Approches

Outline

Credible Intervals

A Bayesian interval estimate is called a credible interval. Recall that for the Bayesian approach to statistics, both the data and the parameter are random thus, the interval estimate is a statement about the posterior probability distribution of the parameter θ.

$$P\{\tilde{\Theta} \in C(\mathbf{X}) | \mathbf{X} = \mathbf{x}\} = \gamma.$$

Here $\tilde{\Theta}$ is the random variable having a distribution equal to the prior probability π. We have choices in defining this interval. For example, we can

- choose the narrowest interval, which involves choosing those values of highest posterior density.
- choosing the equal-tail size interval in which the probability of being below the interval is as likely as being above it.

Credible Intervals

Example. For independent flips of a biased coin with a prior distribution $\pi(p) \sim Beta(\alpha, \beta)$,

$$\pi(p) = c_{\alpha,\beta} p^{(\alpha-1)}(1-p)^{(\beta-1)}, \quad 0 < p < 1.$$

If we perform n Bernoulli trials $\mathbf{x} = (x_1, \ldots, x_n)$, then the joint density

$$\mathbf{f}_{X|\tilde{P}}(\mathbf{x}|p) = p^{\sum_{k=1}^n x_k}(1-p)^{n-\sum_{k=1}^n x_k}.$$

Thus the posterior distribution of the parameter $\tilde{P}$ given the data $\mathbf{x}$,

$$\begin{aligned} f_{\tilde{P}|X}(p|\mathbf{x}) \propto \mathbf{f}_{X|\tilde{P}}(\mathbf{x}|p)\pi(p) &= p^{\sum_{k=1}^n x_k}(1-p)^{n-\sum_{k=1}^n x_k} \cdot c_{\alpha,\beta} p^{(\alpha-1)}(1-p)^{(\beta-1)}. \\ &= c_{\alpha,\beta} p^{\alpha+\sum_{k=1}^n x_k - 1}(1-p)^{\beta+n-\sum_{k=1}^n x_k - 1}. \end{aligned}$$

The posterior also has a beta distribution, parameters

$$\alpha + \#\text{successes} \quad \text{and} \quad \beta + \#\text{failures}.$$

Credible Intervals

Using a $Beta(3,3)$ prior, we look at credible intervals after (top) 6 successes in 10 trials and (bottom) 16 successes in 25 trials.

Exercise. Explain the posterior distribution and give 95%, 98% and 99% credible intervals after 10 and 25 trials using the information below

```
> q<-c(0.005,0.01,0.025,0.5,0.975,0.99,0.995)
> round(qbeta(q,9,7),3)
[1] 0.256 0.282 0.323 0.565 0.787 0.821 0.841
> round(qbeta(q,19,12),3)
[1] 0.384 0.406 0.439 0.615 0.773 0.799 0.815
```

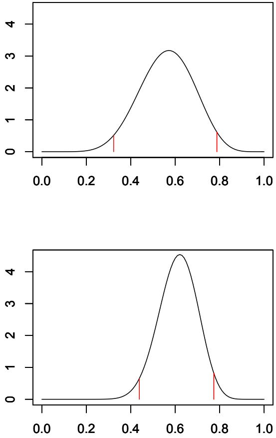

Figure: Posterior densities, indicating 95% credible intervals

Optimizing the Length of a Credible Interval

- Since posterior distribution is a density, we do not face the same issues that were involve in finding confidence intervals for discrete random variables.
- To choose the narrowest interval, we follow a procedure similar to that used in minimizing the length of a confidence interval.

```
> gamma<-0.95
> diff<-function(a) qbeta(gamma+a,9,7)-qbeta(a,9,7)
> (astar<-optimize(diff,interval=c(0.001,0.05),maximum=FALSE))
$minimum
[1] 0.02802784
$objective
[1] 0.4639734
> (u<-qbeta(gamma+astar$minimum,9,7))
[1] 0.7924897
> (l<-qbeta(astar$minimum,9,7))
[1] 0.3285163
```

Credible Intervals

Recall the example of a normal prior on Ψ of normal observations X. We take

- The prior density to be $N(\theta_1, 1/\lambda_0)$
- The observations $X_1, \ldots, X_n$ are independent $N(\theta, \sigma^2)$
- Their mean $\bar{X} \sim N(\theta, \sigma^2/n)$
- The posterior distribution is $N(\theta_1(\bar{x}), \sigma^2/(n + \lambda_0\sigma^2))$ where

$$\theta_1(\bar{x}) = \frac{\lambda_0}{\lambda_0 + n/\sigma^2}\theta_1 + \frac{n/\sigma^2}{\lambda_0 + n/\sigma^2}\bar{x}.$$

Thus a γ- level credible interval is symmetric about $\theta_1(\bar{x})$.

$$\theta_1(\bar{x}) \pm z_{(1-\gamma)/2}\frac{\sigma^2}{n + \lambda_0\sigma^2}.$$

Thus, it is both equal-tailed and narrowest.

Criterion of Choosing Credible Intervals

One approach to choosing a credible interval $C = [a, b]$ $a < b$ is to adopt a loss function. For example, take u to be a strictly increasing function and loss function,

$$L(b-a, \theta) = u(b-a) + cI_{[a,b]^c}(\theta).$$

Thus the loss is the sum of

- a function, u, of the length of the interval plus
- c if the parameter is not in the interval.

Risk, the expected loss, is

$$R(b-a, \theta) = u(b-a) + cP_\theta\{\theta \notin [a, b]\}.$$

The credible interval chosen is the one that minimizes risk.

Comparison of Approaches

Bayesian Approach

- Begins with a prior probability π on the parameter space Θ .
- Uses the data and textcolortealBayes formula to compute the posterior probability.
- The textcolortealcredible interval having textcolortealposterior probability γ is the probability that θ, viewed as a random quantity, lies in the interval.

Classical Approach

- Begins with a family of distributions indexed with a parameter Θ.
- Uses a significance level to construct to create a random interval $[\hat{\theta}_\ell(\mathbf{x}), \hat{\theta}_u(\mathbf{x})]$.
- The creation of the confidence interval can be considered as a single textcolortealBernoulli trial with probability of success γ.