Topic 17

Simple Hypotheses

Introduction to Bayesian Approach

Outline

Overview

As with other aspects of the Bayesian approach to statistics, hypothesis testing is closely aligned with Bayes theorem. For a simple hypothesis, we begin with a prior probability for each of the competing hypotheses.

$$\pi\{\theta_0\} = P\{H_0 \text{ is true}\} \quad \text{and} \quad \pi\{\theta_1\} = P\{H_1 \text{ is true}\}.$$

Naturally, $\pi\{\theta_0\} + \pi\{\theta_1\} = 1$. Although this is easy to state, the choice of a prior ought to be grounded in solid scientific reasoning.

As before, we collect data and with it compute the posterior probabilities of the two parameter values θ_0 and θ_1. This gives us the posterior probabilities that H_0 is true and H_1 is true.

Comparison of Approaches

Bayesian Approach

- Begins with a prior probability that H_0 is true.
- Uses the data and Bayes formula to compute the posterior probability that H_1 is true.
- The decision to reject H_0 is based on minimizing risk using presumed values for losses for both type I and type II errors.

Classical Approach

- Begins with the assumption is that H_0 is true.
- Uses a significance level to construct a critical region to make a decision to reject H_0.
- The decision to reject H_0 is based on whether or not the data land in the critical region. The region is chosen to minimize type II errors.

The question: *What is the probability that H_1 is true?* has *no* meaning in the classical setting.

Formulation

Recall Bayes formula for events A and C,

$$P(C|A) = \frac{P(A|C)P(C)}{P(A|C)P(C) + P(A|C^c)P(C^c)},$$

$$C = \{\tilde{\Theta} = \theta_1\} = \{H_1 \text{ is true}\} \quad \text{and} \quad A = \{X = \mathbf{x}\}.$$

For discrete data, we have the conditional probabilities for the alternative hypothesis.

$$P(A|C) = P_{\theta_1}\{X = \mathbf{x}\} = f_X(\mathbf{x}|\theta_1) = L(\theta_1|\mathbf{x}).$$

Similarly, for the null hypothesis,

$$P(A|C^c) = P_{\theta_0}\{X = \mathbf{x}\} = f_X(\mathbf{x}|\theta_0) = L(\theta_0|\mathbf{x}).$$

The posterior probability that H_1 is true

$$f_{\tilde{\Theta}|X}(\theta_1|\mathbf{x}) = P\{H_1 \text{ is true}|X = \mathbf{x}\} = P\{\tilde{\Theta} = \theta_1|X = \mathbf{x}\}$$

Formulation

Returning to Bayes formula, we make the substitutions,

$$P(C|A) = \frac{P(A|C)P(C)}{P(A|C)P(C) + P(A|C^c)P(C^c)},$$

$$f_{\tilde{\Theta}|X}(\theta_1|\mathbf{x}) = \frac{L(\theta_1|\mathbf{x})\pi\{\theta_1\}}{L(\theta_0|\mathbf{x})\pi\{\theta_0\} + L(\theta_1|\mathbf{x})\pi\{\theta_1\}}.$$

Rewrite the expression above in terms of odds, i. e., as the ratio of probabilities.

$$\frac{f_{\tilde{\Theta}|X}(\theta_1|\mathbf{x})}{f_{\tilde{\Theta}|X}(\theta_0|\mathbf{x})} = \frac{P\{H_1 \text{ is true}|X = \mathbf{x}\}}{P\{H_0 \text{ is true}|X = \mathbf{x}\}} = \frac{P\{\tilde{\Theta} = \theta_1|X = \mathbf{x}\}}{P\{\tilde{\Theta} = \theta_0|X = \mathbf{x}\}} = \frac{L(\theta_1|\mathbf{x})}{L(\theta_0|\mathbf{x})} \cdot \frac{\pi\{\theta_1\}}{\pi\{\theta_0\}}.$$

With this expression we see that the posterior odds are equal to the likelihood ratio times the prior odds.

Formulation

Relying on these odds alone failed to take into account that impact of an incorrect decision.

The decision whether or not to reject H_0 depends on the values assigned for the loss obtained in making such a conclusion. We begin by setting values for the loss.

loss function table		
decision	H_0 is true	H_1 is true
H_0	0	ℓ_{II}
H_1	ℓ_{I}	0

The Bayes procedure is to make the decision that has the smaller posterior expected loss, also known as the risk.

Formulation

If the decision is H_0, the loss $\mathcal{L}_0(\mathbf{x})$ takes on two values

$$\mathcal{L}_0(\mathbf{x}) = \begin{cases} 0 & \text{with probability } P\{H_0 \text{ is true}|X = \mathbf{x}\}, \\ \ell_{\mathrm{II}} & \text{with probability } P\{H_1 \text{ is true}|X = \mathbf{x}\}. \end{cases}$$

In this case, the expected loss

$$E\mathcal{L}_0(\mathbf{x}) = \ell_{\mathrm{II}} P\{H_1 \text{ is true}|X = \mathbf{x}\}$$

is a product of the loss and the probability of incorrectly choosing H_0.

Exercise. If the decision is H_1, the expected loss

$$E\mathcal{L}_1(\mathbf{x}) = \ell_{\mathrm{I}} P\{H_0 \text{ is true}|X = \mathbf{x}\}.$$

Formulation

So, we reject H_0 whenever the risk is greater for H_0 than H_1

$$E\mathcal{L}_0(\mathbf{x}) > E\mathcal{L}_1(\mathbf{x})$$

or

$$1 < \frac{E\mathcal{L}_0(\mathbf{x})}{E\mathcal{L}_1(\mathbf{x})} = \frac{\ell_{\mathrm{II}} P\{H_1 \text{ is true}|X = \mathbf{x}\}}{\ell_{\mathrm{I}} P\{H_0 \text{ is true}|X = \mathbf{x}\}}.$$

Stated in terms of odds,

$$\frac{\ell_{\mathrm{II}}}{\ell_{\mathrm{I}}} < \frac{P\{H_1 \text{ is true}|X = \mathbf{x}\}}{P\{H_0 \text{ is true}|X = \mathbf{x}\}} = \frac{L(\theta_1|\mathbf{x})}{L(\theta_0|\mathbf{x})} \cdot \frac{\pi\{\theta_1\}}{\pi\{\theta_0\}},$$

and

$$\frac{L(\theta_1|\mathbf{x})}{L(\theta_0|\mathbf{x})} > \frac{\ell_{\mathrm{II}}/\pi\{\theta_1\}}{\ell_{\mathrm{I}}/\pi\{\theta_0\}}.$$

Formulation

$$\frac{L(\theta_1|\mathbf{x})}{L(\theta_0|\mathbf{x})} > \frac{\ell_{\mathrm{II}}/\pi\{\theta_1\}}{\ell_{\mathrm{I}}/\pi\{\theta_0\}}$$

Thus, the criterion for rejecting H_0 is a level test on the likelihood ratio, exactly the same type of criterion used in classical statistics. However, the rationale, thus the value for the ratio necessary to reject, can be quite different.

For normal observations with means μ_0 for the null hypothesis and μ_1 for the alternative hypothesis. If the standard deviation has a known value, σ_0, we have the likelihood ratio

$$\frac{L(\mu_1|\mathbf{x})}{L(\mu_0|\mathbf{x})} = \exp -\frac{\mu_0 - \mu_1}{2\sigma_0^2} \sum_{i=1}^{n} (2x_i - \mu_1 - \mu_0) = \exp\left(-\frac{\mu_0 - \mu_1}{2\sigma_0^2} n(2\bar{x} - \mu_1 - \mu_0) \right).$$

Example

Exercise. For the example on the model and mime butterfly species, $\mu_0 = 10$, $\mu_1 = 7$, $\sigma_0 = 3$, and $n = 16$ observations.

- Determine the likelihood ratio for $\bar{x} = 9.1$.
- Choose values for the ratios ℓ_{II}/ℓ_{I} and $\pi\{\theta_1\}/\pi\{\theta_0\}$ for which we fail to reject the null hypothesis. Describe what this choice means.
- Choose values for the ratios for which we reject the null hypothesis. Describe what this choice means.