# Topic 17

## Simple Hypotheses

### Examples

# Outline

# Mimicry

- Mimicry is the similarity of one species to another in a manner that enhances the survivability of one or both species - the model and mimic.
- This similarity can be, for example, in appearance, behavior, sound, or scent.
- One method for producing a mimic species is hybridization. This results in the transferring of adaptations from the model species to the mimic.

The genetic signature of this has recently been discovered in *Heliconius* butterflies with a region of the chromosome that displays an almost perfect genotype by phenotype association across four species in the genus *Heliconius*.

# Mimicry

Consider a model butterfly species with mean wingspan $\mu_0 = 10$ cm and a mimic species with mean wingspan $\mu_1 = 7$ cm. Both species have standard deviation $\sigma_0 = 3$ cm. Collect 16 specimen to decide if the mimic species has migrated into a given region. If we assume, for the null hypothesis, that the habitat under study is populated by the model species, then

- a type I error is falsely concluding that the species is the mimic when indeed the model species is resident and
- a type II error is falsely concluding that the species is the model when indeed the mimic species has invaded.

If our action is to begin an eradication program if the mimic has invaded, then a type I error would result in the eradication of the resident model species and a type II error would result in the letting the invasion by the mimic take its course.

# Mimicry

To begin, we set a significance level. The choice of an $\alpha = 0.05$ test means that we are accepting a $5\%$ chance of having a type I error. If the goal is to design a test that has the lowest type II error probability, then the Neyman-Pearson lemma tells us that the critical region is determined by a threshold level $k_\alpha$ for the likelihood ratio.

$$C = \left\{ \mathbf{x};\ \frac{L(\mu_1|\mathbf{x})}{L(\mu_0|\mathbf{x})} \geq k_\alpha \right\}.$$

Our model is $X = (X_1, \ldots, X_n)$, independent normal observations with unknown mean and known variance $\sigma_0^2$. The hypothesis is

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu = \mu_1.$$

# Normal Observations

We look to determine the critical region.

$$\frac{L(\mu_1|\mathbf{x})}{L(\mu_0|\mathbf{x})} = \frac{\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp -\frac{(x_1-\mu_1)^2}{2\sigma_0^2}\cdots\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp -\frac{(x_n-\mu_1)^2}{2\sigma_0^2}}{\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp -\frac{(x_1-\mu_0)^2}{2\sigma_0^2}\cdots\frac{1}{\sqrt{2\pi\sigma_0^2}}\exp -\frac{(x_n-\mu_1)^2}{2\sigma_0^2}}$$

$$= \frac{\exp -\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu_1)^2}{\exp -\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu_0)^2} = \exp -\frac{1}{2\sigma_0^2}\sum_{i=1}^n \left((x_i-\mu_1)^2-(x_i-\mu_0)^2\right)$$

$$= \exp -\frac{\mu_0-\mu_1}{2\sigma_0^2}\sum_{i=1}^n (2x_i-\mu_1-\mu_0)$$

Because the exponential function is increasing, the critical region are those $\mathbf{x}$ so that

$$\frac{\mu_1-\mu_0}{2\sigma_0^2}\sum_{i=1}^n (2x_i-\mu_1-\mu_0) \quad \text{exceeds some critical value.}$$

# Normal Observations

$$\frac{\mu_1 - \mu_0}{2\sigma_0^2} \sum_{i=1}^{n} (2x_i - \mu_1 - \mu_0) \quad \text{exceeds some critical value.}$$

Because $\mu_1 < \mu_0$, this is equivalent to $\bar{x}$ bounded by some critical value,

$$\bar{x} \leq \tilde{k}_\alpha,$$

where $\tilde{k}_\alpha$ is chosen to satisfy

$$P_{\mu_0}\{\bar{X} \leq \tilde{k}_\alpha\} = \alpha.$$

If we assume that $\mu = \mu_0$, then $\bar{X}$ is $N(\mu_0, \sigma_0/\sqrt{n})$ and consequently the standardized version of $\bar{X}$,

$$Z = \frac{\bar{X} - \mu_0}{\sigma_0/\sqrt{n}},$$

is a standard normal. Set $z_\alpha$ so that $P\{Z \leq -z_\alpha\} = \alpha$.

## Normal Observations

Under the null hypothesis, $\bar{X}$ has a normal distribution with mean $\mu_0 = 10$ and standard deviation $\sigma/\sqrt{n} = 3/4$. This using the distribution function of the normal we can find critical values in R with

- `qnorm(0.05,10,3/4)`, yielding $\tilde{k}_\alpha = 8.767$ for the test statistic $\bar{x}$ or
- `qnorm(0.05)` yielding $\tilde{z}_\alpha = -1.645$ for the test statistic $z$.

Now let's look at data.

```
> x
 [1]  6.8 9.5 6.0 8.5 11.7 9.7 7.6 8.0 8.4 6.7 10.5 9.3 6.2 14.4 12.6 9.7
> mean(x)
[1] 9.1
```

$$\text{Then} \qquad \bar{x} = 9.1 \quad \text{and} \quad z = \frac{9.1 - 10}{3/\sqrt{16}} = -1.2.$$

$\tilde{k}_\alpha = 8.766 < 9.1$ or $-z_\alpha = -1.645 < -1.2$ and we *fail* to reject the null hypothesis.

# Power

Exercise. Give an intuitive explanation why the power should

- increase as a function of $|\mu_1 - \mu_0|$,
- decrease as a function of $\sigma_0^2$, and
- increase as a function of $n$.

Next we determine the type II error probability. We will be guided by the fact that

$$\frac{\bar{X} - \mu_1}{\sigma_0/\sqrt{n}}$$

is a standard normal random variable in the case that $H_1 : \mu = \mu_1$ is *true*.

# Power

Exercise. Let $z_\alpha$ be the $\alpha$-upper tail probability. Show that

$$\frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}} < -z_\alpha \quad \text{if and only if} \quad \frac{\bar{X}-\mu_1}{\sigma_0/\sqrt{n}} < -z_\alpha + \frac{\mu_0-\mu_1}{\sigma_0/\sqrt{n}}.$$

So the power is the probability of rejecting $H_0$ when $H_1$ is true. Assuming $\mu = \mu_1$,

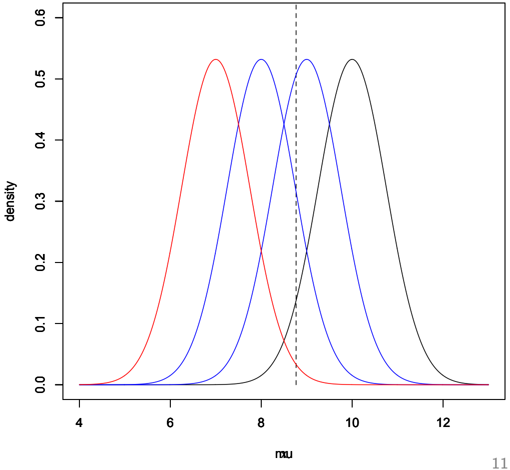
$$1-\beta = P_{\mu_1}\left\{\frac{\bar{X}-\mu_1}{\sigma_0/\sqrt{n}} < -z_\alpha + \frac{\mu_0-\mu_1}{\sigma_0/\sqrt{n}}\right\} = \Phi\left(-z_\alpha + \frac{\mu_0-\mu_1}{\sigma_0/\sqrt{n}}\right).$$

```
> alpha<-0.05;zalpha<-qnorm(1-alpha);mu0<-10;mu<-7;sigma<-3;n<-16
> power<-pnorm(-zalpha+(mu0-mu)/(sigma/sqrt(n)))
> power
[1] 0.9907423
```

The type II error probability is $\beta = 1 - 0.9907 = 0.0093$, a bit under $1\%$.

# Power

- Density of $\bar{X}$ for normal data under the null hypothesis - $\mu_0 = 10$ (black) and $\sigma_0/\sqrt{n} = 3/\sqrt{16} = 3/4$.
- With an $\alpha = 0.05$ level test, the critical value $\tilde{k}_\alpha = \mu_0 - z_\alpha \sigma_0/\sqrt{n} = 8.766$.
- The alternatives shown are
  - $\mu_1 = 9$ (blue) power $0.3777$,
  - $\mu_1 = 8$ (blue) power $0.8466$, and
  - $\mu_1 = 7$ (red) power $0.9907$.



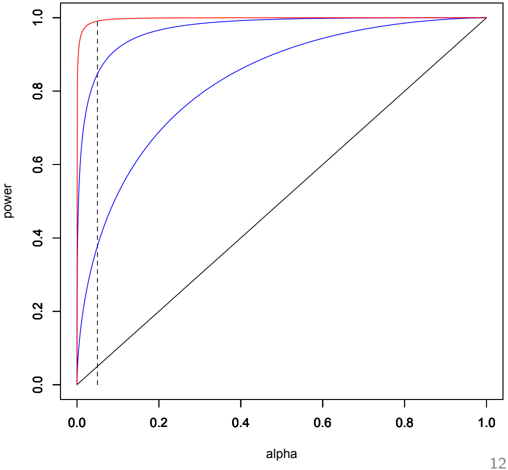
# Receiver Operator Characteristic

The corresponding receiver operator characteristics curves of the power $1-\beta$ versus the significance $\alpha$ using equation

$$1-\beta = \Phi\left(-z_\alpha + \frac{\mu_1 - \mu_0}{\sigma_0/\sqrt{n}}\right).$$

The power for an $\alpha = 0.05$ test indicated by the intersection of vertical dashed line and the curves.

- $\mu_1 = 9$ (blue) power $0.3777$,
- $\mu_1 = 8$ (blue) power $0.8466$, and
- $\mu_1 = 7$ (red) power $0.9907$.



## Power versus Sample Size

Power as a function of the number of observations $n$ for

- An $\alpha = 0.01$ level test is chosen to reflect a stringent criterion for rejecting the null hypothesis that the resident species is the model species.
- The null hypothesis - $\mu_0 = 10$.
- The alternatives shown are $\mu_1 = 9, 8$ (blue) and $\mu_1 = 7$ (red).